ON THE NIELSEN NUMBER OF A FIBER MAP

Dennis McGavran and Jingyal Pak
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Suppose \( \mathcal{F} = \{E, \pi, B, F\} \) is a fiber space such that \( 0 \to \pi_1(F) \to \pi_0(E) \to \pi_0(B) \to 0 \) is exact. Suppose also that the above fundamental groups are abelian. If \( f: E \to E \) is a fiber preserving map such that \( f_\ast(\alpha) = \alpha \) if and only if \( \alpha = 0 \), then it is shown that \( N(f) = N(f') \cdot N(f_h) \) where \( N(h) \) is the Reidemeister number of the map \( h \).

A product formula for the Nielsen number of a fiber map which holds under certain conditions was introduced by R. Brown. Let \( \mathcal{F} = \{E, \pi, L, (p, q), s^1\} \) be a principal \( s^1 \)-bundle over the lens space \( L(p, q) \), where \( \mathcal{F} \) is determined by \( [f_\ast] \in [L(p, q), \pi^0] \simeq H^1(L(p, q), \mathbb{Z}) \simeq \mathbb{Z}_p \). Let \( f: E \to E \) be a fiber preserving map such that \( f_\ast(1) = c_1, f_\ast(\bar{1}_p) = \bar{c}_1 \), where \( 1 \) generates \( \pi_1(s^1) \simeq \mathbb{Z} \) and \( \bar{1}_p \) generates \( \pi_1(L(p, q)) \simeq \mathbb{Z}_p \). Then the Nielsen numbers of the maps involved satisfy

\[
N(f) = N(f_h) \cdot (d, 1 - c_1, s),
\]

where \( d = (j, p) \) and \( s = j/p(c_1 - c_\bar{1}) \).

I. Introduction. Let \( \mathcal{F} = \{E, \pi, B, F\} \) be a fiber space. Any fiber preserving map \( f: E \to E \) induces maps \( f^\ast: B \to B \), and, for each \( b \in B \), \( f^\ast: \pi_1(B) \to \pi_1(B) \), where \( \pi_1(B) \simeq F \). The map \( f \) will be called a fiber map (or bundle map if \( \mathcal{F} \) is a bundle).

Let \( N(g) \) denote the Nielsen number of a map \( g \). The Nielsen number, \( N(g) \), serves as a lower bound on the number of fixed points of a map homotopic to \( g \), and under certain hypotheses, there exists a map homotopic to \( g \) with exactly \( N(g) \) fixed points. R. Brown and E. Fadell ([2] and [3]) proved the following:

**Theorem.** Let \( \mathcal{F} = \{E, \pi, B, F\} \) be a locally trivial fiber space, where \( E, B, \) and \( F \) are connected finite polyhedra. Let \( f: E \to E \) be a fiber map. If one of the following conditions holds:

(i) \( \pi_1(B) = \pi_2(B) = 0 \).

(ii) \( \pi_1(F) = 0 \).

(iii) \( \mathcal{F} \) is trivial and either \( \pi_1(B) = 0 \) or \( f = f' \times f \), for all \( b \in B \) then \( N(f) = N(f') \cdot N(f_b) \) for all \( b \in B \).

These strong restrictions on the spaces involved eliminate some interesting fiber spaces. For example, any circle bundle over \( B \) with \( \pi_1(B) \neq 0 \) is excluded. Furthermore, if \( \pi_1(B) = \pi_2(B) = 0 \), then the total space \( E \) is \( B \times S^1 \).

This paper has two objectives. The first is to try to generalize
the above result to the case of a bundle \( \mathcal{T} = (E, \pi, B, F) \) where \( \pi_1(B) \) is a nontrivial abelian group, and \( \pi_2(B) = 0 \). The second is to investigate the relationships between the Nielsen numbers of the maps \( f, f', \) and \( f_b \) for particular circle bundles.

In this paper all spaces are path-connected.

II. Some general results. The reader may refer to [1] and [2] for definitions and details concerning the Nielsen number \( N(f) \), Reidemeister number \( R(f) \), and Jiang subgroup \( T(f) \) of a map \( f: X \to X \).

