

Pacific Journal of Mathematics

ON THE NORMAL SUBGROUPS OF INTEGRAL ORTHOGONAL GROUPS

DONALD GORDON JAMES

ON THE NORMAL SUBGROUPS OF INTEGRAL ORTHOGONAL GROUPS

D. G. JAMES

Let \mathcal{E} denote the spinorial kernel of an orthogonal group of an indefinite unimodular quadratic form over the integers in a global field. The normal subgroups of \mathcal{E} that arise from the local structure of \mathcal{E} are studied.

Let S be a Dedekind set of spots on a global field F with characteristic not two and \mathfrak{o} the ring of integers of F at S . Let V be a finite dimensional quadratic space over F with dimension at least 5 and associated bilinear form B and quadratic form q . The orthogonal group of V is

$$O(V) = \{\varphi \in \text{End } V \mid q(\varphi(x)) = q(x) \text{ for all } x \in V\}.$$

Assume V supports a unimodular lattice M with orthogonal group $O(M) = \{\varphi \in O(V) \mid \varphi(M) = M\}$. At each spot $\mathfrak{p} \in S$ we can localize and consider the orthogonal group $O(M_{\mathfrak{p}})$ of the isotropic unimodular lattice $M_{\mathfrak{p}}$ over the ring of integers $\mathfrak{o}_{\mathfrak{p}}$ in the local field $F_{\mathfrak{p}}$. The subgroups of $O(M_{\mathfrak{p}})$ normalized by its commutator subgroup have been classified in [1, 2, 3]. We show here how this local structure can be injected into $O(M)$ when S is an indefinite set of spots for V . A rich structure of normal subgroups of the spinorial kernel of $O(M)$ is then provided by the local behaviour at the dyadic spots. Most of our terminology and notation is taken from O'Meara [4].

1. \mathcal{E} -invariant sublattices of M . $M_{\mathfrak{p}}$ is split by a hyperbolic plane $H_{\mathfrak{p}}$ at each spot $\mathfrak{p} \in S$. Write $M_{\mathfrak{p}} = H_{\mathfrak{p}} \perp K_{\mathfrak{p}}$ and $H_{\mathfrak{p}} = \mathfrak{o}_{\mathfrak{p}}u_{\mathfrak{p}} + \mathfrak{o}_{\mathfrak{p}}v_{\mathfrak{p}}$ where $B_{\mathfrak{p}}(u_{\mathfrak{p}}, v_{\mathfrak{p}}) = 1$ and $q_{\mathfrak{p}}(u_{\mathfrak{p}}) = q_{\mathfrak{p}}(v_{\mathfrak{p}}) = 0$. In [2] it is shown that $\mathcal{E}_{\mathfrak{p}}$, the group generated by the Siegel transformations $E(u_{\mathfrak{p}}, x_{\mathfrak{p}})$ and $E(v_{\mathfrak{p}}, x_{\mathfrak{p}})$, is equal to the spinorial kernel of $O(M_{\mathfrak{p}})$. Define

$$\mathcal{E} = \{\varphi \in O(M) \mid \varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in S\}.$$

Note that \mathcal{E} is defined if $\dim V \geq 5$, even when V is anisotropic. In fact, using the global square theorem [4, 65:15] and [2, Theorem 2.9], it is easily seen that \mathcal{E} is the spinorial kernel of $O(M)$.

A sublattice P of M is called \mathcal{E} -invariant if $\varphi(P) = P$ for all $\varphi \in \mathcal{E}$. Let M_* denote the lattice

$$M_* = \{x \in M \mid q(x) \in \mathfrak{o}\}$$

with dual lattice

$$M^* = \{x \in V \mid B(x, M_*) \subseteq \mathfrak{o}\} .$$

Define an ideal $\alpha(P)$ in \mathfrak{o} by

$$\alpha(P) = \sum_{r \in P} B(r, M_*) .$$

Then

$$\alpha(P)^{-1}P \subseteq M^* .$$

LEMMA 1.1. *Let P be sublattice of M and*

$$M_* \subseteq \alpha(P)^{-1}P \subseteq M^* .$$

Then P is \mathcal{E} -invariant.

