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DONALD GORDON JAMES

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D. G. JAMES

Let $\mathscr{E}$ denote the spinorial kernel of an orthogonal group of an indefinite unimodular quadratic form over the integers in a global field. The normal subgroups of $\mathscr{E}$ that arise from the local structure of $\mathscr{E}$ are studied.

Let S be a Dedekind set of spots on a global field F with characteristic not two and $\mathfrak{o}$ the ring of integers of F at S. Let V be a finite dimensional quadratic space over F with dimension at least 5 and associated bilinear form B and quadratic form q. The orthogonal group of V is

$$O(V) = \{\varphi \in \operatorname{End} V \mid q(\varphi(x)) = q(x) \quad \text{for all} \quad x \in V\} .$$

Assume V supports a unimodular lattice M with orthogonal group $O(M) = \{\varphi \in O(V) \mid \varphi(M) = M\}$. At each spot $\mathfrak{p} \in S$ we can localize and consider the orthogonal group $O(M_\mathfrak{p})$ of the isotropic unimodular lattice $M_\mathfrak{p}$ over the ring of integers $\mathfrak{o}_\mathfrak{p}$ in the local field $F_\mathfrak{p}$. The subgroups of $O(M_\mathfrak{p})$ normalized by its commutator subgroup have been classified in [1, 2, 3]. We show here how this local structure can be injected into $O(M)$ when S is an indefinite set of spots for V. A rich structure of normal subgroups of the spinorial kernel of $O(M)$ is then provided by the local behaviour at the dyadic spots. Most of our terminology and notation is taken from O'Meara [4].

1. $\mathscr{E}$-invariant sublattices of M. $M_\mathfrak{p}$ is split by a hyperbolic plane $H_\mathfrak{p}$ at each spot $\mathfrak{p} \in S$. Write $M_\mathfrak{p} = H_\mathfrak{p} \perp K_\mathfrak{p}$ and $H_\mathfrak{p} = \mathfrak{o}_\mathfrak{p} u_\mathfrak{p} + \mathfrak{o}_\mathfrak{p} v_\mathfrak{p}$ where $B_\mathfrak{p}(u_\mathfrak{p}, v_\mathfrak{p}) = 1$ and $q_\mathfrak{p}(u_\mathfrak{p}) = q_\mathfrak{p}(v_\mathfrak{p}) = 0$. In [2] it is shown that $\mathscr{E}_p$, the group generated by the Siegel transformations $E(u_\mathfrak{p}, x_\mathfrak{p})$ and $E(v_\mathfrak{p}, x_\mathfrak{p})$, is equal to the spinorial kernel of $O(M_\mathfrak{p})$. Define

$$\mathscr{E} = \{\varphi \in O(M) \mid \varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p} \quad \text{for all} \quad \mathfrak{p} \in S\} .$$

Note that $\mathscr{E}$ is defined if $\dim V \geqq 5$, even when V is anisotropic. In fact, using the global square theorem [4, 65:15] and [2, Theorem 2.9], it is easily seen that $\mathscr{E}$ is the spinorial kernel of $O(M)$.

A sublattice P of M is called $\mathscr{E}$-invariant if $\varphi(P) = P$ for all $\varphi \in \mathscr{E}$. Let M_* denote the lattice

$$M_* = \{x \in M \mid q(x) \in \mathfrak{o}\}$$

with dual lattice

$$M^* = \{x \in V \mid B(x, M_*) \subseteqq \mathfrak{o}\} .$$

Define an ideal $\mathfrak{a}(P)$ in $\mathfrak{o}$ by

$$\mathfrak{a}(P) = \sum_{r \in P} B(r, M_*) .$$

Then

$$\mathfrak{a}(P)^{-1}P \subseteqq M^* .$$

LEMMA 1.1. *Let P be sublattice of M and*

$$M_* \subseteqq \mathfrak{a}(P)^{-1}P \subseteqq M^* .$$

Then P is $\mathscr{E}$-invariant.