We will be particularly interested in the Reidemeister number. It serves as an upper bound on \( N(f) \) and in many cases \( R(f) = N(f) \). Let \( h: G \to G \) be a homomorphism where \( G \) is an abelian group. It is shown in [1] that \( R(h) = |\text{coker } (1 - h)| \) (| | means the order of a group). The Reidemeister number of a map \( f: X \to X \) is defined to be the Reidemeister number of the induced homomorphism \( f_\# : \pi_1(X) \to \pi_1(X) \). Now let \( \mathcal{T} \) be a fiber space. Let \( F_b = \pi^{-1}(b) \). If \( w: I \to B \) is such that \( w(0) = b \) and \( w(1) = b' \), we may translate \( F_{b'} \) along the path \( w \) to \( F_b \) (see [6]). This gives a homeomorphism \( \bar{w}: F_{b'} \to F_b \).

Given a fiber map \( f: E \to E \), we have the natural map \( f_b : F_b \to F_{f(b)} \), the restriction of \( f \) to \( F_b \). Then by definition \( f_b = \bar{w} \circ f'_b \). For more details on \( f_b : F_b \to F_b \) readers are referred to [2].

Suppose \( \mathcal{T} \) is a fiber space and \( w \) is a loop based at \( b \). Then we have \( \bar{w}: \pi^{-1}(b) \to \pi^{-1}(b) \). The fiber space \( \mathcal{T} \) is said to be orientable if the induced homomorphism \( \bar{w}_* : H_*(\pi^{-1}(b), z) \to H_*(\pi^{-1}(b), z) \) is the identity homomorphism for every loop \( w \) based at \( b \). It is shown in [2] that if \( \mathcal{T} \) is orientable and if the Jiang subgroup \( T(p^{-1}(b), e_b) = \pi_1(p^{-1}(b), e_b) \) for a fixed \( b \in B \) then the Nielsen number of \( f_b \) is independent of the choice of path from \( f'(b) \) to \( b \). Furthermore, the Nielsen number \( N(f_b) \) is independent of the choice of \( b \in B \).

**Lemma 1.** Let \( \mathcal{T} \) be a fiber space with \( \pi_1(F), \pi_1(E), \) and \( \pi_1(B) \) abelian. Suppose \( f: E \to E \) is a fiber map. Then the following diagram commutes:

\[
\begin{array}{ccc}
\pi_1(F_b) & \xrightarrow{i_b} & \pi_1(E) \\
\downarrow{1-f_b} & & \downarrow{1-f} \\
\pi_1(F_b) & \xrightarrow{i_b} & \pi_1(E).
\end{array}
\]

**Proof.** First, by [6], the map \( \bar{w} \) is homotopic in \( E \) to the identity map on \( F_{f'(b)} \). Hence we have

\[
i_2(1 - f_b)(\alpha) = i_2[\alpha - (\bar{w} \circ f'_b)(\alpha)] = i_2(\alpha) - i_2(\bar{w} \circ f'_b)(\alpha) = i_2(\alpha) - (i_2 \circ f'_b)(\alpha) = i_2(\alpha) - (f_b \circ i_b)(\alpha) = (1 - f_b) \circ i_b(\alpha).
\]
Lemma 2 [4]. Suppose we have the following commutative diagram of modules, where the rows are exact:

\[
\begin{array}{ccc}
0 & \longrightarrow & A' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & B' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & & \\
\downarrow & & \downarrow \\
C' & \longrightarrow & C
\end{array}
\]

Then there is an exact sequence

\[
0 \longrightarrow \ker \alpha \overset{\mu_*}{\longrightarrow} \ker \beta \overset{\varepsilon_*}{\longrightarrow} \ker \gamma \\
\overset{\omega}{\longrightarrow} \coker \alpha \overset{\mu'_*}{\longrightarrow} \coker \beta \overset{\varepsilon'_*}{\longrightarrow} \coker \gamma \longrightarrow 0 .
\]