Proof. Locally, $P_{\mathfrak{p}}$ is $\mathcal{E}_{\mathfrak{p}}$ -invariant [2, Theorem 3.1] since $(M_*)_{\mathfrak{p}} = (M_{\mathfrak{p}})_*$, $\alpha(P)_{\mathfrak{p}} = \alpha(P)_{\mathfrak{p}}$ and

$$(M_{\mathfrak{p}})_* \subseteq \alpha(P_{\mathfrak{p}})^{-1}P_{\mathfrak{p}} \subseteq (M_{\mathfrak{p}})^* .$$

Take $\varphi \in \mathcal{E}$ and let $\varphi(P) = Q$. Then $\varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}$ and

$$P_{\mathfrak{p}} = \varphi_{\mathfrak{p}}(P_{\mathfrak{p}}) = \varphi(P)_{\mathfrak{p}} = Q_{\mathfrak{p}}$$

for all $\mathfrak{p} \in S$. Hence $P = Q$ by [4, 81E].

THEOREM 1.2. *Let S be an indefinite set of spots for V and assume $\dim V \geq 7$ if $[\mathfrak{o}_{\mathfrak{p}} : \mathfrak{p}] = 2$ at any dyadic spot. Then a sublattice P of M is \mathcal{E} -invariant if and only if*

$$M_* \subseteq \alpha(P)^{-1}P \subseteq M^* .$$

Proof. We need only show that if P is \mathcal{E} -invariant, then $(M_{\mathfrak{p}})_* \subseteq \alpha(P_{\mathfrak{p}})^{-1}P_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$. Fix $\mathfrak{p} \in S$ and take $\theta_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}$. By the strong approximation theorem for rotations [4, 104: 4], there exists $\varphi \in O(V)$ such that $\|\varphi - \theta_{\mathfrak{p}}\|_{\mathfrak{p}} < \varepsilon$ and $\|\varphi\|_{\mathfrak{q}} = 1$ for $\mathfrak{q} \neq \mathfrak{p}$. Then $\varphi \in O(M)$ and, since \mathcal{E} is the spinorial kernel, $\varphi \in \mathcal{E}$. Hence $\varphi(P) = P$. By making ε sufficiently small, we may assume $(\varphi_{\mathfrak{p}} - \theta_{\mathfrak{p}})(P_{\mathfrak{p}}) \subseteq P_{\mathfrak{p}}$. Thus $\theta_{\mathfrak{p}}(P_{\mathfrak{p}}) = P_{\mathfrak{p}}$ for all $\theta_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}$. Hence $P_{\mathfrak{p}}$ is $\mathcal{E}_{\mathfrak{p}}$ -invariant and by [2, Theorem 3.1] it follows that $\alpha(P_{\mathfrak{p}})(M_{\mathfrak{p}})_* \subseteq P_{\mathfrak{p}}$.

THEOREM 1.3. *For each dyadic spot \mathfrak{p} assume given an $\mathfrak{o}_{\mathfrak{p}}$ -lattice $J_{\mathfrak{p}}$ with $(M_{\mathfrak{p}})_* \subseteq J_{\mathfrak{p}} \subseteq (M_{\mathfrak{p}})^*$. Then there exists an \mathfrak{o} -lattice P such that $P_{\mathfrak{p}} = J_{\mathfrak{p}}$ at each dyadic spot and $M_* \subseteq P \subseteq M^*$.*

Proof. This follows immediately from [4, 81: 14] since $(M_{\mathfrak{p}})_* = (M_{\mathfrak{p}})^*$ at all nondyadic spots.

