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THE SEPTIC CHARACTER OF 2, 3, 5 AND 7

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Necessary and sufficient conditions for 2, 3, 5, and 7 to be seventh powers (mod p) (p **a prime** $\equiv 1 \pmod 7$)) **are determined.**

1. **Introduction.** Let p be a prime $\equiv 1 \pmod 3$. Gauss [5] proved that there are integers x and y such that

$$4p = x^2 + 27y^2,\ x \equiv 1 \pmod 3\ . \tag{1.1}$$

Indeed there are just two solutions $(x, \pm y)$ of (1.1). Jacobi [6] (see also [2], [9], [16]) gave necessary and sufficient conditions for all primes $q \leqq 37$ to be cubes (mod p) in terms of congruence conditions involving a solution of (1.1), which are independent of the particular solution chosen. For example he showed that 3 is a cube (mod p) if and only if $y \equiv 0 \pmod 3$. For p a prime $\equiv 1 \pmod 5$, Dickson [3] proved that the pair of diophantine equations

$$\begin{cases} 16p = x^2 + 50u^2 + 50v^2 + 125w^2\ , \\ xw = v^2 - 4uv - u^2,\ x \equiv 1 \pmod 5\ , \end{cases} \tag{1.2}$$

has exactly four solutions. If one of these is (x, u, v, w) the other three are $(x, -u, -v, w)$, $(x, v, -u, -w)$ and $(x, -v, u, -w)$. Lehmer [7], [8], [10], [11], Muskat [14], [15], and Pepin [17] have given necessary and sufficient conditions for 2, 3, 5, and 7 to be fifth powers (mod p) in terms of congruence conditions on the solutions of (1.2) which do not depend upon the particular solution chosen. For example Lehmer [8] proved that 3 is a fifth power (mod p) if and only if $u \equiv v \equiv 0 \pmod 3$.

In this note, making use of results of Dickson [4], Muskat [14], [15] and Pepin [17], and the authors [12], [13] we obtain the analogous conditions for 2, 3, 5, and 7 to be seventh powers modulo a prime $p \equiv 1 \pmod 7$. The appropriate system to consider is the triple of diophantine equations

$$\begin{cases} 72p = 2x_1^2 + 42(x_2^2 + x_3^2 + x_4^2) + 343(x_5^2 + 3x_6^2)\ , \\ 12x_2^2 - 12x_4^2 + 147x_5^2 - 441x_6^2 + 56x_1x_6 + 24x_2x_3 - 24x_2x_4 \\ \qquad + 48x_3x_4 + 98x_5x_6 = 0\ , \\ 12x_3^2 - 12x_4^2 + 49x_5^2 - 147x_6^2 + 28x_1x_5 + 28x_1x_6 + 48x_2x_3 \\ \qquad + 24x_2x_4 + 24x_3x_4 + 490x_5x_6 = 0,\ x_1 \equiv 1 \pmod 7\ , \end{cases} \tag{1.3}$$

considered by the authors in [12] (see also [20]). It was shown there that (1.3) has six nontrivial solutions in addition to the two trivial solutions $(-6t, \pm 2u, \pm 2u, \mp 2u, 0, 0)$, where t and u are given by

$$p = t^2 + 7u^2,\ t \equiv 1 \pmod 7\ . \tag{1.4}$$

If $(x_1, x_2, x_3, x_4, x_5, x_6)$ is one of the six nontrivial solutions of (1.3) the other five nontrivial solutions are

$$\begin{cases} \left(x_1, -x_3, x_4, x_2, -\frac{1}{2}(x_5 + 3x_6), \frac{1}{2}(x_5 - x_6)\right), \\ \left(x_1, -x_4, x_2, -x_3, -\frac{1}{2}(x_5 - 3x_6), -\frac{1}{2}(x_5 + x_6)\right), \\ (x_1, -x_2, -x_3, -x_4, x_5, x_6) \\ \left(x_1, x_3, -x_4, -x_2, -\frac{1}{2}(x_5 + 3x_6), \frac{1}{2}(x_5 - x_6)\right), \\ \left(x_1, x_4, -x_2, x_3, -\frac{1}{2}(x_5 - 3x_6), -\frac{1}{2}(x_5 + x_6)\right). \end{cases} \tag{1.5}$$

We prove

THEOREM. (a) *2 is a seventh power* (mod p) *if and only if* $x_1 \equiv 0 \pmod 2$.

(b) *3 is a seventh power* (mod p) *if and only if* $x_5 \equiv x_6 \equiv 0 \pmod 3$.

(c) *5 is a seventh power* (mod p) *if and only if either*

$$x_2 \equiv x_3 \equiv -x_4 \pmod 5 \quad \textit{and} \quad x_5 \equiv x_6 \equiv 0 \pmod 5$$

or

$$x_1 \equiv 0 \pmod 5 \quad \textit{and} \quad x_2 + x_3 - x_4 \equiv 0 \pmod 5\ .$$

(d) *7 is a seventh power* (mod p) *if and only if* $x_2 - 19x_3 - 18x_4 \equiv 0 \pmod{49}$.