The homomorphisms \( \mu_* \) and \( \varepsilon_* \) are restrictions of \( \mu \) and \( \varepsilon \), and \( \mu'_* \) and \( \varepsilon'_* \) are induced by \( \mu' \) and \( \varepsilon' \) on quotients. The connecting homomorphism \( \omega : \ker \gamma \to \coker \alpha \) is defined as follows. Let \( c \in \ker \gamma \), choose \( b \in B \) with \( \varepsilon b = c \). Since \( \varepsilon' \beta b = \gamma \varepsilon b = \gamma c = 0 \) there exists \( a' \in A' \) with \( \beta b = \mu a' \). Define \( \omega(c) = [a'] \), the coset of \( a' \) in \( \coker \alpha \). Then \( \omega \) is a well-defined homomorphism. See [4, p. 99] for the proof of the lemma.

Theorem 3. Suppose \( \mathcal{F} = \{E, \pi, B, F\} \) is a fiber space such that

\[
0 \longrightarrow \pi_1(F) \overset{i_*}{\longrightarrow} \pi_1(E) \overset{\pi_*}{\longrightarrow} \pi_1(B) \longrightarrow 0
\]

is an exact sequence of abelian groups. Suppose \( f : E \to E \) is a fiber map and \( w : I \to B \) is a path from \( b \) to \( f'(b) \). Then we have the following exact sequence:

\[
0 \longrightarrow \ker (1 - f_{sb}) \longrightarrow \ker (1 - f_s) \longrightarrow \ker (1 - f') \\
\longrightarrow \coker (1 - f_{sb}) \longrightarrow \coker (1 - f_s) \longrightarrow \coker (1 - f') \longrightarrow 0 .
\]

Proof. The fiber map induces the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_1(F') \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_1(F)
\end{array}
\]

\[
\begin{array}{ccc}
\overset{i_*}{\longrightarrow} & \pi_1(E) \overset{\pi_*}{\longrightarrow} \pi_1(B) \longrightarrow 0 \\
\downarrow (1-f_{sb}) & \downarrow (1-f_s) & \downarrow (1-f'_s) \\
\overset{i_*}{\longrightarrow} & \pi_1(E) \overset{\pi_*}{\longrightarrow} \pi_1(B) \longrightarrow 0
\end{array}
\]

Now the result becomes a simple application of Lemmas 1 and 2.

Corollary 4. \( \ker (1 - f_{sb}) \) is independent of \( w \) and \( b \).

Proof. \( \ker (1 - f_{sb}) \) is isomorphic to the kernel of the map \( \ker (1 - f_s) \overset{\pi_*}{\longrightarrow} \ker (1 - f'_s) \). But this map is the restriction of \( \pi_* : \pi_1(E) \to \pi_1(B) \), which is independent of \( w \) and \( b \).
Suppose \( h: G \to G \) is a homomorphism of abelian groups. We will say that \( h \) satisfies Condition A if \( h(\alpha) = \alpha \) if and only if \( \alpha = 0 \).

**Theorem 5.** Suppose \( \mathcal{F} \) is a fiber space satisfying the hypotheses of Theorem 3. Suppose \( f: E \to E \) is a fiber map such that \( f' \) satisfies Condition A. Then \( R(f) = R(f') \cdot R(f_b) \) for all \( b \in B \).

*Proof.* We have \((1 - f'_i)(\alpha) = 0\) if and only if \( f'_i(\alpha) = \alpha \) if and only if \( \alpha = 0 \). Therefore, \( 1 - f'_i \) is injective and we have the following exact sequence:

\[
0 \to \text{coker } (1 - f_{i_b}) \to \text{coker } (1 - f_i) \to \text{coker } (1 - f'_i) \to 0 .
\]

The theorem follows from the properties of \( R(f) \).

**Corollary 6.** Under the hypotheses of Theorem 5 \( R(f_b) \) is independent of \( w \) and \( b \).

*Proof.* This follows since both \( R(f) \) and \( R(f') \) are independent of \( w \) and \( b \).