REMARK 1.4. If 2 is unramified, the lattices satisfying $(M_2)_* \subseteq P \subseteq (M_2)^*$ were determined in [1]. When 2 ramifies the number of such lattices proliferates (even for the gaussian integers, there are lattices M having more than 20 such P).

2. Normal subgroups of \mathcal{E} . Let \mathcal{E} be an indexing set such that $M_\xi, \xi \in \mathcal{E}$, gives all the lattices on V such that

$$M_* \subseteq M_\xi \subseteq M^* .$$

Let α be an ideal in \mathfrak{o} such that $\alpha M_\xi \subseteq M_*$. Then $\alpha_p M_{\xi_p} \subseteq (M_p)_*$ and we define $\mathcal{E}_p(\alpha_p M_{\xi_p})$ as in [2] as the normal subgroup of \mathcal{E}_p generated by all isometries of the form $\theta_p E(u_p, x_p) \theta_p^{-1}$ or $\theta_p E(v_p, x_p) \theta_p^{-1}$ where $\theta_p \in \mathcal{E}_p$ and $x_p \in K_p \cap \alpha_p M_{\xi_p}$. Define

$$\mathcal{E}(\alpha M_\xi) = \{\varphi \in \mathcal{E} \mid \varphi_p \in \mathcal{E}_p(\alpha_p M_{\xi_p}) \text{ for all } p \in S\}$$

and

$$\mathcal{F}(\alpha M_\xi) = \{\varphi \in \mathcal{E} \mid [\varphi, \mathcal{E}] \subseteq \mathcal{E}(\alpha M_\xi)\} .$$

Then $\mathcal{E}(\alpha M_\xi) \subseteq \mathcal{F}(\alpha M_\xi)$ and any subgroup \mathcal{N} of \mathcal{E} such that

$$\mathcal{E}(\alpha M_\xi) \subseteq \mathcal{N} \subseteq \mathcal{F}(\alpha M_\xi)$$

is a normal subgroup of \mathcal{E} .

We also define the local group $\mathcal{F}_p(\alpha_p M_{\xi_p})$ by

$$\mathcal{F}_p(\alpha_p M_{\xi_p}) = \{\varphi_p \in \mathcal{E}_p \mid [\varphi_p, \mathcal{E}_p] \subseteq \mathcal{E}_p(\alpha_p M_{\xi_p})\} .$$

Note that this definition is more restrictive than that in [2] where φ_p was taken in $O(M_p)$, not \mathcal{E}_p .

THEOREM 2.1. *Let S be an indefinite set of spots for V and $\dim V \geq 5$. Then for any ideal $\alpha \neq \{0\}$ with $\alpha M_\xi \subseteq M_*$,*

$$\mathcal{F}(\alpha M_\xi) / \mathcal{E}(\alpha M_\xi) \cong \prod_{p|\alpha} \mathcal{F}_p(\alpha_p M_{\xi_p}) / \mathcal{E}_p(\alpha_p M_{\xi_p}) .$$

Proof. Observe first that for $p \nmid \alpha$ we have

$$\mathcal{E}_p(\alpha_p M_{\xi_p}) = \mathcal{E}_p = \mathcal{F}_p(\alpha_p M_{\xi_p}) .$$

Define a mapping

$$\Gamma: \mathcal{F}(\alpha M_\xi) \longrightarrow \prod_{p|\alpha} \mathcal{F}_p(\alpha_p M_{\xi_p}) / \mathcal{E}_p(\alpha_p M_{\xi_p})$$

by sending $\varphi \in \mathcal{F}(\alpha M_\xi)$ into $(\dots, \tilde{\varphi}_p, \dots)_{p|\alpha}$ where $\tilde{\varphi}_p$ denotes the coset determined by φ_p in $\mathcal{F}_p(\alpha_p M_{\xi_p}) / \mathcal{E}_p(\alpha_p M_{\xi_p})$. It is necessary to show that Γ is well-defined, namely $\varphi_p \in \mathcal{F}_p(\alpha_p M_{\xi_p})$ for each $p|\alpha$. Let

