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**ON THE NIELSEN NUMBER OF A FIBER MAP**

DENNIS MCGAVRAN AND JINGYAL PAK

## ON THE NIELSEN NUMBER OF A FIBER MAP

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Suppose  $\mathcal{S} = \{E, \pi, B, F\}$  is a fiber space such that  $0 \rightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{\pi_*} \pi_1(B) \rightarrow 0$  is exact. Suppose also that the above fundamental groups are abelian. If  $f: E \rightarrow E$  is a fiber preserving map such that  $f_*(\alpha) = \alpha$  if and only if  $\alpha = 0$ , then it is shown that  $R(f) = R(f') \cdot R(f_b)$  where  $R(h)$  is the Reidemeister number of the map  $h$ .

A product formula for the Nielsen number of a fiber map which holds under certain conditions was introduced by R. Brown. Let  $\mathcal{S} = \{E, \pi, L, (p, q), s^1\}$  be a principal  $s^1$ -bundle over the lens space  $L(p, q)$ , where  $\mathcal{S}$  is determined by  $[f_j] \in [L(p, q), cp^\infty] \simeq H^2(L(p, q), z) \simeq z_p$ . Let  $f: E \rightarrow E$  be a fiber preserving map such that  $f_{b*}(1) = c_2$ ,  $f'_*(\bar{l}_p) = \bar{c}_1$ , where  $1$  generates  $\pi_1(s^1) \simeq z$  and  $\bar{l}_p$  generates  $\pi_1(L(p, q)) \simeq z_p$ . Then the Nielsen numbers of the maps involved satisfy

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s),$$

where  $d = (j, p)$  and  $s = j/p(c_1 - c_2)$ .

**I. Introduction.** Let  $\mathcal{S} = \{E, \pi, B, F\}$  be a fiber space. Any fiber preserving map  $f: E \rightarrow E$  induces maps  $f': B \rightarrow B$ , and, for each  $b \in B$ ,  $f_b: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$ , where  $\pi^{-1}(b) \simeq F$ . The map  $f$  will be called a fiber map (or bundle map if  $\mathcal{S}$  is a bundle).

Let  $N(g)$  denote the Nielsen number of a map  $g$ . The Nielsen number,  $N(g)$ , serves as a lower bound on the number of fixed points of a map homotopic to  $g$ , and under certain hypotheses, there exists a map homotopic to  $g$  with exactly  $N(g)$  fixed points. R. Brown and E. Fadell ([2] and [3]) proved the following:

**THEOREM.** *Let  $\mathcal{S} = \{E, \pi, B, F\}$  be a locally trivial fiber space, where  $E, B$ , and  $F$  are connected finite polyhedra. Let  $f: E \rightarrow E$  be a fiber map. If one of the following conditions holds:*

(i)  $\pi_1(B) = \pi_2(B) = 0$ .

(ii)  $\pi_1(F) = 0$ .

(iii)  $\mathcal{S}$  is trivial and either  $\pi_1(B) = 0$  or  $f = f' \times f_b$  for all  $b \in B$  then  $N(f) = N(f') \cdot N(f_b)$  for all  $b \in B$ .

These strong restrictions on the spaces involved eliminate some interesting fiber spaces. For example, any circle bundle over  $B$  with  $\pi_1(B) \neq 0$  is excluded. Furthermore, if  $\pi_1(B) = \pi_2(B) = 0$ , then the total space  $E$  is  $B \times S^1$ .

This paper has two objectives. The first is to try to generalize

the above result to the case of a bundle  $\mathcal{S} = \{E, \pi, B, F\}$  where  $\pi_1(B)$  is a nontrivial abelian group, and  $\pi_2(B) = 0$ . The second is to investigate the relationships between the Nielsen numbers of the maps  $f, f'$ , and  $f_b$  for particular circle bundles.

In this paper all spaces are path-connected.

II. Some general results. The reader may refer to [1] and [2] for definitions and details concerning the Nielsen number  $N(f)$ , Reidemeister number  $R(f)$ , and Jiang subgroup  $T(f)$  of a map  $f: X \rightarrow X$ .