Proof. Locally, $P_\mathfrak{p}$ is $\mathscr{E}_\mathfrak{p}$-invariant [2, Theorem 3.1] since $(M_*)_\mathfrak{p} = (M_\mathfrak{p})_*$, $\mathfrak{a}(P_\mathfrak{p}) = \mathfrak{a}(P)_\mathfrak{p}$ and

$$(M_\mathfrak{p})_* \subseteqq \mathfrak{a}(P_\mathfrak{p})^{-1}P_\mathfrak{p} \subseteqq (M_\mathfrak{p})^* .$$

Take $\varphi \in \mathscr{E}$ and let $\varphi(P) = Q$. Then $\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}$ and

$$P_\mathfrak{p} = \varphi_\mathfrak{p}(P_\mathfrak{p}) = \varphi(P)_\mathfrak{p} = Q_\mathfrak{p}$$

for all $\mathfrak{p} \in S$. Hence $P = Q$ by [4, 81E].

THEOREM 1.2. *Let S be an indefinite set of spots for V and assume* $\dim V \geqq 7$ *if* $[\mathfrak{o}_\mathfrak{p}: \mathfrak{p}] = 2$ *at any dyadic spot. Then a sublattice P of M is $\mathscr{E}$-invariant if and only if*

$$M_* \subseteqq \mathfrak{a}(P)^{-1}P \subseteqq M^* .$$

Proof. We need only show that if P is $\mathscr{E}$-invariant, then $(M_\mathfrak{p})_* \subseteqq \mathfrak{a}(P_\mathfrak{p})^{-1}P_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Fix $\mathfrak{p} \in S$ and take $\theta_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}$. By the strong approximation theorem for rotations [4, 104: 4], there exists $\varphi \in O(V)$ such that $\|\varphi - \theta_\mathfrak{p}\|_\mathfrak{p} < \varepsilon$ and $\|\varphi\|_\mathfrak{q} = 1$ for $\mathfrak{q} \neq \mathfrak{p}$. Then $\varphi \in O(M)$ and, since $\mathscr{E}$ is the spinorial kernel, $\varphi \in \mathscr{E}$. Hence $\varphi(P) = P$. By making ε sufficiently small, we may assume $(\varphi_\mathfrak{p} - \theta_\mathfrak{p})(P_\mathfrak{p}) \subseteqq P_\mathfrak{p}$. Thus $\theta_\mathfrak{p}(P_\mathfrak{p}) = P_\mathfrak{p}$ for all $\theta_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}$. Hence $P_\mathfrak{p}$ is $\mathscr{E}_\mathfrak{p}$-invariant and by [2, Theorem 3.1] it follows that $\mathfrak{a}(P_\mathfrak{p})(M_\mathfrak{p})_* \subseteqq P_\mathfrak{p}$.

THEOREM 1.3. *For each dyadic spot $\mathfrak{p}$ assume given an $\mathfrak{o}_\mathfrak{p}$-lattice $J_\mathfrak{p}$ with $(M_\mathfrak{p})_* \subseteqq J_\mathfrak{p} \subseteqq (M_\mathfrak{p})^*$. Then there exists an $\mathfrak{o}$-lattice P such that $P_\mathfrak{p} = J_\mathfrak{p}$ at each dyadic spot and $M_* \subseteqq P \subseteqq M^*$.*

Proof. This follows immediately from [4, 81: 14] since $(M_\mathfrak{p})_* = (M_\mathfrak{p})^* = M_\mathfrak{p}$ at all nondyadic spots.

REMARK 1.4. If 2 is unramified, the lattices satisfying $(M_2)_* \subseteqq P \subseteqq (M_2)^*$ were determined in [1]. When 2 ramifies the number of such lattices proliferates (even for the gaussian integers, there are lattices M having more than 20 such P).