$\psi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}$ and take $\theta \in \mathcal{E}$ such that $\|\psi_{\mathfrak{p}}^{-1}\theta - I\|_{\mathfrak{p}} < \varepsilon$. Then $[\varphi, \theta]$ is in $\mathcal{E}(\alpha M_{\xi})$ and hence $[\varphi_{\mathfrak{p}}, \theta_{\mathfrak{p}}]$ is in $\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. For sufficiently small ε , it follows from Corollary 3.3 that $\psi_{\mathfrak{p}}^{-1}\theta_{\mathfrak{p}}$ is in $\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ and hence $[\varphi_{\mathfrak{p}}, \psi_{\mathfrak{p}}] \in \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. Thus Γ is a well-defined group homomorphism.

φ is in the kernel of Γ if and only if $\varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ for each $\mathfrak{p}|\alpha$. But $\varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}} = \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ for $\mathfrak{p} \nmid \alpha$. Hence the kernel of Γ is $\mathcal{E}(\alpha M_{\xi})$.

It remains to show that Γ is surjective. For each $\mathfrak{p}|\alpha$ fix $\psi_{\mathfrak{p}}$ in $\mathcal{F}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. By the strong approximation theorem, there exists $\varphi \in \mathcal{E}$ such that

$$\|\psi_{\mathfrak{p}}^{-1}\varphi - I\|_{\mathfrak{p}} < \varepsilon \text{ for } \mathfrak{p}|\alpha.$$

It now suffices to show $\varphi \in \mathcal{F}(\alpha M_{\xi})$, for if ε is sufficiently small, $\psi_{\mathfrak{p}}^{-1}\varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ by Corollary 3.3 and hence $\psi_{\mathfrak{p}}$ and $\varphi_{\mathfrak{p}}$ determine the same coset in $\mathcal{F}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})/\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. We must show $[\varphi_{\mathfrak{p}}, \theta_{\mathfrak{p}}] \in \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ for any $\theta \in \mathcal{E}$ and all $\mathfrak{p} \in S$. For $\mathfrak{p} \nmid \alpha$ this is trivial. For $\mathfrak{p}|\alpha$ and ε sufficiently small, $\chi_{\mathfrak{p}} = \psi_{\mathfrak{p}}^{-1}\varphi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$ and hence

$$[\varphi_{\mathfrak{p}}, \theta_{\mathfrak{p}}] = [\psi_{\mathfrak{p}}\chi_{\mathfrak{p}}, \theta_{\mathfrak{p}}] = [\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\theta_{\mathfrak{p}}\chi_{\mathfrak{p}}^{-1}] [\chi_{\mathfrak{p}}, \theta_{\mathfrak{p}}]$$

is in $\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. This completes the proof.

REMARK 2.2. We restricted ourselves to Dedekind domains coming from global fields so that the strong approximation theorem for rotations could be used. If, however, we assume M has hyperbolic rank at least one, so that globally $M = H \perp K$ with $H = ou + ov$ a hyperbolic plane and K is free, a strong approximation theorem can still be established for any Dedekind domain. Define all $\mathcal{E}_{\mathfrak{p}}$ with respect to the localization of H . Then, from [2, Theorem 2.9], any $\theta_{\mathfrak{p}}$ in the spinorial kernel is of the form

$$\theta_{\mathfrak{p}} = \prod_{i=1}^{n(\mathfrak{p})} E(u, x_i(\mathfrak{p}))E(v, y_i(\mathfrak{p}))$$

with $x_i(\mathfrak{p})$ and $y_i(\mathfrak{p})$ in $(K_{\mathfrak{p}})_{*}$. By approximating to the coefficients of $x_i(\mathfrak{p})$ and $y_i(\mathfrak{p})$ in the Dedekind domain \mathfrak{o} , we can approximate to $\theta_{\mathfrak{p}}$ by an isometry in $O(M)$ for a finite number of \mathfrak{p} .

