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# ON THE NIELSEN NUMBER OF A FIBER MAP

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**Suppose $\mathscr{F} = \{E, \pi, B, F\}$ is a fiber space such that $0 \to \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{\pi_\#} \pi_1(B) \to 0$ is exact. Suppose also that the above fundamental groups are abelian. If $f: E \to E$ is a fiber preserving map such that $f_\#(\alpha) = \alpha$ if and only if $\alpha = 0$, then it is shown that $R(f) = R(f') \cdot R(f_b)$ where $R(h)$ is the Reidemeister number of the map $h$.**

**A product formula for the Nielsen number of a fiber map which holds under certain conditions was introduced by R. Brown. Let $\mathscr{F} = \{E, \pi, L, (p, q), s^1\}$be a principal $s^1$-bundle over the lens space $L(p, q)$, where $\mathscr{F}$ is determined by $[f_j] \in [L(p, q), cp^\infty] \simeq H^2(L(p, q), z) \simeq z_p$. Let $f: E \to E$ be a fiber preserving map such that $f_{b\#}(1) = c_2, f'_\#(\bar{l}_p) = \bar{c}_1$, where 1 generates $\pi_1(s^1) \simeq z$ and $\bar{l}_p$ generates $\pi_1(L(p, q)) \simeq z_p$. Then the Nielsen numbers of the maps involved satisfy**

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s) ,$$

**where $d = (j, p)$ and $s = j/p(c_1 - c_2)$.**

I. Introduction. Let $\mathscr{F} = \{E, \pi, B, F\}$ be a fiber space. Any fiber preserving map $f: E \to E$ induces maps $f': B \to B$, and, for each $b \in B$, $f_b: \pi^{-1}(b) \to \pi^{-1}(b)$, where $\pi^{-1}(b) \simeq F$. The map $f$ will be called a fiber map (or bundle map if $\mathscr{F}$ is a bundle).

Let $N(g)$ denote the Nielsen number of a map $g$. The Nielsen number, $N(g)$, serves as a lower bound on the number of fixed points of a map homotopic to $g$, and under certain hypotheses, there exists a map homotopic to $g$ with exactly $N(g)$ fixed points. R. Brown and E. Fadell ([2] and [3]) proved the following:

THEOREM. *Let $\mathscr{F} = \{E, \pi, B, F\}$ be a locally trivial fiber space, where $E$, $B$, and $F$ are connected finite polyhedra. Let $f: E \to E$ be a fiber map. If one of the following conditions holds:*

(i) $\pi_1(B) = \pi_2(B) = 0$ .

(ii) $\pi_1(F) = 0$ .

(iii) *$\mathscr{F}$ is trivial and either $\pi_1(B) = 0$ or $f = f' \times f_b$ for all $b \in B$ then $N(f) = N(f') \cdot N(f_b)$ for all $b \in B$.*

These strong restrictions on the spaces involved eliminate some interesting fiber spaces. For example, any circle bundle over $B$ with $\pi_1(B) \neq 0$ is excluded. Furthermore, if $\pi_1(B) = \pi_2(B) = 0$, then the total space $E$ is $B \times S^1$.

This paper has two objectives. The first is to try to generalize the above result to the case of a bundle $\mathscr{T} = \{E, \pi, B, F\}$ where $\pi_1(B)$ is a nontrivial abelian group, and $\pi_2(B) = 0$. The second is to investigate the relationships between the Nielsen numbers of the maps $f, f'$, and $f_b$ for particular circle bundles.

In this paper all spaces are path-connected.

**II. Some general results.** The reader may refer to [1] and [2] for definitions and details concerning the Nielsen number $N(f)$, Reidemeister number $R(f)$, and Jiang subgroup $T(f)$ of a map $f: X \to X$.

