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ON THE NIELSEN NUMBER OF A FIBER MAP

DENNIS MCGAVRAN AND JINGYAL PAK

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Suppose $\mathcal{F} = \{E, \pi, B, F\}$ is a fiber space such that $0 \rightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{\pi_*} \pi_1(B) \rightarrow 0$ is exact. Suppose also that the above fundamental groups are abelian. If $f: E \rightarrow E$ is a fiber preserving map such that $f_*(\alpha) = \alpha$ if and only if $\alpha = 0$, then it is shown that $R(f) = R(f') \cdot R(f_b)$ where $R(h)$ is the Reidemeister number of the map h .

A product formula for the Nielsen number of a fiber map which holds under certain conditions was introduced by R. Brown. Let $\mathcal{S} = \{E, \pi, L, (p, q), s^1\}$ be a principal s^1 -bundle over the lens space $L(p, q)$, where \mathcal{S} is determined by $[f_j] \in [L(p, q), c\mathbb{P}^n] \simeq H^2(L(p, q), \mathbb{Z}) \simeq \mathbb{Z}_p$. Let $f: E \rightarrow E$ be a fiber preserving map such that $f_{i_*}(1) = c_2$, $f'_*(\bar{l}_p) = \bar{c}_1$, where 1 generates $\pi_1(s^1) \simeq \mathbb{Z}$ and \bar{l}_p generates $\pi_1(L(p, q)) \simeq \mathbb{Z}_p$. Then the Nielsen numbers of the maps involved satisfy

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s),$$

where $d = (j, p)$ and $s = j/p(c_1 - c_2)$.

I. Introduction. Let $\mathcal{F} = \{E, \pi, B, F\}$ be a fiber space. Any fiber preserving map $f: E \rightarrow E$ induces maps $f': B \rightarrow B$, and, for each $b \in B$, $f_b: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$, where $\pi^{-1}(b) \simeq F$. The map f will be called a fiber map (or bundle map if \mathcal{F} is a bundle).

Let $N(g)$ denote the Nielsen number of a map g . The Nielsen number, $N(g)$, serves as a lower bound on the number of fixed points of a map homotopic to g , and under certain hypotheses, there exists a map homotopic to g with exactly $N(g)$ fixed points. R. Brown and E. Fadell ([2] and [3]) proved the following:

THEOREM. *Let $\mathcal{F} = \{E, \pi, B, F\}$ be a locally trivial fiber space, where E, B , and F are connected finite polyhedra. Let $f: E \rightarrow E$ be a fiber map. If one of the following conditions holds:*

- (i) $\pi_1(B) = \pi_2(B) = 0$.
- (ii) $\pi_1(F) = 0$.
- (iii) \mathcal{F} is trivial and either $\pi_1(B) = 0$ or $f = f' \times f_b$ for all $b \in B$ then $N(f) = N(f') \cdot N(f_b)$ for all $b \in B$.

These strong restrictions on the spaces involved eliminate some interesting fiber spaces. For example, any circle bundle over B with $\pi_1(B) \neq 0$ is excluded. Furthermore, if $\pi_1(B) = \pi_2(B) = 0$, then the total space E is $B \times S^1$.

This paper has two objectives. The first is to try to generalize

the above result to the case of a bundle $\mathcal{F} = \{E, \pi, B, F\}$ where $\pi_1(B)$ is a nontrivial abelian group, and $\pi_2(B) = 0$. The second is to investigate the relationships between the Nielsen numbers of the maps f, f' , and f_b for particular circle bundles.

In this paper all spaces are path-connected.

II. Some general results. The reader may refer to [1] and [2] for definitions and details concerning the Nielsen number $N(f)$, Reidemeister number $R(f)$, and Jiang subgroup $T(f)$ of a map $f: X \rightarrow X$.

We will be particularly interested in the Reidemeister number. It serves as an upper bound on $N(f)$ and in many cases $R(f) = N(f)$. Let $h: G \rightarrow G$ be a homomorphism where G is an abelian group. It is shown in [1] that $R(h) = |\text{coker}(1 - h)|$ ($|\cdot|$ means the order of a group). The Reidemeister number of a map $f: X \rightarrow X$ is defined to be the Reidemeister number of the induced homomorphism $f_*: \pi_1(X) \rightarrow \pi_1(X)$. Now let \mathcal{F} be a fiber space. Let $F_b = \pi^{-1}(b)$. If $w: I \rightarrow B$ is such that $w(0) = b$ and $w(1) = b'$, we may translate $F_{b'}$ along the path w to F_b (see [6]). This gives a homeomorphism $\bar{w}: F_{b'} \rightarrow F_b$. Given a fiber map $f: E \rightarrow E$, we have the natural map $f'_b: F_b \rightarrow F_{f'(b)}$, the restriction of f to F_b . Then by definition $f_b = \bar{w} \circ f'_b$. For more details on $f_b: F_b \rightarrow F_b$ readers are referred to [2].