**Example 1.** Let \( \mathcal{F} \) be a principal \( T^k \)-bundle over a \((2n + 1)\)-dimensional lens space \( L(p) \), \( p \equiv 1 \). We know from [5] that \( L = L(d) \times T^k \) where \( d \) divides \( p \). Let \( f: E \to E \) be a bundle map. It follows easily from results in [1] that \( N(f_i) = R(f_i) \). It is also shown in [1] that \( N(f'_n) = R(f'_n) \) for \( n = 1 \), and the proof can be easily generalized to higher dimensions. Furthermore, by showing that \( T(f) = \pi_1(L(d) \times T^k) \), where \( T(f) \) is the Jiang subgroup of \( f \), one can show that \( N(f) = R(f) \). Now such a bundle satisfies the hypothesis of Theorem 3. Hence, if \( f'_i: \pi_1(L(p)) \to \pi_1(L(p)) \) satisfies the hypothesis of Theorem 5, we have \( N(f) = N(f'_n) \cdot N(f_i) \) for all \( b \in B \).

**Example 2.** If \( G \) is a compact connected semi-simple Lie group, then \( \mathcal{F} = \{E, \pi, G, S^i\} \) satisfies the hypothesis of Theorem 3. If \( f: E \to E \) is a fiber map then \( N(f) = N(f'_n) \cdot N(f_i) \) follows from [3] since the second integral cohomology group of \( G \) vanishes. Assume \( N(f'_n) \neq 0 \neq N(f_i) \). Then since \( G \) and \( S^i \) are H-spaces \( T(f') = \pi_1(G) \) and \( T(f_i) = \pi_1(S^i) \); and we have \( N(f'_n) = R(f'_n) \) and \( N(f_i) = R(f_i) \). It follows that \( R(f) = R(f'_n) \cdot R(f_i) \) independent of Condition A.

**Lemma 7.** Suppose \( h: Z_p \to Z_p \) is such that \( h(\overline{l}) = \overline{m} \). Then Condition A holds iff \( (1 - m, p) = 1 \).

*Proof.* Suppose \( (1 - m, p) = 1 \). If \( h(\overline{n}) = \overline{m\overline{n}} = \overline{n}, 1 \leq n < p \), then \( mn \equiv n \pmod{p} \). Hence \( p \) divides \((1 - m)n\), which is impossible if \( (1 - m, p) = 1 \).
Now suppose \( h(\alpha) = \alpha \) iff \( \alpha = 0 \). Suppose \((1 - m, p) = d\). Let \(1 - m = c \cdot d, \ p = c \cdot d\). Then \( h(\bar{e}_2) = \bar{m} \cdot \bar{e}_2\). Now

\[
mc_2 - c_2 = c_2(m - 1) = -c_2c_1d = -c_1p.
\]

Thus \( h(\bar{e}_2) = \bar{e}_2 \) and \( d = 1 \).

**EXAMPLE 1 (con't).** We have \( \pi_1(L(p)) \cong \mathbb{Z}_p \). Suppose \( f'_*(\bar{l}) = \bar{m} \). Then \( N(f') = (1 - m, p) \). Hence Theorem 5 is applicable if and only if \( N(f') = 1 \).

**III. A general solution to Example 1.** Let \( \mathcal{F} = (E, \pi, (L(p, q), s^1) \) be a principal \( s^1 \)-bundle over a 3-dimensional lens space \( L(p, q) \). If \( \mathcal{F} \) is induced by \( \bar{f}_* \in [L(p, q), CP^\infty] \cong H^1(L(p, q), \mathbb{Z}) \cong \mathbb{Z}_p \), then \( E \cong L(d, q) \times s^1 \), where \( d = (j, p) \) (see [7]). Let \( j = j'd, \ p = p'd \).