2. Normal subgroups of $\mathscr{E}$.

Let Ξ be an indexing set such that M_ξ, $\xi \in \Xi$, gives all the lattices on V such that

$$M_* \subseteqq M_\xi \subseteqq M^* .$$

Let $\mathfrak{a}$ be an ideal in $\mathfrak{o}$ such that $\mathfrak{a}M_\xi \subseteqq M_*$. Then $\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}} \subseteqq (M_\mathfrak{p})_*$ and we define $\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})$ as in [2] as the normal subgroup of $\mathscr{E}_\mathfrak{p}$ generated by all isometries of the form $\theta_\mathfrak{p} E(u_\mathfrak{p}, x_\mathfrak{p})\theta_\mathfrak{p}^{-1}$ or $\theta_\mathfrak{p} E(v_\mathfrak{p}, x_\mathfrak{p})\theta_\mathfrak{p}^{-1}$ where $\theta_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}$ and $x_\mathfrak{p} \in K_\mathfrak{p} \cap \mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}$. Define

$$\mathscr{E}(\mathfrak{a}M_\xi) = \{\varphi \in \mathscr{E} \mid \varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}) \quad \text{for all} \quad \mathfrak{p} \in S\}$$

and

$$\mathscr{F}(\mathfrak{a}M_\xi) = \{\varphi \in \mathscr{E} \mid [\varphi, \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi)\} .$$

Then $\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$ and any subgroup $\mathscr{N}$ of $\mathscr{E}$ such that

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$$

is a normal subgroup of $\mathscr{E}$.

We also define the local group $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})$ by

$$\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}) = \{\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p} \mid [\varphi_\mathfrak{p}, \mathscr{E}_\mathfrak{p}] \subseteqq \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})\} .$$

Note that this definition is more restrictive than that in [2] where $\varphi_\mathfrak{p}$ was taken in $O(M_\mathfrak{p})$, not $\mathscr{E}_\mathfrak{p}$.

THEOREM 2.1. *Let S be an indefinite set of spots for V and* $\dim V \geqq 5$. *Then for any ideal $\mathfrak{a} \neq \{0\}$ with $\mathfrak{a}M_\xi \subseteqq M_*$,*

$$\mathscr{F}(\mathfrak{a}M_\xi)/\mathscr{E}(\mathfrak{a}M_\xi) \cong \prod_{\mathfrak{p}\mid\mathfrak{a}} \mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}) .$$

Proof. Observe first that for $\mathfrak{p}\nmid\mathfrak{a}$ we have

$$\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}) = \mathscr{E}_\mathfrak{p} = \mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}}) .$$

Define a mapping

$$\Gamma\colon \mathscr{F}(\mathfrak{a}M_\xi) \longrightarrow \prod_{\mathfrak{p}\mid\mathfrak{a}} \mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})$$

by sending $\varphi \in \mathscr{F}(\mathfrak{a}M_\xi)$ into $(\cdots, \tilde{\varphi}_\mathfrak{p}, \cdots)_{\mathfrak{p}\mid\mathfrak{a}}$ where $\tilde{\varphi}_\mathfrak{p}$ denotes the coset determined by $\varphi_\mathfrak{p}$ in $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})$. It is necessary to show that Γ is well-defined, namely $\varphi_\mathfrak{p} \in \mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p} M_{\xi\mathfrak{p}})$ for each $\mathfrak{p}\mid\mathfrak{a}$. Let

$\psi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}$ and take $\theta \in \mathscr{E}$ such that $\|\psi_\mathfrak{p}^{-1}\theta - I\|_\mathfrak{p} < \varepsilon$. Then $[\varphi, \theta]$ is in $\mathscr{E}(\mathfrak{a}M_\xi)$ and hence $[\varphi_\mathfrak{p}, \theta_\mathfrak{p}]$ is in $\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. For sufficiently small ε, it follows from Corollary 3.3 that $\psi_\mathfrak{p}^{-1}\theta_\mathfrak{p}$ is in $\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ and hence $[\varphi_\mathfrak{p}, \psi_\mathfrak{p}] \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. Thus Γ is a well-defined group homomorphism.