Suppose \mathcal{F} is a fiber space and w is a loop based at b . Then we have $\bar{w}: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$. The fiber space \mathcal{F} is said to be orientable if the induced homomorphism $\bar{w}_*: H_*(\pi^{-1}(b), z) \rightarrow H_*(\pi^{-1}(b), z)$ is the identity homomorphism for every loop w based at b . It is shown in [2] that if \mathcal{F} is orientable and if the Jiang subgroup $T(p^{-1}(b), e_0) = \pi_1(p^{-1}(b), e_0)$ for a fixed $b \in B$ then the Nielsen number of f_b is independent of the choice of path from $f'(b)$ to b . Furthermore, the Nielsen number $N(f_b)$ is independent of the choice of $b \in B$.

LEMMA 1. *Let \mathcal{F} be a fiber space with $\pi_1(F)$, $\pi_1(E)$, and $\pi_1(B)$ abelian. Suppose $f: E \rightarrow E$ is a fiber map. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(F_b) & \xrightarrow{i_{\#}} & \pi_1(E) \\ \downarrow 1-f_{b\#} & & \downarrow 1-f_{\#} \\ \pi_1(F_b) & \xrightarrow{i_{\#}} & \pi_1(E) . \end{array}$$

Proof. First, by [6], the map \bar{w} is homotopic in E to the identity map on $F_{f'(b)}$. Hence we have

$$\begin{aligned} i_{\#} \circ (1 - f_{b\#})(\alpha) &= i_{\#}[\alpha - (\bar{w} \circ f'_b)_{\#}(\alpha)] \\ &= i_{\#}(\alpha) - i_{\#}(\bar{w} \circ f'_b)_{\#}(\alpha) = i_{\#}(\alpha) - (i_{\#} \circ f'_{b\#})(\alpha) \\ &= i_{\#}(\alpha) - (f_{\#} \circ i_{\#})(\alpha) = (1 - f_{\#}) \circ i_{\#}(\alpha) . \end{aligned}$$

LEMMA 2 [4]. *Suppose we have the following commutative diagram of modules, where the rows are exact:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' \longrightarrow 0 . \end{array}$$

Then there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \alpha & \xrightarrow{\mu_*} & \ker \beta & \xrightarrow{\varepsilon_*} & \ker \gamma \\ & & \xrightarrow{\omega} & \operatorname{coker} \alpha & \xrightarrow{\mu'_*} & \operatorname{coker} \beta & \xrightarrow{\varepsilon'_*} & \operatorname{coker} \gamma \longrightarrow 0 . \end{array}$$

The homomorphisms μ_* and ε_* are restrictions of μ and ε , and μ'_* and ε'_* are induced by μ' and ε' on quotients. The connecting homomorphism $\omega: \ker \gamma \rightarrow \operatorname{coker} \alpha$ is defined as follows. Let $c \in \ker \gamma$, choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon' \beta b = \gamma \varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $\omega(c) = [a']$, the coset of a' in $\operatorname{coker} \alpha$. Then ω is a well-defined homomorphism. See [4, p. 99] for the proof of the lemma.

THEOREM 3. *Suppose $\mathcal{F} = \{E, \pi, B, F\}$ is a fiber space such that*

$$0 \longrightarrow \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{\pi_\#} \pi_1(B) \longrightarrow 0$$

is an exact sequence of abelian groups. Suppose $f: E \rightarrow E$ is a fiber map and $w: I \rightarrow B$ is a path from b to $f'(b)$. Then we have the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(1 - f_{b\#}) & \longrightarrow & \ker(1 - f_\#) & \longrightarrow & \ker(1 - f'_\#) \\ & & \longrightarrow & \operatorname{coker}(1 - f_{b\#}) & \longrightarrow & \operatorname{coker}(1 - f_\#) & \longrightarrow & \operatorname{coker}(1 - f'_\#) \longrightarrow 0 . \end{array}$$

Proof. The fiber map induces the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(F) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{\pi_\#} & \pi_1(B) \longrightarrow 0 \\ & & (1 - f_{b\#}) \downarrow & & (1 - f_\#) \downarrow & & (1 - f'_\#) \downarrow \\ 0 & \longrightarrow & \pi_1(F) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{\pi_\#} & \pi_1(B) \longrightarrow 0 . \end{array}$$

Now the result becomes a simple application of Lemmas 1 and 2.