We will be particularly interested in the Reidemeister number. It serves as an upper bound on $N(f)$ and in many cases $R(f) = N(f)$. Let $h: G \to G$ be a homomorphism where $G$ is an abelian group. It is shown in [1] that $R(h) = |\operatorname{coker}(1-h)|$ ($|\ |$ means the order of a group). The Reidemeister number of a map $f: X \to X$ is defined to be the Reidemeister number of the induced homomorphism $f_\#: \pi_1(X) \to \pi_1(X)$. Now let $\mathscr{T}$ be a fiber space. Let $F_b = \pi^{-1}(b)$. If $w: I \to B$ is such that $w(0) = b$ and $w(1) = b'$, we may translate $F_{b'}$ along the path $w$ to $F_b$ (see [6]). This gives a homeomorphism $\bar{w}: F_{b'} \to F_b$. Given a fiber map $f: E \to E$, we have the natural map $f_b'': F_b \to F_{f'(b)}$, the restriction of $f$ to $F_b$. Then by definition $f_b = \bar{w} \circ f_b''$. For more details on $f_b: F_b \to F_b$ readers are referred to [2].

Suppose $\mathscr{T}$ is a fiber space and $w$ is a loop based at $b$. Then we have $\bar{w}: \pi^{-1}(b) \to \pi^{-1}(b)$. The fiber space $\mathscr{T}$ is said to be orientable if the induced homomorphism $\bar{w}_*: H_*(\pi^{-1}(b), z) \to H_*(\pi^{-1}(b), z)$ is the identity homomorphism for every loop $w$ based at $b$. It is shown in [2] that if $\mathscr{T}$ is orientable and if the Jiang subgroup $T(p^{-1}(b), e_0) = \pi_1(p^{-1}(b), e_0)$ for a fixed $b \in B$ then the Nielsen number of $f_b$ is independent of the choice of path from $f'(b)$ to $b$. Furthermore, the Nielsen number $N(f_b)$ is independent of the choice of $b \in B$.

LEMMA 1. *Let $\mathscr{T}$ be a fiber space with $\pi_1(F)$, $\pi_1(E)$, and $\pi_1(B)$ abelian. Suppose $f: E \to E$ is a fiber map. Then the following diagram commutes*:

$$\begin{array}{ccc}
\pi_1(F_b) & \xrightarrow{\quad i_\# \quad} & \pi_1(E) \\
\Big\downarrow{\scriptstyle 1-f_{b\#}} & & \Big\downarrow{\scriptstyle 1-f_\#} \\
\pi_1(F_b) & \xrightarrow{\quad i_\# \quad} & \pi_1(E)\ .
\end{array}$$

*Proof.* First, by [6], the map $\bar{w}$ is homotopic in $E$ to the identity map on $F_{f'(b)}$. Hence we have

$$\begin{aligned}
i_\# \circ (1 - f_{b\#})(\alpha) &= i_\#[\alpha - (\bar{w} \circ f_b'')_\#(\alpha)] \\
&= i_\#(\alpha) - i_\#(\bar{w} \circ f_b'')_\#(\alpha) = i_\#(\alpha) - (i_\# \circ f_{b\#}'')(\alpha) \\
&= i_\#(\alpha) - (f_\# \circ i_\#)(\alpha) = (1 - f_\#) \circ i_\#(\alpha)\ .
\end{aligned}$$

LEMMA 2 [4]. *Suppose we have the following commutative diagram of modules, where the rows are exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & \longrightarrow & 0 . \end{array}$$

*Then there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \ker \alpha \xrightarrow{\mu_*} \ker \beta \xrightarrow{\varepsilon_*} \ker \gamma \\ \xrightarrow{\omega} \operatorname{coker} \alpha \xrightarrow{\mu'_*} \operatorname{coker} \beta \xrightarrow{\varepsilon'_*} \operatorname{coker} \gamma \longrightarrow 0 . \end{aligned}$$

The homomorphisms $\mu_*$ and $\varepsilon_*$ are restrictions of $\mu$ and $\varepsilon$, and $\mu'_*$ and $\varepsilon'_*$ are induced by $\mu'$ and $\varepsilon'$ on quotients. The connecting homomorphism $\omega\colon \ker \gamma \to \operatorname{coker} \alpha$ is defined as follows. Let $c \in \ker \gamma$, choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon'\beta b = \gamma\varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $\omega(c) = [a']$, the coset of $a'$ in $\operatorname{coker} \alpha$. Then $\omega$ is a well-defined homomorphism. See [4, p. 99] for the proof of the lemma.