We will be particularly interested in the Reidemeister number. It serves as an upper bound on  $N(f)$  and in many cases  $R(f) = N(f)$ . Let  $h: G \rightarrow G$  be a homomorphism where  $G$  is an abelian group. It is shown in [1] that  $R(h) = |\text{coker}(1 - h)|$  ( $|\cdot|$  means the order of a group). The Reidemeister number of a map  $f: X \rightarrow X$  is defined to be the Reidemeister number of the induced homomorphism  $f_*: \pi_1(X) \rightarrow \pi_1(X)$ . Now let  $\mathcal{S}$  be a fiber space. Let  $F_b = \pi^{-1}(b)$ . If  $w: I \rightarrow B$  is such that  $w(0) = b$  and  $w(1) = b'$ , we may translate  $F_{b'}$  along the path  $w$  to  $F_b$  (see [6]). This gives a homeomorphism  $\bar{w}: F_{b'} \rightarrow F_b$ . Given a fiber map  $f: E \rightarrow E$ , we have the natural map  $f'_b: F_b \rightarrow F_{f'(b)}$ , the restriction of  $f$  to  $F_b$ . Then by definition  $f_b = \bar{w} \circ f'_b$ . For more details on  $f_b: F_b \rightarrow F_b$  readers are referred to [2].

Suppose  $\mathcal{S}$  is a fiber space and  $w$  is a loop based at  $b$ . Then we have  $\bar{w}: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$ . The fiber space  $\mathcal{S}$  is said to be orientable if the induced homomorphism  $\bar{w}_*: H_*(\pi^{-1}(b), \mathbb{Z}) \rightarrow H_*(\pi^{-1}(b), \mathbb{Z})$  is the identity homomorphism for every loop  $w$  based at  $b$ . It is shown in [2] that if  $\mathcal{S}$  is orientable and if the Jiang subgroup  $T(p^{-1}(b), e_0) = \pi_1(p^{-1}(b), e_0)$  for a fixed  $b \in B$  then the Nielsen number of  $f_b$  is independent of the choice of path from  $f'(b)$  to  $b$ . Furthermore, the Nielsen number  $N(f_b)$  is independent of the choice of  $b \in B$ .

LEMMA 1. *Let  $\mathcal{S}$  be a fiber space with  $\pi_1(F)$ ,  $\pi_1(E)$ , and  $\pi_1(B)$  abelian. Suppose  $f: E \rightarrow E$  is a fiber map. Then the following diagram commutes:*

$$\begin{CD} \pi_1(F_b) @>i_*>> \pi_1(E) \\ @V1-f_{b*}VV @VV1-f_*V \\ \pi_1(F_b) @>i_*>> \pi_1(E) . \end{CD}$$

*Proof.* First, by [6], the map  $\bar{w}$  is homotopic in  $E$  to the identity map on  $F_{f'(b)}$ . Hence we have

$$\begin{aligned} i_* \circ (1 - f_{b*})(\alpha) &= i_*[\alpha - (\bar{w} \circ f'_b)_*(\alpha)] \\ &= i_*(\alpha) - i_*(\bar{w} \circ f'_b)_*(\alpha) = i_*(\alpha) - (i_* \circ f'_{b*})(\alpha) \\ &= i_*(\alpha) - (f_* \circ i_*)(\alpha) = (1 - f_*) \circ i_*(\alpha) . \end{aligned}$$

LEMMA 2 [4]. Suppose we have the following commutative diagram of modules, where the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & \longrightarrow & 0. \end{array}$$

Then there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \alpha & \xrightarrow{\mu_*} & \ker \beta & \xrightarrow{\varepsilon_*} & \ker \gamma \\ & & \xrightarrow{\omega} & \text{coker } \alpha & \xrightarrow{\mu'_*} & \text{coker } \beta & \xrightarrow{\varepsilon'_*} & \text{coker } \gamma & \longrightarrow & 0. \end{array}$$