φ is in the kernel of Γ if and only if $\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ for each $\mathfrak{p}|\mathfrak{a}$. But $\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p} = \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ for $\mathfrak{p}\nmid\mathfrak{a}$. Hence the kernel of Γ is $\mathscr{E}(\mathfrak{a}M_\xi)$.

It remains to show that Γ is surjective. For each $\mathfrak{p}|\mathfrak{a}$ fix $\psi_\mathfrak{p}$ in $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. By the strong approximation theorem, there exists $\varphi \in \mathscr{E}$ such that

$$\|\psi_\mathfrak{p}^{-1}\varphi - I\|_\mathfrak{p} < \varepsilon \quad \text{for} \quad \mathfrak{p}|\mathfrak{a}\,.$$

It now suffices to show $\varphi \in \mathscr{F}(\mathfrak{a}M_\xi)$, for if ε is sufficiently small, $\psi_\mathfrak{p}^{-1}\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ by Corollary 3.3 and hence $\psi_\mathfrak{p}$ and $\varphi_\mathfrak{p}$ determine the same coset in $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. We must show $[\varphi_\mathfrak{p}, \theta_\mathfrak{p}] \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ for any $\theta \in \mathscr{E}$ and all $\mathfrak{p} \in S$. For $\mathfrak{p}\nmid\mathfrak{a}$ this is trivial. For $\mathfrak{p}|\mathfrak{a}$ and ε sufficiently small, $\chi_\mathfrak{p} = \psi_\mathfrak{p}^{-1}\varphi_\mathfrak{p} \in \mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$ and hence

$$[\varphi_\mathfrak{p}, \theta_\mathfrak{p}] = [\psi_\mathfrak{p}\chi_\mathfrak{p}, \theta_\mathfrak{p}] = [\psi_\mathfrak{p}, \chi_\mathfrak{p}\theta_\mathfrak{p}\chi_\mathfrak{p}^{-1}]\,[\chi_\mathfrak{p}, \theta_\mathfrak{p}]$$

is in $\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. This completes the proof.

REMARK 2.2. We restricted ourselves to Dedekind domains coming from global fields so that the strong approximation theorem for rotations could be used. If, however, we assume M has hyperbolic rank at least one, so that globally $M = H \perp K$ with $H = \mathfrak{o}u + \mathfrak{o}v$ a hyperbolic plane and K is free, a strong approximation theorem can still be established for any Dedekind domain. Define all $\mathscr{E}_\mathfrak{p}$ with respect to the localization of H. Then, from [2, Theorem 2.9], any $\theta_\mathfrak{p}$ in the spinorial kernel is of the form

$$\theta_\mathfrak{p} = \prod_{i=1}^{n(\mathfrak{p})} E(u, x_i(\mathfrak{p}))E(v, y_i(\mathfrak{p}))$$

with $x_i(\mathfrak{p})$ and $y_i(\mathfrak{p})$ in $(K_\mathfrak{p})_*$. By approximating to the coefficients of $x_i(\mathfrak{p})$ and $y_i(\mathfrak{p})$ in the Dedekind domain $\mathfrak{o}$, we can approximate to $\theta_\mathfrak{p}$ by an isometry in $O(M)$ for a finite number of $\mathfrak{p}$.

3. **The structure of $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$.** We close with some comments on the structure of the abelian group $\mathscr{F}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})/\mathscr{E}_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}M_{\xi\mathfrak{p}})$. For simplicity, since only the local situation is considered in this section, the suffix $\mathfrak{p}$ is dropped and we write $\mathfrak{o}$ for $\mathfrak{o}_\mathfrak{p}$, M for $M_\mathfrak{p}$, V for $V_\mathfrak{p}$, and so on.