3. The structure of $\mathcal{F}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})/\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. We close with some comments on the structure of the abelian group $\mathcal{F}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})/\mathcal{E}_{\mathfrak{p}}(\alpha_{\mathfrak{p}} M_{\xi_{\mathfrak{p}}})$. For simplicity, since only the local situation is considered in this section, the suffix \mathfrak{p} is dropped and we write \mathfrak{o} for $\mathfrak{o}_{\mathfrak{p}}$, M for $M_{\mathfrak{p}}$, V for $V_{\mathfrak{p}}$, and so on.

For $\xi \in \mathcal{E}$, let

$$M^{\xi} = \{x \in V \mid B(x, M_{\xi}) \subseteq \mathfrak{o}\}$$

be the dual lattice of M_ξ . Write $M_* = H \perp K_*$ and $M_\xi = H \perp K_\xi$. Then $K_* \subseteq K_\xi$ and [we can choose a basis x_1, \dots, x_n for K_* such that

$$K_* = \mathfrak{o}x_1 + \dots + \mathfrak{o}x_n$$

and

$$K_\xi = \mathfrak{p}^{-k_1}x_1 + \dots + \mathfrak{p}^{-k_n}x_n$$

with the integers k_i invariants for K_* , K_ξ and

$$0 \leq k_1 \leq k_2 \leq \dots \leq k_n .$$

Let y_1, \dots, y_n be the dual basis of x_1, \dots, x_n . Then

$$K^* = \mathfrak{o}y_1 + \dots + \mathfrak{o}y_n$$

and

$$K^\xi = \mathfrak{p}^{k_1}y_1 + \dots + \mathfrak{p}^{k_n}y_n .$$

For each $1 \leq i \leq n$, define

$$\begin{aligned} K_{\xi^{(i)}} &= K_* + \mathfrak{p}^{k_i}K_\xi \\ &= \mathfrak{o}x_1 + \dots + \mathfrak{o}x_i + \mathfrak{p}^{k_i - k_{i+1}}x_{i+1} + \dots + \mathfrak{p}^{k_i - k_n}x_n \end{aligned}$$

and let $K^{\xi^{(i)}}$, $1 \leq i \leq n$, be the dual lattices. Write $M_{\xi^{(i)}} = H \perp K_{\xi^{(i)}}$ and $M^{\xi^{(i)}} = H \perp K^{\xi^{(i)}}$. Then

$$M_\xi \supseteq M_{\xi^{(1)}} \supseteq M_{\xi^{(2)}} \supseteq \dots \supseteq M_{\xi^{(n)}} = M_*$$

and

$$M^\xi \subseteq M^{\xi^{(1)}} \subseteq M^{\xi^{(2)}} \subseteq \dots \subseteq M^{\xi^{(n)}} = M^* .$$

Define congruence subgroups by

$$\begin{aligned} O(\mathfrak{a}M_\xi) &= \{\varphi \in \mathcal{E} \mid \varphi(r) \equiv r \pmod{\mathfrak{a}M_\xi} \text{ for all } r \in M_*\} , \\ O(\mathfrak{a}M_\xi)^i &= \{\varphi \in O(\mathfrak{a}M_\xi) \mid \varphi(r) \equiv r \pmod{2\mathfrak{a}\mathfrak{p}^{-k_i}M_{\xi^{(i)}}} \text{ for all } r \in 2M^{\xi^{(i)}}\} \end{aligned}$$

for $1 \leq i \leq n$, and

$$O(\mathfrak{a}M_\xi)^{\#} = \bigcap_{i=1}^n O(\mathfrak{a}M_\xi)^i .$$

Since $\mathfrak{p}^{k_i}M_\xi \subseteq M_{\xi^{(i)}}$, $1 \leq i \leq n$, it can be checked that

$$\mathcal{E}(\mathfrak{a}M_\xi) \subseteq O(\mathfrak{a}M_\xi)^{\#} \subseteq O(\mathfrak{a}M_\xi) \subseteq \mathcal{F}(\mathfrak{a}M_\xi)$$

provided that $\mathfrak{a}q(M_\xi) \subseteq \mathfrak{o}$.