The homomorphisms  $\mu_*$  and  $\varepsilon_*$  are restrictions of  $\mu$  and  $\varepsilon$ , and  $\mu'_*$  and  $\varepsilon'_*$  are induced by  $\mu'$  and  $\varepsilon'$  on quotients. The connecting homomorphism  $\omega: \ker \gamma \rightarrow \text{coker } \alpha$  is defined as follows. Let  $c \in \ker \gamma$ , choose  $b \in B$  with  $\varepsilon b = c$ . Since  $\varepsilon' \beta b = \gamma \varepsilon b = \gamma c = 0$  there exists  $a' \in A'$  with  $\beta b = \mu' a'$ . Define  $\omega(c) = [a']$ , the coset of  $a'$  in  $\text{coker } \alpha$ . Then  $\omega$  is a well-defined homomorphism. See [4, p. 99] for the proof of the lemma.

THEOREM 3. Suppose  $\mathcal{F} = \{E, \pi, B, F\}$  is a fiber space such that

$$0 \longrightarrow \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{\pi_\#} \pi_1(B) \longrightarrow 0$$

is an exact sequence of abelian groups. Suppose  $f: E \rightarrow E$  is a fiber map and  $w: I \rightarrow B$  is a path from  $b$  to  $f'(b)$ . Then we have the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(1 - f_{b\#}) & \longrightarrow & \ker(1 - f_\#) & \longrightarrow & \ker(1 - f'_\#) \\ & & \longrightarrow & \text{coker}(1 - f_{b\#}) & \longrightarrow & \text{coker}(1 - f_\#) & \longrightarrow & \text{coker}(1 - f'_\#) & \longrightarrow & 0. \end{array}$$

*Proof.* The fiber map induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(F) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{\pi_\#} & \pi_1(B) & \longrightarrow & 0 \\ & & (1 - f_{b\#}) \downarrow & & (1 - f_\#) \downarrow & & (1 - f'_\#) \downarrow & & \\ 0 & \longrightarrow & \pi_1(F) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{\pi_\#} & \pi_1(B) & \longrightarrow & 0. \end{array}$$

Now the result becomes a simple application of Lemmas 1 and 2.

COROLLARY 4.  $\ker(1 - f_{b\#})$  is independent of  $w$  and  $b$ .

*Proof.*  $\ker(1 - f_{b\#})$  is isomorphic to the kernel of the map  $\ker(1 - f_\#) \xrightarrow{\pi_\#} \ker(1 - f'_\#)$ . But this map is the restriction of  $\pi_\#: \pi_1(E) \rightarrow \pi_1(B)$ , which is independent of  $w$  and  $b$ .

Suppose  $h: G \rightarrow G$  is a homomorphism of abelian groups. We will say that  $h$  satisfies Condition A if  $h(\alpha) = \alpha$  if and only if  $\alpha = 0$ .

**THEOREM 5.** *Suppose  $\mathcal{S}$  is a fiber space satisfying the hypotheses of Theorem 3. Suppose  $f: E \rightarrow E$  is a fiber map such that  $f'_\#$  satisfies Condition A. Then  $R(f) = R(f') \cdot R(f_b)$  for all  $b \in B$ .*

*Proof.* We have  $(1 - f'_\#)(\alpha) = 0$  if and only if  $f'_\#(\alpha) = \alpha$  if and only if  $\alpha = 0$ . Therefore,  $1 - f'_\#$  is injective and we have the following exact sequence:

$$0 \rightarrow \text{coker}(1 - f_{b\#}) \rightarrow \text{coker}(1 - f_\#) \rightarrow \text{coker}(1 - f'_\#) \rightarrow 0.$$

The theorem follows from the properties of  $R(f)$ .

**COROLLARY 6.** *Under the hypotheses of Theorem 5  $R(f_b)$  is independent of  $w$  and  $b$ .*

*Proof.* This follows since both  $R(f)$  and  $R(f')$  are independent of  $w$  and  $b$ .