**THEOREM 8.** Let \( \mathcal{F} \) be as above and \( f: E \to E \) a fiber map such that, for a particular choice of \( b \in B \) and \( w \), \( f_*(1) = c \cdot s_1 \) and \( f'_*(\bar{l}_p) = \bar{c}_1 \), where 1 generates \( \pi_1(s^1) \cong \mathbb{Z} \) and \( \bar{l}_p \) generates \( \pi_1(L(p, q)) \cong \mathbb{Z}_p \). Let \( s = j/p(c_1 - c_2) \). Then

\[
N(f) = N(f'_*) \cdot (d, 1 - c_1, s).
\]

**Proof.** We first examine the structure of \( L(d, q) \times s^1 \) as an \( s^1 \)-bundle over \( L(p, q) \) (see [7]). \( L(p, q) \) and \( L(d, q) \) are obtained from \( s^1 \) as the orbit space of a free \( \mathbb{Z}_p \)-action and \( \mathbb{Z}_f \)-action, respectively. Given \((r_1, \theta), (r_2, \theta_2) \in s^1 \), let \( \langle (r_1, \theta), (r_2, \theta_2) \rangle \) represent its equivalence class as an element in \( L(p, q) \). In \( L(d, q) \times I, I = [0, 2\pi] \), identify \( \{\langle (r_1, \theta), (r_2, \theta_2) \rangle, 2\pi\} \) with \( \{\langle (r_1, \theta_1 + j\nu), \theta_2 + j(\nu), 0\rangle, 0 \} \) to obtain \( E \), where \( \nu = 2\pi/p \). Define \( h: E \to L(d, q) \times S^1 \) by

\[
h(\langle (r_1, \theta), (r_2, \theta_2), \nu, t \rangle) = \left\{ \left( r_1, \theta_1 + \frac{t}{2\pi} j\nu, r_2, \theta_2 + \frac{t}{2\pi} j(\nu), t \right) \right\}.
\]

Then \( h \) is a homeomorphism. Let \( \pi_1(L(d, q) \times S^1) \) be generated by \((\bar{l}_d, 0) \) and \((0, 1) \). Then \( \langle (\bar{l}_d, 0), (0, 0) \rangle \) is represented by the loop \( \bar{\sigma}_1 = \langle (1, t(2\pi/d)), (0, 0) \rangle \), \( 0 \leq t \leq 1 \), and \((0, 1) \) is represented by \( \bar{\sigma}_2 = \langle (1, 0), (0, 0) \rangle, t \), \( 0 \leq t \leq 2\pi \). Then in \( E \), \( \sigma_1 = \langle (1, t(2\pi/d)), (0, 0) \rangle \) and \( \sigma_2 = \langle (1, -t/(2\pi))j\nu), (0, 0) \rangle, t \rangle \) represent \( \langle (l_d, 0) \) and \((0, 1) \) respectively. \( \bar{l}_p \) is represented by the loop \( \gamma = \langle (1, tv), (0, 0) \rangle \leq t \leq 1 \). Now the projection map \( \pi: E \to L(p, q) \) is given by

\[
\pi(\langle (r_1, \theta), (r_2, \theta_2) \rangle, \nu, t) = \langle (r_1, \theta), (r_2, \theta_2) \rangle.
\]

We have

\[
\pi \circ \sigma_1 = \left\langle \left( 1, \frac{t2\pi}{d} \right), (0, 0) \right\rangle \leq t \leq 1 = \langle (1, tp\nu), (0, 0) \rangle.
\]
Hence
\[ \pi_\delta(l, 0) = \bar{p}' . \]
Also
\[ \pi_\sigma \sigma_\delta = \left\{ \left( 1, -\frac{t}{2\pi} j'v \right), (0, 0) \right\} \quad 0 \leq t \leq 2\pi \]
so
\[ \pi_\delta(0, 1) = -\bar{j}' . \]

One fiber in \( E \) consists of
\[ \bigcup_{0 \leq t \leq 2\pi} \left\{ \left( 1, n_j'v, (0, 0) \right), t \right\} . \]
Hence, in \( L(d, q) \times S' \), this fiber is
\[ \bigcup_{0 \leq t \leq 2\pi} \left\{ \left( 1, \left( n + \frac{t}{2\pi} \right) j'v, (0, 0) \right), t \right\} = \left\{ \left( 1, n_j'v, (0, 0) \right), 2\pi \bar{j}' \right\} \]
where \( 0 \leq \tau \leq p' \) and \( 2\pi \bar{j}' \) represents the equivalence class of \( 2\pi \tau (\text{mod} \ 2\pi) \). Hence \( \iota_\delta(1) = (j', p') \).