COROLLARY 4. $\ker(1 - f_{b\#})$ is independent of w and b .

Proof. $\ker(1 - f_{b\#})$ is isomorphic to the kernel of the map $\ker(1 - f_\#) \xrightarrow{\pi_{\#\#}} \ker(1 - f'_\#)$. But this map is the restriction of $\pi_\#: \pi_1(E) \rightarrow \pi_1(B)$, which is independent of w and b .

Suppose $h: G \rightarrow G$ is a homomorphism of abelian groups. We will say that h satisfies Condition A if $h(\alpha) = \alpha$ if and only if $\alpha = 0$.

THEOREM 5. *Suppose \mathcal{F} is a fiber space satisfying the hypotheses of Theorem 3. Suppose $f: E \rightarrow E$ is a fiber map such that $f'_\#$ satisfies Condition A. Then $R(f) = R(f') \cdot R(f_b)$ for all $b \in B$.*

Proof. We have $(1 - f'_\#)(\alpha) = 0$ if and only if $f'_\#(\alpha) = \alpha$ if and only if $\alpha = 0$. Therefore, $1 - f'_\#$ is injective and we have the following exact sequence:

$$0 \rightarrow \text{coker}(1 - f_{b\#}) \rightarrow \text{coker}(1 - f_\#) \rightarrow \text{coker}(1 - f'_\#) \rightarrow 0.$$

The theorem follows from the properties of $R(f)$.

COROLLARY 6. *Under the hypotheses of Theorem 5 $R(f_b)$ is independent of w and b .*

Proof. This follows since both $R(f)$ and $R(f')$ are independent of w and b .

EXAMPLE 1. Let \mathcal{F} be a principal T^k -bundle over a $(2n + 1)$ -dimensional lens space $L(p)$, $p \geq 1$. We know from [5] that $L = L(d) \times T^k$ where d divides p . Let $f: E \rightarrow E$ be a bundle map. It follows easily from results in [1] that $N(f_b) = R(f_b)$. It is also shown in [1] that $N(f') = R(f')$ for $n = 1$, and the proof can be easily generalized to higher dimensions. Furthermore, by showing that $T(f) = \pi_1(L(d) \times T^k)$, where $T(f)$ is the Jiang subgroup of f , one can show that $N(f) = R(f)$. Now such a bundle satisfies the hypothesis of Theorem 3. Hence, if $f'_\#: \pi_1(L(p)) \rightarrow \pi_1(L(p))$ satisfies the hypothesis of Theorem 5, we have $N(f) = N(f') \cdot N(f_b)$ for all $b \in B$.

EXAMPLE 2. If G is a compact connected semi-simple Lie group, then $\mathcal{F} = \{E, \pi, G, S^1\}$ satisfies the hypothesis of Theorem 3. If $f: E \rightarrow E$ is a fiber map then $N(f) = N(f') \cdot N(f_b)$ follows from [3] since the second integral cohomology group of G vanishes. Assume $N(f') \neq 0 \neq N(f_b)$. Then since G and S^1 are H -spaces $T(f') = \pi_1(G)$ and $T(f_b) = \pi_1(S^1)$; and we have $N(f') = R(f')$ and $N(f_b) = R(f_b)$. It follows that $R(f) = R(f') \cdot R(f_b)$ independent of Condition A.

LEMMA 7. *Suppose $h: Z_p \rightarrow Z_p$ is such that $h(\bar{l}) = \bar{m}$. Then Condition A holds iff $(1 - m, p) = 1$.*

Proof. Suppose $(1 - m, p) = 1$. If $h(\bar{n}) = \bar{m}\bar{n} = \bar{n}$, $1 \leq n < p$, then $m\bar{n} \equiv n \pmod{p}$. Hence p divides $(1 - m)n$, which is impossible if $(1 - m, p) = 1$.

Now suppose $h(\alpha) = \alpha$ iff $\alpha = 0$. Suppose $(1 - m, p) = d$. Let $1 - m = c_1 d$, $p = c_2 d$. Then $h(\bar{c}_2) = \bar{m} \bar{c}_2$. Now

$$m c_2 - c_2 = c_2(m - 1) = -c_2 c_1 d = -c_1 p.$$

Thus $h(\bar{c}_2) = \bar{c}_2$ and $d = 1$.