**EXAMPLE 1.** Let  $\mathcal{S}$  be a principal  $T^k$ -bundle over a  $(2n + 1)$ -dimensional lens space  $L(p)$ ,  $p \geq 1$ . We know from [5] that  $L = L(d) \times T^k$  where  $d$  divides  $p$ . Let  $f: E \rightarrow E$  be a bundle map. It follows easily from results in [1] that  $N(f_b) = R(f_b)$ . It is also shown in [1] that  $N(f') = R(f')$  for  $n = 1$ , and the proof can be easily generalized to higher dimensions. Furthermore, by showing that  $T(f) = \pi_1(L(d) \times T^k)$ , where  $T(f)$  is the Jiang subgroup of  $f$ , one can show that  $N(f) = R(f)$ . Now such a bundle satisfies the hypothesis of Theorem 3. Hence, if  $f'_\#: \pi_1(L(p)) \rightarrow \pi_1(L(p))$  satisfies the hypothesis of Theorem 5, we have  $N(f) = N(f') \cdot N(f_b)$  for all  $b \in B$ .

**EXAMPLE 2.** If  $G$  is a compact connected semi-simple Lie group, then  $\mathcal{S} = \{E, \pi, G, S^1\}$  satisfies the hypothesis of Theorem 3. If  $f: E \rightarrow E$  is a fiber map then  $N(f) = N(f') \cdot N(f_b)$  follows from [3] since the second integral cohomology group of  $G$  vanishes. Assume  $N(f') \neq 0 \neq N(f_b)$ . Then since  $G$  and  $S^1$  are  $H$ -spaces  $T(f') = \pi_1(G)$  and  $T(f_b) = \pi_1(S^1)$ ; and we have  $N(f') = R(f')$  and  $N(f_b) = R(f_b)$ . It follows that  $R(f) = R(f') \cdot R(f_b)$  independent of Condition A.

**LEMMA 7.** *Suppose  $h: Z_p \rightarrow Z_p$  is such that  $h(\bar{l}) = \bar{m}$ . Then Condition A holds iff  $(1 - m, p) = 1$ .*

*Proof.* Suppose  $(1 - m, p) = 1$ . If  $h(\bar{n}) = \bar{m}\bar{n} = \bar{n}$ ,  $1 \leq n < p$ , then  $mn \equiv n \pmod{p}$ . Hence  $p$  divides  $(1 - m)n$ , which is impossible if  $(1 - m, p) = 1$ .

Now suppose  $h(\alpha) = \alpha$  iff  $\alpha = 0$ . Suppose  $(1 - m, p) = d$ . Let  $1 - m = c_1 d$ ,  $p = c_2 d$ . Then  $h(\bar{c}_2) = \bar{m}\bar{c}_2$ . Now

$$m c_2 - c_2 = c_2(m - 1) = -c_2 c_1 d = -c_1 p.$$

Thus  $h(\bar{c}_2) = \bar{c}_2$  and  $d = 1$ .

EXAMPLE 1 (con't). We have  $\pi_1(L(p)) \simeq Z_p$ . Suppose  $f'_*(\bar{l}) = \bar{m}$ . Then  $N(f') = (1 - m, p)$ . Hence Theorem 5 is applicable if and only if  $N(f') = 1$ .

III. A general solution to Example 1. Let  $\mathcal{S} = \{E, \pi, L(p, q), s^1\}$  be a principal  $s^1$ -bundle over a 3-dimensional lens space  $L(p, q)$ . If  $\mathcal{S}$  is induced by  $[f_j] \in [L(p, q), CP^\infty] \simeq H^2(L(p, q), Z) \simeq Z_p$ , then  $E \simeq L(d, q) \times s^1$ , where  $d = (j, p)$  (see [7]). Let  $j = j'd$ ,  $p = p'd$ .