THEOREM 3.1. For $\alpha \subseteq 4\mathfrak{p}$, the subgroup $O(\alpha M_\varepsilon)^*$ is generated by $\mathcal{E}(\alpha M_\varepsilon)$ and the isometries $\Phi(\varepsilon)$ with $\varepsilon \equiv 1 \pmod{\alpha}$.

Proof. The isometries $\Phi(\varepsilon)$ are defined in [2]. Let $\varphi \in O(\alpha M_\varepsilon)^*$; changing φ by the given isometries, we will reduce it to the identity mapping. Assume for some $m \leq n$ that $\varphi(y_j) = y_j$, $1 \leq j \leq m-1$, (consider $\varphi \in O(V)$). Let

$$\varphi(u + y_m) = \varepsilon u + \beta v + y_m + \sum_{i=1}^n \alpha_i x_i.$$

Since $\varphi(y_j) = y_j$, it follows that $\alpha_j = 0$ for $1 \leq j \leq m-1$. For $j \geq m$, $u + y_m \in M^{\varepsilon(j)}$ and hence $\alpha_j x_j \in \alpha \mathfrak{p}^{-kj} M_{\varepsilon(j)}$. Thus $\alpha_j \in \alpha \mathfrak{p}^{-kj}$, so that $\alpha_j x_j \in \alpha M_\varepsilon$, $m \leq j \leq n$, and hence $s = \sum_{i=1}^n \alpha_i x_i$ is in αM_ε . Now let

$$\psi = E(u, -x_m)E(u, (1 - \varepsilon)x_m)E(v, \varepsilon^{-1}s)\varphi E(u, x_m).$$

If ψ is generated by the given isometries, so is φ (since $\varepsilon - 1 \in \alpha \mathfrak{p}^{-km}$). Since now $\psi(y_j) = y_j$ for $1 \leq j \leq m$, the proof is concluded by induction.

REMARK 3.2. The assumption $\alpha \subseteq 4\mathfrak{p}$ in the theorem is used to ensure $\alpha \mathfrak{p}^{-ki} \subseteq \mathfrak{p}$ (in particular ε is now a unit). When $4\mathfrak{p} \subseteq \alpha$, more care is needed (see [3, Tables I, II]).

COROLLARY 3.3. $O(4\alpha M_\varepsilon)^* \subseteq \mathcal{E}(\alpha M_\varepsilon)$ when $\alpha \subseteq \mathfrak{p}$.

Proof. We need only show $\Phi(\varepsilon) \in \mathcal{E}(\alpha M_\varepsilon)$ when $\varepsilon \equiv 1 \pmod{4\alpha}$. By Hensel's lemma $\varepsilon = \eta^2$ with $\eta \equiv 1 \pmod{\alpha}$. Now $\Phi(\eta^2)$ is in $\mathcal{E}(\alpha M_*)$ by [2, Proposition 1.2].

COROLLARY 3.4. $O(\alpha M_\varepsilon) = \mathcal{E}(\alpha M_\varepsilon)$ when \mathfrak{p} is non-dyadic.

Proof. For non-dyadic \mathfrak{p} , $M_\varepsilon = M = M^* = M_*$. Clearly $O(\mathfrak{p}M) = \mathcal{E}$ by the definition of $O(\alpha M)$. If $\alpha \subseteq \mathfrak{p}$, then $\varepsilon \equiv 1 \pmod{\alpha}$ and by Hensel's lemma $\varepsilon = \eta^2$ with $\eta \equiv 1 \pmod{\alpha}$. Again, $\Phi(\varepsilon)$ is in $\mathcal{E}(\alpha M)$.