For $\xi \in \Xi$, let

$$M^\xi = \{x \in V \mid B(x, M_\xi) \subseteq \mathfrak{o}\}$$

be the dual lattice of M_ξ. Write $M_* = H \perp K_*$ and $M_\xi = H \perp K_\xi$. Then $K_* \subseteqq K_\xi$ and we can choose a basis $x_1, \cdots, x_n$ for K_* such that

$$K_* = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$$

and

$$K_\xi = \mathfrak{p}^{-k_1}x_1 + \cdots + \mathfrak{p}^{-k_n}x_n$$

with the integers k_i invariants for K_*, K_ξ and

$$0 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_n .$$

Let $y_1, \cdots, y_n$ be the dual basis of $x_1, \cdots, x_n$. Then

$$K^* = \mathfrak{o}y_1 + \cdots + \mathfrak{o}y_n$$

and

$$K^\xi = \mathfrak{p}^{k_1}y_1 + \cdots + \mathfrak{p}^{k_n}y_n .$$

For each $1 \leqq i \leqq n$, define

$$\begin{aligned} K_{\xi(i)} &= K_* + \mathfrak{p}^{k_i}K_\xi \\ &= \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_i + \mathfrak{p}^{k_i-k_{i+1}}x_{i+1} + \cdots + \mathfrak{p}^{k_i-k_n}x_n \end{aligned}$$

and let $K^{\xi(i)}$, $1 \leqq i \leqq n$, be the dual lattices. Write $M_{\xi(i)} = H \perp K_{\xi(i)}$ and $M^{\xi(i)} = H \perp K^{\xi(i)}$. Then

$$M_\xi \supseteqq M_{\xi(1)} \supseteqq M_{\xi(2)} \supseteqq \cdots \supseteqq M_{\xi(n)} = M_*$$

and

$$M^\xi \subseteqq M^{\xi(1)} \subseteqq M^{\xi(2)} \subseteqq \cdots \subseteqq M^{\xi(n)} = M^* .$$

Define congruence subgroups by

$$\begin{aligned} O(\mathfrak{a}M_\xi) &= \{\varphi \in \mathscr{E} \mid \varphi(r) \equiv r \bmod \mathfrak{a}M_\xi \quad \text{for all} \quad r \in M_*\} , \\ O(\mathfrak{a}M_\xi)^i &= \{\varphi \in O(\mathfrak{a}M_\xi) \mid \varphi(r) \equiv r \bmod 2\mathfrak{a}\mathfrak{p}^{-k_i}M_{\xi(i)} \quad \text{for all} \quad r \in 2M^{\xi(i)}\} \end{aligned}$$

for $1 \leqq i \leqq n$, and

$$O(\mathfrak{a}M_\xi)^\# = \bigcap_{i=1}^{n} O(\mathfrak{a}M_\xi)^i .$$

Since $\mathfrak{p}^{k_i}M_\xi \subseteqq M_{\xi(i)}$, $1 \leqq i \leqq n$, it can be checked that

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq O(\mathfrak{a}M_\xi)^\# \subseteqq O(\mathfrak{a}M_\xi) \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$$

provided that $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$.

THEOREM 3.1. *For $\mathfrak{a} \subseteqq 4\mathfrak{p}$, the subgroup $O(\mathfrak{a}M_\xi)^\#$ is generated by $\mathscr{E}(\mathfrak{a}M_\xi)$ and the isometries $\Phi(\varepsilon)$ with $\varepsilon \equiv 1 \bmod \mathfrak{a}$.*

Proof. The isometries $\Phi(\varepsilon)$ are defined in [2]. Let $\varphi \in O(\mathfrak{a}M_\xi)^\#$; changing φ by the given isometries, we will reduce it to the identity mapping. Assume for some $m \leqq n$ that $\varphi(y_j) = y_j$, $1 \leqq j \leqq m - 1$, (consider $\varphi \in O(V)$). Let

$$\varphi(u + y_m) = \varepsilon u + \beta v + y_m + \sum_{i=1}^{n} \alpha_i x_i \, .$$