THEOREM 8. Let  $\mathcal{S}$  be as above and  $f: E \rightarrow E$  a fiber map such that, for a particular choice of  $b \in B$  and  $w$ ,  $f_{i*}(1) = c_2$  and  $f'_*(\bar{l}_p) = \bar{c}_1$ , where 1 generates  $\pi_1(s^1) \simeq Z$  and  $\bar{l}_p$  generates  $\pi_1(L(p, q)) \simeq Z_p$ . Let  $s = j/p(c_1 - c_2)$ . Then

$$N(f) = N(f_b) \cdot (d, 1 - c_1, s).$$

*Proof.* We first examine the structure of  $L(d, q) \times s^1$  as an  $s^1$ -bundle over  $L(p, q)$  (see [7]).  $L(p, q)$  and  $L(d, q)$  are obtained from  $s^3$  as the orbit space of a free  $Z_p$ -action and  $Z_d$ -action, respectively. Given  $((r_1, \theta_1), (r_2, \theta_2)) \in s^3$ , let  $\langle (r_1, \theta_1), (r_2, \theta_2) \rangle$  represent its equivalence class as an element in  $L(p, q)$ . In  $L(d, q) \times I$ ,  $I = [0, 2\pi]$ , identify  $\{\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, 2\pi\}$  with  $\{\langle (r_1, \theta_1 + j'v), (r_2, \theta_2 + j'qv) \rangle, 0\}$  to obtain  $E$ , where  $v = 2\pi/p$ . Define  $h: E \rightarrow L(d, q) \times S^1$  by

$$h\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t = \left\langle \left\langle \left( r_1, \theta_1 + \frac{t}{2\pi} j'v \right), \left( r_2, \theta_2 + \frac{t}{2\pi} j'qv \right) \right\rangle, t \right\rangle.$$

Then  $h$  is a homeomorphism. Let  $\pi_1(L(d, q) \times S^1)$  be generated by  $(\bar{l}_d, 0)$  and  $(0, 1)$ . Then  $(\bar{l}_d, 0)$  is represented by the loop  $\bar{\sigma}_1 = \langle \langle (1, t(2\pi/d)), (0, 0) \rangle, 0 \rangle$ ,  $0 \leq t \leq 1$ , and  $(0, 1)$  is represented by  $\bar{\sigma}_2 = \langle \langle (1, 0), (0, 0) \rangle, t \rangle$ ,  $0 \leq t \leq 2\pi$ . Then in  $E$ ,  $\sigma_1 = \langle \langle (1, t(2\pi/d)), (0, 0) \rangle, 0 \rangle$  and  $\sigma_2 = \langle \langle (1, -t/(2\pi)j'v), (0, 0) \rangle, t \rangle$  represent  $(\bar{l}_d, 0)$  and  $(0, 1)$  respectively.  $\bar{l}_p$  is represented by the loop  $\gamma = \langle \langle (1, tv), (0, 0) \rangle, 0 \leq t \leq 1$ . Now the projection map  $\pi: E \rightarrow L(p, q)$  is given by

$$\pi\langle (r_1, \theta_1), (r_2, \theta_2) \rangle, t = \langle \langle (r_1, \theta_1), (r_2, \theta_2) \rangle \rangle.$$

We have

$$\pi \circ \sigma_1 = \left\langle \left\langle \left( 1, t \frac{2\pi}{d} \right), (0, 0) \right\rangle \right\rangle \quad 0 \leq t \leq 1 = \langle \langle (1, tp'v), (0, 0) \rangle \rangle.$$

Hence

$$\pi_{\#}(\bar{l}_d, 0) = \bar{p}' .$$

Also

$$\pi \circ \sigma_2 = \left\langle \left\langle \left( 1, -\frac{t}{2\pi} j'v \right), (0, 0) \right\rangle \right\rangle \quad 0 \leq t \leq 2\pi$$

so

$$\pi_{\#}(0, 1) = -\bar{j}' .$$

One fiber in  $E$  consists of

$$\bigcup_{\substack{n=0 \\ 0 \leq t \leq 2\pi}}^{p'} \{ \langle (1, nj'v), (0, 0) \rangle, t \} .$$

Hence, in  $L(d, q) \times S^1$ , this fiber is

$$\bigcup_{\substack{n=0 \\ 0 \leq t \leq 2\pi}}^{p'} \left\{ \left\langle \left( 1, \left( n + \frac{t}{2\pi} \right) j'v \right), (0, 0) \right\rangle, t \right\} = \{ \langle (1, \tau j'v), (0, 0) \rangle, \bar{2\tau\pi} \}$$

where  $0 \leq \tau \leq p'$  and  $\bar{2\tau\pi}$  represents the equivalence class of  $2\pi\tau \pmod{2\pi}$ . Hence  $i_{\#}(1) = (\bar{j}', p')$ .