In view of (1.5) it is clear that none of the conditions given in the theorem depends upon the particular nontrivial solution of (1.3) chosen. Moreover, in connection with (d) we remark that any solution of (1.3) satisfies $x_2 + 2x_3 + 3x_4 \equiv 0 \pmod 7$ (see [12]) so that $x_2 - 19x_3 - 18x_4 \equiv 0 \pmod 7$.

We remark that since this paper was written a paper has appeared by Helen Popova Alderson [1] giving necessary and sufficient conditions for 2 and 3 to be seventh powers (mod p). Her conditions are not as simple as (a) and (b) above.

2. *Proof of* (a). Let g be a primitive root (mod p), where p is an odd prime. Let $e > 1$ be an odd divisor of $p - 1$ and set $p -$

$1 = ef$. The cyclotomic number $(h, k)_e$ is defined to be the number of solutions s, t of the trinomial congruence

$$g^{es+h} + 1 \equiv g^{et+k} \pmod{p}, \quad 0 \leqq s, t \leqq f - 1 .$$

It is well-known [8], [18] that 2 is an eth power $(\bmod p)$ if and only if $(0, 0)_e \equiv 1 \pmod{2}$. From [4], [13] we have $49(0, 0)_7 = p - 20 - 12t + 3x_1$, so that 2 is a seventh power $(\bmod p)$ if and only if $x_1 \equiv 0 \pmod{2}$.

Alternatively this result can be proved using a result of Pepin [17] (see also [14]) or by using the representation of x_1 in terms of a Jacobsthal sum (see [7] and [12]).

3. *Proof of* (b). The Dickson-Hurwitz sum $B_e(i, j)$ is defined by

$$B_e(i, j) = \sum_{h=0}^{e-1} (h, i - jh)_e .$$

In [13] it was shown that

$$(3.1)\quad \begin{cases} 84B_7(0, 1) = 12x_1 + 12p - 24 , \\ 84B_7(1, 1) = -2x_1 + 42x_2 + 49x_5 + 147x_6 + 12p - 24 , \\ 84B_7(2, 1) = -2x_1 + 42x_3 + 49x_5 - 147x_6 + 12p - 24 , \\ 84B_7(3, 1) = -2x_1 + 42x_4 - 98x_5 + 12p - 24 , \\ 84B_7(4, 1) = -2x_1 - 42x_4 - 98x_5 + 12p - 24 , \\ 84B_7(5, 1) = -2x_1 - 42x_3 + 49x_5 - 147x_6 + 12p - 24 , \\ 84B_7(6, 1) = -2x_1 - 42x_2 + 49x_5 + 147x_6 + 12p - 24 , \end{cases}$$

for some nontrivial solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ of (1.3). Muskat [14], Pepin [17] have shown that 3 is a seventh power $(\bmod p)$ if and only if

$$B_7(1, 1) \equiv B_7(2, 1) \equiv B_7(4, 1) \pmod{3} ,$$
$$B_7(3, 1) \equiv B_7(5, 1) \equiv B_7(6, 1) \pmod{3} .$$

This condition using (3.1) is easily shown to be equivalent to $x_5 \equiv x_6 \equiv 0 \pmod{3}$. In verifying this it is necessary to observe that if $x_5 \equiv x_6 \equiv 0 \pmod{3}$ then $x_1 \equiv x_5 \equiv x_6 \equiv 0 \pmod{3}$, $x_2 \equiv x_3 \equiv -x_4 \pmod{3}$ follow from (1.3).

4. *Proof of* (c). Muskat [14] has shown that 5 is a seventh power $(\bmod p)$ if and only if either

$$B_7(1, 1) \equiv B_7(2, 1) \equiv B_7(4, 1) \pmod{5}$$
$$B_7(3, 1) \equiv B_7(5, 1) \equiv B_7(6, 1) \pmod{5}$$

or

$$B_7(1, 1) + B_7(2, 1) + B_7(4, 1) \equiv B_7(3, 1) + B_7(5, 1) + B_7(6, 1) \equiv 0 \pmod 5,$$

which by (3.1) is equivalent to

$$x_2 \equiv x_3 \equiv -x_4 \pmod 5 \quad \text{and} \quad x_5 \equiv x_6 \equiv 0 \pmod 5,$$

or

$$x_1 \equiv 0 \pmod 5 \text{ and } x_2 + x_3 - x_4 \equiv 0 \pmod 5.$$

5. *Proof of* (d). Muskat [15] has shown that 7 is a seventh power (mod p) if and only if

$$B_7(1, 1) - B_7(6, 1) - 19(B_7(2, 1) - B_7(5, 1)) - 18(B_7(3, 1) - B_7(4, 1)) \equiv 0 \pmod{49},$$

which by (3.1) is easily seen to be equivalent to

$$x_2 - 19x_3 - 18x_4 \equiv 0 \pmod{49}.$$

6. Application of theorem to primes $p \equiv 1 \pmod 7$, $p < 1000$. One of us (K.S.W.) has prepared a table of solutions [19] of (1.3) for all primes $p \equiv 1 \pmod 7$, $p < 1000$. For these primes the table shows that