We have the following commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_\delta(S') \xrightarrow{i_\delta} \pi_\delta(L(d, q) \times S') \xrightarrow{\pi_\delta} \pi_\delta(L(p, q)) \longrightarrow 0 \\
(1-f_\delta) & | & | \\
0 & \longrightarrow & \pi_\delta(S') \xrightarrow{i_\delta} \pi_\delta(L(d, q) \times S') \xrightarrow{\pi_\delta} \pi_\delta(L(p, q)) \longrightarrow 0 .
\end{array}
\]
We must compute the cokernel of \( (1 - f_\delta) \) since \( N(f) = |\coker (1 - f_\delta)|. \)
Let
\[ (1-f_\delta)(\bar{l}, 0) = (\bar{a}, 0) \quad (1-f_\delta)(0, 1) = (\bar{a}, u) . \]
Commutativity of the right hand square implies that \( a = 1 - c' \), while commutativity of the left hand square implies \( u = 1 - c' \). Now
\[ (1-f_\delta) \circ \pi_\delta(0, 1) = -(1-c')j' \]
\[ \pi_\delta(1-f_\delta)(0, 1) = p's - j'u = p's - j'(1-c) . \]
Hence
\[ p's - j'(1-c) \equiv -(1-c)j'(\text{mod} \ p) . \]
Therefore,
\[ j'(c_2 - c) + p's = kp . \]
We must have \( p' \mid j'(c_2 - c) \) so
\[ s = kd + \frac{j'}{p'}(c_1 - c_2). \]

Hence we may assume
\[ s = \frac{j'}{p'}(c_1 - c_2) = \frac{j}{p}(c_1 - c_2). \]

Therefore, \( \text{Im}(1 - f_d) \) is generated by \((1 - c_1, 0), (\bar{s}, 0), \) and \((0, 1 - c_2)\). Now the group \( \pi_1(L(d, q) \times S^1) \cong z_d \oplus z \), and the subgroup generated by \((1 - c_1, 0)\) and \((\bar{s}, 0)\) is the subgroup generated by \(((1 - c_1, s), 0)\). Consequently, the cokernel of \( (1 - f_d) \) is isomorphic to \( z_d/(1 - c_1, s)z_d \oplus z/(1 - c_2)z \). Which, in turn, is isomorphic to \( z/(1 - c_2)z \). Therefore,
\[ |\text{coker} (1 - f_d)| = N(f) = (d, 1 - c_1, s) \cdot |1 - c_2| = (d, 1 - c_1, s) \cdot N(f_d). \]

**Note.** (1) Since \( \mathcal{L} \) is orientable and \( T(\pi^{-1}(b), e_b) = \pi_1(\pi^{-1}(b), e_b) \), the above formula is independent of \( w \) and \( b \).

(2) In the above argument we could replace \( L(p, q) \) with the generalized lens space as in [5].

(3) If \( p \) is a prime the product formula follows from results of R. Brown and E. Fadell [3].

(4) Theorem 8 also indicates that a product theorem of the type obtained by R. Brown and E. Fadell is hard to expect in general.

**Corollary 9.** Let \( \mathcal{L} \) be as in Theorem 8. Suppose \( f: E \to E \) is a bundle map such that for some \( b \in L(p, q) \) \( f_b: \pi^{-1}(b) \to \pi^{-1}(b) \) is homotopic to a fixed-point free map. Then there exists a map \( g: E \to E \), homotopic to \( f \), which is fixed-point free.

**Proof.** Let \( \tilde{f}_b \) be the fixed-point free map on \( \pi^{-1}(b) \) which is homotopic to \( f_b \). Clearly \( N(\tilde{f}_b) = 0 \) and since the Nielsen number is a homotopy invariant, \( N(f_b) = 0 \). Thus from Theorem 8, \( N(f) = 0 \), and the corollary follows from the converse of the Lefschetz fixed-point theorem of F. Wecken [8].

**References**


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