EXAMPLE 1 (con't). We have $\pi_1(L(p)) \simeq Z_p$. Suppose $f'_\#(\bar{l}) = \bar{m}$. Then $N(f') = (1 - m, p)$. Hence Theorem 5 is applicable if and only if $N(f') = 1$.

III. A general solution to Example 1. Let $\mathcal{S} = \{E, \pi, L(p, q), s^1\}$ be a principal s^1 -bundle over a 3-dimensional lens space $L(p, q)$. If \mathcal{S} is induced by $[f_j] \in [L(p, q), CP^\infty] \simeq H^2(L(p, q), Z) \simeq Z_p$, then $E \simeq L(d, q) \times s^1$, where $d = (j, p)$ (see [7]). Let $j = j'd$, $p = p'd$.

THEOREM 8. Let \mathcal{S} be as above and $f: E \rightarrow E$ a fiber map such that, for a particular choice of $b \in B$ and w , $f_{b\#}(1) = c_2$ and $f'_\#(\bar{l}_p) = \bar{c}_1$, where 1 generates $\pi_1(s^1) \simeq Z$ and \bar{l}_p generates $\pi_1(L(p, q)) \simeq Z_p$. Let $s = j/p(c_1 - c_2)$. Then

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s).$$

Proof. We first examine the structure of $L(d, q) \times s^1$ as an s^1 -bundle over $L(p, q)$ (see [7]). $L(p, q)$ and $L(d, q)$ are obtained from s^3 as the orbit space of a free Z_p -action and Z_d -action, respectively. Given $((r_1, \theta_1), (r_2, \theta_2)) \in s^3$, let $\langle (r_1, \theta_1), (r_2, \theta_2) \rangle$ represent its equivalence class as an element in $L(p, q)$. In $L(d, q) \times I$, $I = [0, 2\pi]$, identify $\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, 2\pi\}$ with $\{\langle (r_1, \theta_1 + j'v), (r_2, \theta_2 + j'qv) \rangle, 0\}$ to obtain E , where $v = 2\pi/p$. Define $h: E \rightarrow L(d, q) \times S^1$ by

$$h\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t\} = \left\{ \left\langle \left(r_1, \theta_1 + \frac{t}{2\pi} j'v \right), \left(r_2, \theta_2 + \frac{t}{2\pi} j'qv \right) \right\rangle, t \right\}.$$

Then h is a homeomorphism. Let $\pi_1(L(d, q) \times S^1)$ be generated by $(\bar{l}_d, 0)$ and $(0, 1)$. Then $(\bar{l}_d, 0)$ is represented by the loop $\bar{\sigma}_1 = \{\langle (1, t(2\pi/d)), (0, 0) \rangle, 0\}$, $0 \leq t \leq 1$, and $(0, 1)$ is represented by $\bar{\sigma}_2 = \{\langle (1, 0), (0, 0) \rangle, t\}$, $0 \leq t \leq 2\pi$. Then in E , $\sigma_1 = \{\langle (1, t(2\pi/d)), (0, 0) \rangle, 0\}$ and $\sigma_2 = \{\langle (1, -t/(2\pi)j'v), (0, 0) \rangle, t\}$ represent $(\bar{l}_d, 0)$ and $(0, 1)$ respectively. \bar{l}_p is represented by the loop $\gamma = \{\langle (1, tv), (0, 0) \rangle, 0 \leq t \leq 1\}$. Now the projection map $\pi: E \rightarrow L(p, q)$ is given by

$$\pi\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t\} = \langle (r_1, \theta_1), (r_2, \theta_2) \rangle.$$

We have

$$\pi \circ \sigma_1 = \left\langle \left\langle \left(1, t \frac{2\pi}{d} \right), (0, 0) \right\rangle, 0 \leq t \leq 1 \right\rangle = \langle (1, tp'v), (0, 0) \rangle.$$

Hence

$$\pi_{\#}(\bar{l}_d, 0) = \bar{p}' .$$

Also

$$\pi \circ \sigma_2 = \left\langle \left\langle \left(1, -\frac{t}{2\pi} j'v \right), (0, 0) \right\rangle \right\rangle \quad 0 \leq t \leq 2\pi$$

so

$$\pi_{\#}(0, 1) = -\bar{j}' .$$

One fiber in E consists of

$$\bigcup_{\substack{n=0 \\ 0 \leq t \leq 2\pi}}^{p'} \{ \langle (1, nj'v), (0, 0) \rangle, t \} .$$