THEOREM 3. *Suppose $\mathscr{T} = \{E, \pi, B, F\}$ is a fiber space such that*

$$0 \longrightarrow \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{\pi_\#} \pi_1(B) \longrightarrow 0$$

*is an exact sequence of abelian groups. Suppose $f\colon E \to E$ is a fiber map and $w\colon I \to B$ is a path from $b$ to $f'(b)$. Then we have the following exact sequence:*

$$\begin{aligned} 0 \longrightarrow \ker (1 - f_{b\#}) \longrightarrow \ker (1 - f_\#) \longrightarrow \ker (1 - f'_\#) \\ \longrightarrow \operatorname{coker} (1 - f_{b\#}) \longrightarrow \operatorname{coker} (1 - f_\#) \longrightarrow \operatorname{coker} (1 - f'_\#) \longrightarrow 0 . \end{aligned}$$

*Proof.* The fiber map induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(F) & \xrightarrow{\quad i_\# \quad} & \pi_1(E) & \xrightarrow{\quad \pi_\# \quad} & \pi_1(B) & \longrightarrow & 0 \\ & & {\scriptstyle(1-f_{b\#})}\downarrow & & {\scriptstyle(1-f_\#)}\downarrow & & {\scriptstyle(1-f'_\#)}\downarrow & & \\ 0 & \longrightarrow & \pi_1(F) & \xrightarrow{\quad i_\# \quad} & \pi_1(E) & \xrightarrow{\quad \pi_\# \quad} & \pi_1(B) & \longrightarrow & 0 . \end{array}$$

Now the result becomes a simple application of Lemmas 1 and 2.

COROLLARY 4. $\ker (1 - f_{b\#})$ *is independent of $w$ and $b$.*

*Proof.* $\ker (1 - f_{b\#})$ is isomorphic to the kernel of the map $\ker (1 - f_\#) \xrightarrow{\pi_{\#*}} \ker (1 - f'_\#)$. But this map is the restriction of $\pi_\#\colon \pi_1(E) \to \pi_1(B)$, which is independent of $w$ and $b$.

Suppose $h: G \to G$ is a homomorphism of abelian groups. We will say that $h$ satisfies Condition A if $h(\alpha) = \alpha$ if and only if $\alpha = 0$.

THEOREM 5. *Suppose $\mathscr{T}$ is a fiber space satisfying the hypotheses of Theorem 3. Suppose $f: E \to E$ is a fiber map such that $f'_\#$ satisfies Condition* A. *Then $R(f) = R(f') \cdot R(f_b)$ for all $b \in B$.*

*Proof.* We have $(1 - f'_\#)(\alpha) = 0$ if and only if $f'_\#(\alpha) = \alpha$ if and only if $\alpha = 0$. Therefore, $1 - f'_\#$ is injective and we have the following exact sequence:

$$0 \to \operatorname{coker}(1 - f_{b\#}) \to \operatorname{coker}(1 - f_\#) \to \operatorname{coker}(1 - f'_\#) \to 0 .$$

The theorem follows from the properties of $R(f)$.

COROLLARY 6. *Under the hypotheses of Theorem* 5 *$R(f_b)$ is independent of $w$ and $b$.*

*Proof.* This follows since both $R(f)$ and $R(f')$ are independent of $w$ and $b$.