For dyadic \mathfrak{p} , under the hypothesis of Theorem 3.1, $O(\alpha M_\varepsilon)^*/\mathcal{E}(\alpha M_\varepsilon)$ is an abelian group with exponent dividing 2. We have indicated in [3, VI] how to find the cardinality of this quotient group (when 2 is unramified). For non-dyadic \mathfrak{p} the group $\mathcal{F}(\alpha M)/O(\alpha M)$ has either 1 or 2 elements, depending on whether $-I$ is in \mathcal{E} . For dyadic \mathfrak{p} some comments on the group are made in [3, VII]. The quotient group $O(\alpha M_\varepsilon)/O(\alpha M_\varepsilon)^*$ will now be determined explicitly in two special cases.

EXAMPLE 3.5. Let $M = H \perp N \perp \mathfrak{p}w$ where $q(N) \subseteq \mathfrak{p}$. Let $h =$

$[(1 + \text{ord } 2)/2]$ so that

$$M_* = H \perp N \perp \mathfrak{p}^h w \quad \text{and} \quad M^* = H \perp N \perp \mathfrak{p}^{-h} w .$$

The \mathcal{E} -invariant dual lattices are

$$M_\varepsilon = H \perp N \perp \mathfrak{p}^{-k} w \quad \text{and} \quad M^\varepsilon = H \perp N \perp \mathfrak{p}^k w$$

where $-h \leq k \leq h$. We show, for $\alpha \subseteq 2\mathfrak{p}^{k+1+h}$, that

$$O(\alpha M_\varepsilon) = O(\alpha M_\varepsilon)^\sharp .$$

Let $\varphi \in O(\alpha M_\varepsilon)$. Then $\varphi(w) = \varepsilon w + s$ where $s \in \alpha \mathfrak{p}^{-h}(H \perp N)$. For all $t \in H \perp N$, since $\varphi(t) \equiv t \pmod{\alpha M_\varepsilon}$, it follows that $B(s, t)$ is in $\alpha \mathfrak{p}^{-k}$ and hence $s \in \alpha \mathfrak{p}^{-k}(H \perp N)$. Now $q(w) = q(\varphi(w))$ gives

$$\varepsilon \equiv 1 \pmod{\alpha^2 \mathfrak{p}^{-2k}} .$$

Consequently, $\varphi(w) \equiv w \pmod{\alpha \mathfrak{p}^{-k} M_*}$ and $\varphi \in O(\alpha M_\varepsilon)^\sharp$.

EXAMPLE 3.6. Let $M = H \perp N \perp (\mathfrak{o}w + \mathfrak{o}z)$ where $q(N) \subseteq \mathfrak{o}$ and $\mathfrak{o}w + \mathfrak{o}z = \langle A(1, 0) \rangle$. Let $h = [(1 + \text{ord } 2)/2]$ so that

$$M_* = H \perp N \perp (\mathfrak{p}^h w + \mathfrak{o}z) \quad \text{and} \quad M^* = H \perp N \perp (\mathfrak{o}w + \mathfrak{p}^{-h} z) .$$

Fix $0 \leq i, j \leq h$; then

$$M_\varepsilon = H \perp N \perp (\mathfrak{p}^i w + \mathfrak{p}^{-j} z) \quad \text{and} \quad M^\varepsilon = H \perp N \perp (\mathfrak{p}^j w + \mathfrak{p}^{-i} z)$$

are a pair of \mathcal{E} -invariant lattices (but not all are of this form). If $\alpha \subseteq 4\mathfrak{p}$,

$$O(\alpha M_\varepsilon)/O(\alpha M_\varepsilon)^\sharp \cong \begin{cases} \mathfrak{o}/\mathfrak{p}^{i+j} & \text{if } i + j \leq h \\ \mathfrak{o}/\mathfrak{p}^{2h-i-j} & \text{if } i + j \geq h . \end{cases}$$

We omit the computational details of the proof except to mention that representatives of the various cosets can be obtained from the isometry defined by

$$\psi_\varepsilon(w) = \varepsilon^{-1} w + \frac{1}{2}(\varepsilon - \varepsilon^{-1})z, \quad \psi_\varepsilon(z) = \varepsilon z$$

by suitably choosing the unit ε .