We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_{\#}} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_{\#}} & \pi_1(L(p, q)) \longrightarrow 0 \\ & & (1-f_{\#}) \downarrow & & \downarrow (1-f_{\#}) & & \downarrow (1-f'_{\#}) \\ 0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{i_{\#}} & \pi_1(L(d, q) \times S^1) & \xrightarrow{\pi_{\#}} & \pi_1(L(p, q)) \longrightarrow 0 . \end{array}$$

We must compute the cokernel of  $(1 - f_{\#})$  since  $N(f) = | \text{coker } (1 - f_{\#}) |$ .  
Let

$$(1 - f_{\#})(\bar{l}_d, 0) = (\bar{a}, 0) \quad (1 - f_{\#})(0, 1) = (\bar{s}, u) .$$

Commutativity of the right hand square implies that  $a = 1 - c_1$ , while commutativity of the left hand square implies  $u = 1 - c_2$ . Now

$$\begin{aligned} (1 - f'_{\#}) \circ \pi_{\#}(0, 1) &= \overline{-(1 - c_1)j'} \\ \pi_{\#} \circ (1 - f_{\#})(0, 1) &= \overline{p's - j'u} = \overline{p's - j'(1 - c_2)} . \end{aligned}$$

Hence

$$p's - j'(1 - c_2) \equiv -(1 - c_1)j' \pmod{p} .$$

Therefore,

$$j'(c_2 - c_1) + p's = kp .$$

We must have  $p' \mid j'(c_2 - c_1)$  so

$$s = kd + \frac{j'}{p'}(c_1 - c_2).$$

Hence we may assume

$$s = \frac{j'}{p'}(c_1 - c_2) = \frac{j}{p}(c_1 - c_2).$$

Therefore,  $\text{Im}(1 - f_{\#})$  is generated by  $(\overline{1 - c_1}, 0)$ ,  $(\bar{s}, 0)$ , and  $(0, 1 - c_2)$ . Now the group  $\pi_1(L(d, q) \times S^1) \simeq z_d \oplus z$ , and the subgroup generated by  $(\overline{1 - c_1}, 0)$  and  $(\bar{s}, 0)$  is the subgroup generated by  $(\overline{(1 - c_1, s)}, 0)$ . Consequently, the cokernel of  $(1 - f_{\#})$  is isomorphic to  $z_d/(1 - c_1, s)z_d \oplus z/(1 - c_2)z$ . Which, in turn, is isomorphic to  $z_{(d, 1 - c_1, s)} \oplus z_{(1 - c_2)}$ . Therefore,

$$|\text{coker}(1 - f_{\#})| = N(f) = (d, 1 - c_1, s) \cdot |1 - c_2| = (d, 1 - c_1, s) \cdot N(f_b).$$

*Note.* (1) Since  $\mathcal{S}$  is orientable and  $T(\pi^{-1}(b), e_0) = \pi_1(\pi^{-1}(b), e_0)$ , the above formula is independent of  $w$  and  $b$ .

(2) In the above argument we could replace  $L(p, q)$  with the generalized lens space as in [5].

(3) If  $p$  is a prime the product formula follows from results of R. Brown and E. Fadell [3].

(4) Theorem 8 also indicates that a product theorem of the type obtained by R. Brown and E. Fadell is hard to expect in general.