EXAMPLE 1. Let $\mathscr{T}$ be a principal $T^k$-bundle over a $(2n + 1)$-dimensional lens space $L(p)$, $p \geqq 1$. We know from [5] that $L = L(d) \times T^k$ where $d$ divides $p$. Let $f: E \to E$ be a bundle map. It follows easily from results in [1] that $N(f_b) = R(f_b)$. It is also shown in [1] that $N(f') = R(f')$ for $n = 1$, and the proof can be easily generalized to higher dimensions. Furthermore, by showing that $T(f) = \pi_1(L(d) \times T^k)$, where $T(f)$ is the Jiang subgroup of $f$, one can show that $N(f) = R(f)$. Now such a bundle satisfies the hypothesis of Theorem 3. Hence, if $f'_\#: \pi_1(L(p)) \to \pi_1(L(p))$ satisfies the hypothesis of Theorem 5, we have $N(f) = N(f') \cdot N(f_b)$ for all $b \in B$.

EXAMPLE 2. If $G$ is a compact connected semi-simple Lie group, then $\mathscr{T} = \{E, \pi, G, S^1\}$ satisfies the hypothesis of Theorem 3. If $f: E \to E$ is a fiber map then $N(f) = N(f') \cdot N(f_b)$ follows from [3] since the second integral cohomology group of $G$ vanishes. Assume $N(f') \neq 0 \neq N(f_b)$. Then since $G$ and $S^1$ are $H$-spaces $T(f') = \pi_1(G)$ and $T(f_b) = \pi_1(S^1)$; and we have $N(f') = R(f')$ and $N(f_b) = R(f_b)$. It follows that $R(f) = R(f') \cdot R(f_b)$ independent of Condition A.

LEMMA 7. *Suppose $h: Z_p \to Z_p$ is such that $h(\bar{1}) = \bar{m}$. Then Condition* A *holds iff $(1 - m, p) = 1$.*

*Proof.* Suppose $(1 - m, p) = 1$. If $h(\bar{n}) = \bar{m}\bar{n} = \bar{n}$, $1 \leqq n < p$, then $mn \equiv n (\operatorname{mod} p)$. Hence $p$ divides $(1 - m)n$, which is impossible if $(1 - m, p) = 1$.

Now suppose $h(\alpha)=\alpha$ iff $\alpha = 0$. Suppose $(1 - m, p) = d$. Let $1 - m = c_1 d$, $p = c_2 d$. Then $h(\bar{c}_2) = \bar{m}\bar{c}_2$. Now

$$mc_2 - c_2 = c_2(m - 1) = -c_2 c_1 d = -c_1 p \, .$$

Thus $h(\bar{c}_2) = \bar{c}_2$ and $d = 1$.

EXAMPLE 1 (con't). We have $\pi_1(L(p)) \simeq Z_p$. Suppose $f'_\#(\bar{l}) = \bar{m}$. Then $N(f') = (1 - m, p)$. Hence Theorem 5 is applicable if and only if $N(f') = 1$.

**III. A general solution to Example 1.** Let $\mathscr{T} = \{E, \pi, L(p, q), s^1\}$ be a principal $s^1$-bundle over a 3-dimensional lens space $L(p, q)$. If $\mathscr{T}$ is induced by $[f_j] \in [L(p, q), CP^\infty] \simeq H^2(L(p, q), Z) \simeq Z_p$, then $E \simeq L(d, q) \times s^1$, where $d = (j, p)$(see [7]). Let $j = j'd$, $p = p'd$.

THEOREM 8. *Let $\mathscr{T}$ be as above and $f\colon E \to E$ a fiber map such that, for a particular choice of $b \in B$ and $w$, $f_{b\#}(1) = c_2$ and $f'_\#(\bar{l}_p) = \bar{c}_1$, where 1 generates $\pi_1(s^1) \simeq Z$ and $\bar{l}_p$ generates $\pi_1(L(p, q)) \simeq Z_p$. Let $s = j/p(c_1 - c_2)$. Then*