Since $\varphi(y_j) = y_j$, it follows that $\alpha_j = 0$ for $1 \leqq j \leqq m - 1$. For $j \geqq m$, $u + y_m \in M^{\xi(j)}$ and hence $\alpha_j x_j \in \mathfrak{a}\mathfrak{p}^{-k_j} M_{\xi(j)}$. Thus $\alpha_j \in \mathfrak{a}\mathfrak{p}^{-k_j}$, so that $\alpha_j x_j \in \mathfrak{a}M_\xi$, $m \leqq j \leqq n$, and hence $s = \sum_{i=1}^{n} \alpha_i x_i$ is in $\mathfrak{a}M_\xi$. Now let

$$\psi = E(u, -x_m)E(u, (1 - \varepsilon)x_m)E(v, \varepsilon^{-1}s)\varphi E(u, x_m) \, .$$

If ψ is generated by the given isometries, so is φ (since $\varepsilon - 1 \in \mathfrak{a}\mathfrak{p}^{-k_m}$). Since now $\psi(y_j) = y_j$ for $1 \leqq j \leqq m$, the proof is concluded by induction.

REMARK 3.2. The assumption $\mathfrak{a} \subseteqq 4\mathfrak{p}$ in the theorem is used to ensure $\mathfrak{a}\mathfrak{p}^{-k_i} \subseteqq \mathfrak{p}$ (in particular ε is now a unit). When $4\mathfrak{o} \subseteqq \mathfrak{a}$, more care is needed (see [3, Tables I, II]).

COROLLARY 3.3. *$O(4\mathfrak{a}M_\xi)^\# \subseteqq \mathscr{E}(\mathfrak{a}M_\xi)$ when $\mathfrak{a} \subseteqq \mathfrak{p}$.*

Proof. We need only show $\Phi(\varepsilon) \in \mathscr{E}(\mathfrak{a}M_\xi)$ when $\varepsilon \equiv 1 \bmod 4\mathfrak{a}$. By Hensel's lemma $\varepsilon = \eta^2$ with $\eta \equiv 1 \bmod \mathfrak{a}$. Now $\Phi(\eta^2)$ is in $\mathscr{E}(\mathfrak{a}M_*)$ by [2, Proposition 1.2].

COROLLARY 3.4. *$O(\mathfrak{a}M_\xi) = \mathscr{E}(\mathfrak{a}M_\xi)$ when $\mathfrak{p}$ is nondyadic.*

Proof. For nondyadic $\mathfrak{p}$, $M_\xi = M = M^* = M_*$. Clearly $O(\mathfrak{o}M) = \mathscr{E}$ by the definition of $O(\mathfrak{a}M)$. If $\mathfrak{a} \subseteqq \mathfrak{p}$, then $\varepsilon \equiv 1 \bmod \mathfrak{a}$ and by Hensel's lemma $\varepsilon = \eta^2$ with $\eta \equiv 1 \bmod \mathfrak{a}$. Again, $\Phi(\varepsilon)$ is in $\mathscr{E}(\mathfrak{a}M)$.

For dyadic $\mathfrak{p}$, under the hypothesis of Theorem 3.1, $O(\mathfrak{a}M_\xi)^\#/\mathscr{E}(\mathfrak{a}M_\xi)$ is an abelian group with exponent dividing 2. We have indicated in [3, VI] how to find the cardinality of this quotient group (when 2 is unramified). For nondyadic $\mathfrak{p}$ the group $\mathscr{T}(\mathfrak{a}M)/O(\mathfrak{a}M)$ has either 1 or 2 elements, depending on whether $-I$ is in $\mathscr{E}$. For dyadic $\mathfrak{p}$ some comments on the group are made in [3, VII]. The quotient group $O(\mathfrak{a}M_\xi)/O(\mathfrak{a}M_\xi)^\#$ will now be determined explicitly in two special cases.