REFERENCES

1. D. G. James, *Orthogonal groups of dyadic unimodular quadratic forms*, Math. Ann., **201** (1973), 65-74.
2. ———, *Orthogonal groups of dyadic unimodular quadratic forms II*, to appear.
3. ———, *The structure of orthogonal groups of 2-adic unimodular quadratic forms*, J. Number Theory, **5** (1973), 444-455.

4. O. T. O'Meara, *Introduction to Quadratic Forms*, Springer, Berlin-Göttingen-Heidelberg, 1963.

Received August 25, 1973. This research was partially supported by the National Science Foundation.

THE PENNSYLVANIA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

David R. Adams, <i>On the exceptional sets for spaces of potentials</i>	1
Philip Bacon, <i>Axioms for the Čech cohomology of paracompacta</i>	7
Selwyn Ross Caradus, <i>Perturbation theory for generalized Fredholm operators</i>	11
Kuang-Ho Chen, <i>Phragmén-Lindelöf type theorems for a system of nonhomogeneous equations</i>	17
Frederick Knowles Dashiell, Jr., <i>Isomorphism problems for the Baire classes</i>	29
M. G. Deshpande and V. K. Deshpande, <i>Rings whose proper homomorphic images are right subdirectly irreducible</i>	45
Mary Rodriguez Embry, <i>Self adjoint strictly cyclic operator algebras</i>	53
Paul Erdős, <i>On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions</i>	59
Richard Joseph Fleming and James E. Jamison, <i>Hermitian and adjoint abelian operators on certain Banach spaces</i>	67
Stanley P. Gudder and L. Haskins, <i>The center of a poset</i>	85
Richard Howard Herman, <i>Automorphism groups of operator algebras</i>	91
Worthen N. Hunsacker and Somashekhar Amrith Nainpally, <i>Local compactness of families of continuous point-compact relations</i>	101
Donald Gordon James, <i>On the normal subgroups of integral orthogonal groups</i>	107
Eugene Carlyle Johnsen and Thomas Frederick Storer, <i>Combinatorial structures in loops. II. Commutative inverse property cyclic neofields of prime-power order</i>	115
Ka-Sing Lau, <i>Extreme operators on Choquet simplexes</i>	129
Philip A. Leonard and Kenneth S. Williams, <i>The septic character of 2, 3, 5 and 7</i>	143
Dennis McGavran and Jingyal Pak, <i>On the Nielsen number of a fiber map</i>	149
Stuart Edward Mills, <i>Normed Köthe spaces as intermediate spaces of L_1 and L_∞</i>	157
Philip Olin, <i>Free products and elementary equivalence</i>	175
Louis Jackson Ratliff, Jr., <i>Locally quasi-unmixed Noetherian rings and ideals of the principal class</i>	185
Seiya Sasao, <i>Homotopy types of spherical fibre spaces over spheres</i>	207
Helga Schirmer, <i>Fixed point sets of polyhedra</i>	221
Kevin James Sharpe, <i>Compatible topologies and continuous irreducible representations</i>	227
Frank Siwiec, <i>On defining a space by a weak base</i>	233
James McLean Sloss, <i>Global reflection for a class of simple closed curves</i>	247
M. V. Subba Rao, <i>On two congruences for primality</i>	261
Raymond D. Terry, <i>Oscillatory properties of a delay differential equation of even order</i>	269
Joseph Dinneen Ward, <i>Chebyshev centers in spaces of continuous functions</i>	283
Robert Breckenridge Warfield, Jr., <i>The uniqueness of elongations of Abelian groups</i>	289
V. M. Warfield, <i>Existence and adjoint theorems for linear stochastic differential equations</i>	305