Hence, in $L(d, q) \times S^1$, this fiber is

$$\bigcup_{\substack{n=0 \\ 0 \leq t \leq 2\pi}}^{p'} \left\{ \left\langle \left(1, \left(n + \frac{t}{2\pi} \right) j'v \right), (0, 0) \right\rangle, t \right\} = \{ \langle (1, \tau j'v), (0, 0) \rangle, \bar{2}\bar{\tau}\bar{\pi} \}$$

where $0 \leq \tau \leq p'$ and $\bar{2}\bar{\tau}\bar{\pi}$ represents the equivalence class of $2\pi\tau \pmod{2\pi}$. Hence $i_{\#}(1) = (\bar{j}', p')$.

We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_{\#}} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_{\#}} & \pi_1(L(p, q)) \longrightarrow 0 \\ & & (1-f_{\#}) \downarrow & & \downarrow (1-f_{\#}) & & \downarrow (1-f'_{\#}) \\ 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_{\#}} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_{\#}} & \pi_1(L(p, q)) \longrightarrow 0 . \end{array}$$

We must compute the cokernel of $(1 - f_{\#})$ since $N(f) = |\text{coker}(1 - f_{\#})|$. Let

$$(1 - f_{\#})(\bar{l}_d, 0) = (\bar{a}, 0) \quad (1 - f_{\#})(0, 1) = (\bar{s}, u) .$$

Commutativity of the right hand square implies that $a = 1 - c_1$, while commutativity of the left hand square implies $u = 1 - c_2$. Now

$$\begin{aligned} (1 - f'_{\#}) \circ \pi_{\#}(0, 1) &= \overline{-(1 - c_1)j'} \\ \pi_{\#} \circ (1 - f_{\#})(0, 1) &= \overline{p's - j'u} = \overline{p's - j'(1 - c_2)} . \end{aligned}$$

Hence

$$p's - j'(1 - c_2) \equiv -(1 - c_1)j' \pmod{p} .$$

Therefore,

$$j'(c_2 - c_1) + p's = kp .$$

We must have $p' \mid j'(c_2 - c_1)$ so

$$s = kd + \frac{j'}{p'}(c_1 - c_2) .$$

Hence we may assume

$$s = \frac{j'}{p'}(c_1 - c_2) = \frac{j}{p}(c_1 - c_2) .$$

Therefore, $\text{Im}(1 - f_{\#})$ is generated by $(\overline{1 - c_1}, 0)$, $(\overline{s}, 0)$, and $(0, 1 - c_2)$. Now the group $\pi_1(L(d, q) \times S^1) \simeq z_d \oplus z$, and the subgroup generated by $(\overline{1 - c_1}, 0)$ and $(\overline{s}, 0)$ is the subgroup generated by $(\overline{(1 - c_1, s)}, 0)$. Consequently, the cokernel of $(1 - f_{\#})$ is isomorphic to $z_d/(1 - c_1, s)z_d \oplus z/(1 - c_2)z$. Which, in turn, is isomorphic to $z_{(d, 1 - c_1, s)} \oplus z_{(1 - c_2)}$. Therefore,

$$|\text{coker}(1 - f_{\#})| = N(f) = (d, 1 - c_1, s) \cdot |1 - c_2| = (d, 1 - c_1, s) \cdot N(f_b) .$$

Note. (1) Since \mathcal{S} is orientable and $T(\pi^{-1}(b), e_0) = \pi_1(\pi^{-1}(b), e_0)$, the above formula is independent of w and b .

(2) In the above argument we could replace $L(p, q)$ with the generalized lens space as in [5].

(3) If p is a prime the product formula follows from results of R. Brown and E. Fadell [3].

(4) Theorem 8 also indicates that a product theorem of the type obtained by R. Brown and E. Fadell is hard to expect in general.

COROLLARY 9. *Let \mathcal{S} be as in Theorem 8. Suppose $f: E \rightarrow E$ is a bundle map such that for some $b \in L(p, q)$ $f_b: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$ is homotopic to a fixed-point free map. Then there exists a map $g: E \rightarrow E$, homotopic to f , which is fixed-point free.*

Proof. Let \tilde{f}_b be the fixed-point free map on $\pi^{-1}(b)$ which is homotopic to f_b . Clearly $N(\tilde{f}_b) = 0$ and since the Nielsen number is a homotopy invariant, $N(f_b) = 0$. Thus from Theorem 8, $N(f) = 0$, and the corollary follows from the converse of the Lefschetz fixed-point theorem of F. Wecken [8].

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