**COROLLARY 9.** *Let  $\mathcal{S}$  be as in Theorem 8. Suppose  $f: E \rightarrow E$  is a bundle map such that for some  $b \in L(p, q)$   $f_b: \pi^{-1}(b) \rightarrow \pi^{-1}(b)$  is homotopic to a fixed-point free map. Then there exists a map  $g: E \rightarrow E$ , homotopic to  $f$ , which is fixed-point free.*

*Proof.* Let  $\tilde{f}_b$  be the fixed-point free map on  $\pi^{-1}(b)$  which is homotopic to  $f_b$ . Clearly  $N(\tilde{f}_b) = 0$  and since the Nielsen number is a homotopy invariant,  $N(f_b) = 0$ . Thus from Theorem 8,  $N(f) = 0$ , and the corollary follows from the converse of the Lefschetz fixed-point theorem of F. Wecken [8].

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David R. Adams, <i>On the exceptional sets for spaces of potentials</i> .....	1
Philip Bacon, <i>Axioms for the Čech cohomology of paracompacta</i> .....	7
Selwyn Ross Caradus, <i>Perturbation theory for generalized Fredholm operators</i> .....	11
Kuang-Ho Chen, <i>Phragmén-Lindelöf type theorems for a system of nonhomogeneous equations</i> .....	17
Frederick Knowles Dashiell, Jr., <i>Isomorphism problems for the Baire classes</i> .....	29
M. G. Deshpande and V. K. Deshpande, <i>Rings whose proper homomorphic images are right subdirectly irreducible</i> .....	45
Mary Rodriguez Embry, <i>Self adjoint strictly cyclic operator algebras</i> .....	53
Paul Erdős, <i>On the distribution of numbers of the form <math>\sigma(n)/n</math> and on some related questions</i> .....	59
Richard Joseph Fleming and James E. Jamison, <i>Hermitian and adjoint abelian operators on certain Banach spaces</i> .....	67
Stanley P. Gudder and L. Haskins, <i>The center of a poset</i> .....	85
Richard Howard Herman, <i>Automorphism groups of operator algebras</i> .....	91
Worthen N. Hunsacker and Somashekhar Amrith Nainpally, <i>Local compactness of families of continuous point-compact relations</i> .....	101
Donald Gordon James, <i>On the normal subgroups of integral orthogonal groups</i> .....	107
Eugene Carlyle Johnsen and Thomas Frederick Storer, <i>Combinatorial structures in loops. II. Commutative inverse property cyclic neofields of prime-power order</i> .....	115
Ka-Sing Lau, <i>Extreme operators on Choquet simplexes</i> .....	129
Philip A. Leonard and Kenneth S. Williams, <i>The septic character of 2, 3, 5 and 7</i> .....	143
Dennis McGavran and Jingyal Pak, <i>On the Nielsen number of a fiber map</i> .....	149
Stuart Edward Mills, <i>Normed Köthe spaces as intermediate spaces of <math>L_1</math> and <math>L_\infty</math></i> .....	157
Philip Olin, <i>Free products and elementary equivalence</i> .....	175
Louis Jackson Ratliff, Jr., <i>Locally quasi-unmixed Noetherian rings and ideals of the principal class</i> .....	185
Seiya Sasao, <i>Homotopy types of spherical fibre spaces over spheres</i> .....	207
Helga Schirmer, <i>Fixed point sets of polyhedra</i> .....	221
Kevin James Sharpe, <i>Compatible topologies and continuous irreducible representations</i> .....	227
Frank Siwiec, <i>On defining a space by a weak base</i> .....	233
James McLean Sloss, <i>Global reflection for a class of simple closed curves</i> .....	247
M. V. Subba Rao, <i>On two congruences for primality</i> .....	261
Raymond D. Terry, <i>Oscillatory properties of a delay differential equation of even order</i> .....	269
Joseph Dinneen Ward, <i>Chebyshev centers in spaces of continuous functions</i> .....	283
Robert Breckenridge Warfield, Jr., <i>The uniqueness of elongations of Abelian groups</i> .....	289
V. M. Warfield, <i>Existence and adjoint theorems for linear stochastic differential equations</i> .....	305