EXAMPLE 3.5. Let $M = H \perp N \perp \mathfrak{o}w$ where $q(N) \subseteqq \mathfrak{o}$. Let $h =$

$[(1 + \text{ord}\, 2)/2]$ so that

$$M_* = H \perp N \perp \mathfrak{p}^h w \quad \text{and} \quad M^* = H \perp N \perp \mathfrak{p}^{-h} w\ .$$

The $\mathscr{E}$-invariant dual lattices are

$$M_\xi = H \perp N \perp \mathfrak{p}^{-k} w \quad \text{and} \quad M^\xi = H \perp N \perp \mathfrak{p}^{k} w$$

where $-h \leqq k \leqq h$. We show, for $\mathfrak{a} \subseteqq 2\mathfrak{p}^{k+1+h}$, that

$$O(\mathfrak{a}M_\xi) = O(\mathfrak{a}M_\xi)^\# \ .$$

Let $\varphi \in O(\mathfrak{a}M_\xi)$. Then $\varphi(w) = \varepsilon w + s$ where $s \in \mathfrak{a}\mathfrak{p}^{-h}(H \perp N)$. For all $t \in H \perp N$, since $\varphi(t) \equiv t \bmod \mathfrak{a}M_\xi$, it follows that $B(s, t)$ is in $\mathfrak{a}\mathfrak{p}^{-k}$ and hence $s \in \mathfrak{a}\mathfrak{p}^{-k}(H \perp N)$. Now $q(w) = q(\varphi(w))$ gives

$$\varepsilon \equiv 1 \bmod \mathfrak{a}^2\mathfrak{p}^{-2k}\ .$$

Consequently, $\varphi(w) \equiv w \bmod \mathfrak{a}\mathfrak{p}^{-k}M_*$ and $\varphi \in O(\mathfrak{a}M_\xi)^\#$.

EXAMPLE 3.6. Let $M = H \perp N \perp (\mathfrak{o}w + \mathfrak{o}z)$ where $q(N) \subseteqq \mathfrak{o}$ and $\mathfrak{o}w + \mathfrak{o}z = \langle A(1, 0)\rangle$. Let $h = [(1 + \text{ord}\, 2)/2]$ so that

$$M_* = H \perp N \perp (\mathfrak{p}^h w + \mathfrak{o}z) \quad \text{and} \quad M^* = H \perp N \perp (\mathfrak{o}w + \mathfrak{p}^{-h} z)\ .$$

Fix $0 \leqq i, j \leqq h$; then

$$M_\xi = H \perp N \perp (\mathfrak{p}^i w + \mathfrak{p}^{-j} z) \quad \text{and} \quad M^\xi = H \perp N \perp (\mathfrak{p}^j w + \mathfrak{p}^{-i} z)$$

are a pair of $\mathscr{E}$-invariant lattices (but not all are of this form). If $\mathfrak{a} \subseteqq 4\mathfrak{p}$,

$$O(\mathfrak{a}M_\xi)/O(\mathfrak{a}M_\xi)^\# \cong \begin{cases} \mathfrak{o}/\mathfrak{p}^{i+j} & \text{if} \quad i + j \leqq h \\ \mathfrak{o}/\mathfrak{p}^{2h-i-j} & \text{if} \quad i + j \geqq h\ . \end{cases}$$

We omit the computational details of the proof except to mention that representatives of the various cosets can be obtained from the isometry defined by

$$\psi_\varepsilon(w) = \varepsilon^{-1} w + \frac{1}{2}(\varepsilon - \varepsilon^{-1}) z,\ \psi_\varepsilon(z) = \varepsilon z$$

by suitably choosing the unit ε.

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