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s) \, .$$

*Proof.* We first examine the structure of $L(d, q) \times s^1$ as an $s^1$-bundle over $L(p, q)$ (see [7]). $L(p, q)$ and $L(d, q)$ are obtained from $s^3$ as the orbit space of a free $Z_p$-action and $Z_d$-action, respectively. Given $((r_1, \theta_1), (r_2, \theta_2)) \in s^3$, let $\langle (r_1, \theta_1), (r_2, \theta_2) \rangle$ represent its equivalence class as an element in $L(p, q)$. In $L(d, q) \times I$, $I = [0, 2\pi]$, identify $\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, 2\pi\}$ with $\{\langle (r_1, \theta_1 + j'v), (r_2, \theta_2 + j'qv) \rangle, 0\}$ to obtain $E$, where $v = 2\pi/p$. Define $h\colon E \to L(d, q) \times S^1$ by

$$h\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t\} = \left\{ \left\langle \left( r_1, \theta_1 + \frac{t}{2\pi} j'v \right), \left( r_2, \theta_2 + \frac{t}{2\pi} j'qv \right) \right\rangle, t \right\} .$$

Then $h$ is a homeomorphism. Let $\pi_1(L(d, q) \times S^1)$ be generated by $(\bar{l}_d, 0)$ and $(0, 1)$. Then $(\bar{l}_d, 0)$ is represented by the loop $\bar{\sigma}_{\bar{l}} = \{\langle (1, t(2\pi/d)), (0, 0) \rangle, 0\}$, $0 \leqq t \leqq 1$, and $(0, 1)$ is represented by $\bar{\sigma}_2 = \{\langle (1, 0), (0, 0) \rangle, t\}$, $0 \leqq t \leqq 2\pi$. Then in $E$, $\sigma_1 = \{\langle (1, t(2\pi/d)), (0, 0) \rangle, 0\}$ and $\sigma_2 = \{\langle (1, -t/(2\pi) j'v), (0, 0) \rangle, t\}$ represent $(\bar{l}_d, 0)$ and $(0, 1)$ respectively. $\bar{l}_p$ is represented by the loop $\gamma = \langle\!\langle (1, tv), (0, 0) \rangle\!\rangle$ $0 \leqq t \leqq 1$. Now the projection map $\pi\colon E \to L(p, q)$ is given by

$$\pi\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t\} = \langle\!\langle (r_1, \theta_1), (r_2, \theta_2) \rangle\!\rangle \, .$$

We have

$$\pi \circ \sigma_1 = \left\langle\!\!\left\langle \left(1, t\frac{2\pi}{d}\right), (0, 0) \right\rangle\!\!\right\rangle \quad 0 \leqq t \leqq 1 = \langle\!\langle (1, tp'v), (0, 0) \rangle\!\rangle \, .$$

Hence

$$\pi_\#(\bar{l}_d, 0) = \bar{p}' .$$

Also

$$\pi \circ \sigma_2 = \left\langle\!\left\langle\left(1, \ -\frac{t}{2\pi}j'v\right), (0, 0)\right\rangle\!\right\rangle \qquad 0 \leqq t \leqq 2\pi$$

so

$$\pi_\#(0, 1) = -\bar{j}' .$$

One fiber in $E$ consists of

$$\bigcup_{\substack{n=0\\ 0\leqq t\leqq 2\pi}}^{p'} \{\langle(1, nj'v), (0, 0)\rangle, t\} .$$

Hence, in $L(d, q) \times S^1$, this fiber is

$$\bigcup_{\substack{n=0\\ 0\leqq t\leqq 2\pi}}^{p'} \left\{\left\langle\left(1, \left(n + \frac{t}{2\pi}\right)j'v\right), (0, 0)\right\rangle, t\right\} = \{\langle(1, \tau j'v), (0, 0)\rangle, \overline{2\tau\pi}\}$$

where $0 \leqq \tau \leqq p'$ and $\overline{2\pi\tau}$ represents the equivalence class of $2\pi\tau(\bmod 2\pi)$. Hence $i_\#(1) = (\bar{j}', p')$.