(a) $x_1 \equiv 0 \pmod 2$ only for $p = 631, 673, 693$,
(b) $x_5 \equiv x_6 \equiv 0 \pmod 3$ only for $p = 757, 883$,
(c) (i) $x_2 \equiv x_3 \equiv -x_4 \pmod 5$ and $x_5 \equiv x_6 \equiv 0 \pmod 5$ not satisfied,
(ii) $x_1 \equiv 0 \pmod 5$ and $x_2 + x_3 - x_4 \equiv 0$ only for $p = 71, 827, 883$,
(d) $x_2 - 19x_3 - 18x_4 \equiv 0 \pmod{49}$ only for $p = 43, 281$,

so that by the theorem, for primes $p \equiv 1 \pmod 7$, $p < 1000$,

2 is a seventh power (mod p) only for $p = 631, 673, 953$,
3 is a seventh power (mod p) only for $p = 757, 883$,
5 is a seventh power (mod p) only for $p = 71, 827, 883$,
7 is a seventh power (mod p) only for $p = 43, 281$.

Indeed we can show directly that

$$2 \equiv 196^7 \pmod{631},\ 2 \equiv 128^7 \pmod{673},\ 2 \equiv 120^7 \pmod{953},$$
$$3 \equiv 81^7 \pmod{757},\ 3 \equiv 207^7 \pmod{883},$$
$$5 \equiv 58^7 \pmod{71},\ 5 \equiv 561^7 \pmod{827},\ 5 \equiv 432^7 \pmod{883},$$
$$7 \equiv 28^7 \pmod{43},\ 7 \equiv 264^7 \pmod{281}.$$

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Pacific Journal of Mathematics

Vol. 52, No. 1 January, 1974

David R. Adams, *On the exceptional sets for spaces of potentials* 1
Philip Bacon, *Axioms for the Čech cohomology of paracompacta* 7
Selwyn Ross Caradus, *Perturbation theory for generalized Fredholm operators* 11
Kuang-Ho Chen, *Phragmén-Lindelöf type theorems for a system of nonhomogeneous equations* 17
Frederick Knowles Dashiell, Jr., *Isomorphism problems for the Baire classes* 29
M. G. Deshpande and V. K. Deshpande, *Rings whose proper homomorphic images are right subdirectly irreducible* 45
Mary Rodriguez Embry, *Self adjoint strictly cyclic operator algebras* 53
Paul Erdős, *On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions* 59
Richard Joseph Fleming and James E. Jamison, *Hermitian and adjoint abelian operators on certain Banach spaces* 67
Stanley P. Gudder and L. Haskins, *The center of a poset* 85
Richard Howard Herman, *Automorphism groups of operator algebras* 91
Worthen N. Hunsacker and Somashekhar Amrith Naimpally, *Local compactness of families of continuous point-compact relations* 101
Donald Gordon James, *On the normal subgroups of integral orthogonal groups* 107
Eugene Carlyle Johnsen and Thomas Frederick Storer, *Combinatorial structures in loops. II. Commutative inverse property cyclic neofields of prime-power order* 115
Ka-Sing Lau, *Extreme operators on Choquet simplexes* 129
Philip A. Leonard and Kenneth S. Williams, *The septic character of 2, 3, 5 and 7* 143
Dennis McGavran and Jingyal Pak, *On the Nielsen number of a fiber map* 149
Stuart Edward Mills, *Normed Köthe spaces as intermediate spaces of L_1 and L_∞* 157
Philip Olin, *Free products and elementary equivalence* 175
Louis Jackson Ratliff, Jr., *Locally quasi-unmixed Noetherian rings and ideals of the principal class* 185
Seiya Sasao, *Homotopy types of spherical fibre spaces over spheres* 207
Helga Schirmer, *Fixed point sets of polyhedra* 221
Kevin James Sharpe, *Compatible topologies and continuous irreducible representations* 227
Frank Siwiec, *On defining a space by a weak base* 233
James McLean Sloss, *Global reflection for a class of simple closed curves* 247
M. V. Subba Rao, *On two congruences for primality* 261
Raymond D. Terry, *Oscillatory properties of a delay differential equation of even order* 269
Joseph Dinneen Ward, *Chebyshev centers in spaces of continuous functions* 283
Robert Breckenridge Warfield, Jr., *The uniqueness of elongations of Abelian groups* 289
V. M. Warfield, *Existence and adjoint theorems for linear stochastic differential equations* 305