We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_\#} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_\#} & \pi_1(L(p, q)) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (1-f_{b\#})} & & \Big\downarrow{\scriptstyle (1-f_\#)} & & \Big\downarrow{\scriptstyle (1-f'_\#)} & & \\
0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_\#} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_\#} & \pi_1(L(p, q)) & \longrightarrow & 0 .
\end{array}$$

We must compute the cokernel of $(1 - f_\#)$ since $N(f) = |\operatorname{coker}(1 - f_\#)|$. Let

$$(1 - f_\#)(\bar{l}_d, 0) = (\bar{a}, 0) \qquad (1 - f_\#)(0, 1) = (\bar{s}, u) .$$

Commutativity of the right hand square implies that $a = 1 - c_1$, while commutativity of the left hand square implies $u = 1 - c_2$. Now

$$\begin{aligned}
(1 - f'_\#)\circ\pi_\#(0, 1) &= \overline{-(1 - c_1)j'} \\
\pi_\#\circ(1 - f_\#)(0, 1) &= \overline{p's - j'u} = \overline{p's - j'(1 - c_2)} .
\end{aligned}$$

Hence

$$p's - j'(1 - c_2) \equiv -(1 - c_1)j'(\bmod p) .$$

Therefore,

$$j'(c_2 - c_1) + p's = kp .$$

We must have $p' \mid j'(c_2 - c_1)$ so

$$s = kd + \frac{j'}{p'}(c_1 - c_2) .$$

Hence we may assume

$$s = \frac{j'}{p'}(c_1 - c_2) = \frac{j}{p}(c_1 - c_2) .$$

Therefore, $\mathrm{Im}(1 - f_\sharp)$ is generated by $(\overline{1 - c_1}, 0)$, $(\bar{s}, 0)$, and $(0, 1 - c_2)$. Now the group $\pi_1(L(d, q) \times S^1) \simeq z_d \oplus z$, and the subgroup generated by $(\overline{1 - c_1}, 0)$ and $(\bar{s}, 0)$ is the subgroup generated by $(\overline{(1 - c_1, s)}, 0)$. Consequently, the cokernel of $(1 - f_\sharp)$ is isomorphic to $z_d/(1 - c_1, s)z_d \oplus z/(1-c_2)z$. Which, in turn, is isomorphic to $z_{(d,1-c_1,s)} \oplus z_{(1-c_2)}$. Therefore,

$$|\operatorname{coker}(1 - f_\sharp)| = N(f) = (d, 1 - c_1, s)\cdot|1 - c_2| = (d, 1 - c_1, s)\cdot N(f_b) .$$

*Note.* (1) Since $\mathscr{T}$ is orientable and $T(\pi^{-1}(b), e_0) = \pi_1(\pi^{-1}(b), e_0)$, the above formula is independent of $w$ and $b$.

(2) In the above argument we could replace $L(p, q)$ with the generalized lens space as in [5].

(3) If $p$ is a prime the product formula follows from results of R. Brown and E. Fadell [3].

(4) Theorem 8 also indicates that a product theorem of the type obtained by R. Brown and E. Fadell is hard to expect in general.

COROLLARY 9. *Let $\mathscr{T}$ be as in Theorem 8. Suppose $f\colon E \to E$ is a bundle map such that for some $b \in L(p, q)$ $f_b\colon \pi^{-1}(b) \to \pi^{-1}(b)$ is homotopic to a fixed-point free map. Then there exists a map $g\colon E \to E$, homotopic to $f$, which is fixed-point free.*

*Proof.* Let $\tilde{f}_b$ be the fixed-point free map on $\pi^{-1}(b)$ which is homotopic to $f_b$. Clearly $N(\tilde{f}_b) = 0$ and since the Nielsen number is a homotopy invariant, $N(f_b) = 0$. Thus from Theorem 8, $N(f) = 0$, and the corollary follows from the converse of the Lefschetz fixed-point theorem of F. Wecken [8].

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