Pacific Journal of Mathematics

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Vol. 52, No. 1 January 1974

PACIFIC JOURNAL OF MATHEMATICS
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Let (Δ, Σ, μ) be a totally σ-finite measure space and let $M(\Delta)$ be the set of all complex-valued μ-measurable functions on Δ. This paper is concerned with determining whether certain classes of normed Köthe spaces (Banach function spaces) are intermediate spaces of $L_1 = L_1(\mu)$ and $L_\infty = L_\infty(\mu)$. It is proven that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are associate Orlicz spaces and that for every nontrivial Young's function Φ there is an equivalent Young's function Φ_1 such that the Orlicz space $L_{M\Phi_1}$ is an intermediate space of L_1 and L_∞. The notion of a universal Köthe space is presented and it is proven that if Λ is a universal Köthe space then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent universally rearrangement invariant norm ρ_1 for which L_{ρ_1} is an intermediate space of L_1 and L_∞.

1. Introduction. Let X_1 and X_2 be two Banach spaces contained in a linear Hausdorff space Y such that the injection of $X_i (i = 1, 2)$ into Y is continuous. Denote the norm of X_i by $\|\cdot\|_i$. The space $X_1 \cap X_2$ is the set of all elements which are in both X_1 and X_2, and the space $X_1 + X_2$ is the set of all $f \in Y$ of the form $f = f_1 + f_2$ with $f_1 \in X_1$ and $f_2 \in X_2$. The spaces $X_1 \cap X_2$ and $X_1 + X_2$ are Banach spaces under the norms $\|f\|_{X_1 \cap X_2} = \max\{\|f\|_1, \|f\|_2\}$ and $\|f\|_{X_1+X_2} = \inf\{\|f_1\|_1 + \|f_2\|_2 : f = f_1 + f_2, f_i \in X_i\}$ (see [1, p. 165, Prop. 3.2.1]). A Banach space $X \subset Y$ satisfying $X_1 \cap X_2 \subset X \subset X_1 + X_2$ and $\|f\|_{X_1+X_2} \leqq \|f\|_X \leqq \|f\|_{X_1 \cap X_2}$ is called an *intermediate space* of X_1 and X_2.

Much work has been done on intermediate spaces and the related topic of interpolation theory. (See [1], [2], [12].) In particular, it has been shown that the Lebesgue spaces L_p and the Lorentz spaces L_{pq} ([6] and [7]) are intermediate spaces of L_1 and L_∞. In this paper we investigate what other classes of normed Köthe spaces are intermediate spaces of L_1 and L_∞. In §7 we introduce the notion of a universal Köthe space, which we prove to be equivalent to Luxemburg's notion of a universally rearrangement invariant Köthe space [9]. We have been able to show that if Λ is a universal Köthe space, then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent norm ρ_1 which is universally rearrangement invariant and L_{ρ_1} is an intermediate space of L_1 and L_∞.

Section 2 contains preliminaries and §3 deals with Orlicz spaces. We show that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are Orlicz spaces and prove that they are associate Orlicz spaces. It is shown that for any nontrivial

Young's function Π, there is an equivalent Young's function Π_1 such that $L_{M\Pi_1}$ is an intermediate space of L_1 and L_∞. This means that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are the smallest and the largest Orlicz spaces, respectively. Section 4 deals with the monotonic rearrangement of a measurable function. Sections 5 and 6 deal with universal and universally rearrangement invariant function norms.

2. **Preliminaries.** Let (Δ, Σ, μ) be a σ-finite measure space where Δ is a point set, Σ is a σ-algebra of measurable sets, and μ is a totally σ-finite measure. Let M^+ be the set of all nonnegative μ-measurable functions on Δ. We allow that a function can assume the value $+\infty$ at some or all points $x \in \Delta$.

A mapping ρ on M^+ to the extended reals is called a *function norm* if ρ satisfies the following conditions for all f and g in M^+:

(i) $\rho(f) \geqq 0$ and $\rho(f) = 0$ if and only if $f = 0$ a.e. (almost everywhere).

(ii) $\rho(af) = a\rho(f)$ for $a \geqq 0$.

(iii) $\rho(f + g) \leqq \rho(f) + \rho(g)$.

(iv) $f(x) \leqq g(x)$ a.e. implies $\rho(f) \leqq \rho(g)$.

In addition, we assume that ρ satisfies:

(v) (*Fatou property*) $f_0, f_1, \cdots \in M^+$ and $f_n \uparrow f_0$ (pointwise a.e.) implies $\rho(f_n) \uparrow \rho(f_0)$.

(vi) (Saturated) there are no sets $E \in \Sigma$ such that $\rho(\chi_B) = \infty$ for every measurable $B \subset E$ with $\mu(B) > 0$ (χ_B is the characteristic function for the set B).

The domain of definition of ρ is extended to $M = M(\Delta, \mu)$, the set of all complex-valued, μ-measurable functions on Δ, by defining $\rho(f) = \rho(|f|)$ for $f \in M$. We denote by $L_\rho = L_\rho(\Delta, \Sigma, \mu)$ the set of all $f \in M$ satisfying $\rho(f) < \infty$. If we assume μ-almost equal functions are identified in the usual way, the spaces L_ρ are complete normed linear spaces. Such spaces are commonly called *normed Köthe spaces* or *Banach function spaces*. (For theory of normed Köthe spaces see [10].) Examples of normed Köthe spaces are Orlicz spaces, the spaces of Ellis and Halperin [3], and the Lorentz spaces [6, 7].

The *associate norm* ρ' of any function norm ρ is defined by

$$\rho'(f) = \sup\left\{\int_\Delta |fg|d\mu \colon \rho(g) \leqq 1\right\}.$$

The *associate space*, denoted $(L_\rho)'$ or $L_{\rho'}$, is defined to be $L_{\rho'} = \{f \in M\colon \rho'(f) < \infty\}$. The associate norm ρ' has the Fatou property (even if ρ did not) and hence is a normed Köthe space. (For the details see [10].)

Let (Δ, Σ, μ) be as outlined earlier, and let Δ_n be a fixed increasing sequence of sets of finite measure whose union is Δ. Let $\Omega =$

$\left\{f\colon \int |f\chi_{\Delta_n}|d\mu < \infty \text{ for all } n\right\}$ be the space of locally integrable function on Δ. For any subset $\Gamma \subset \Omega$ we define the *Köthe space* $\Lambda(\Gamma)$ associated with Γ to be $\Lambda = \Lambda(\Gamma) = \left\{f \in \Omega\colon \int_\Delta |fg|d\mu < \infty \text{ for all } g \in \Gamma\right\}$. The *associate Köthe space* Λ' is defined to be $\Lambda' = \Lambda(\Lambda(\Gamma)) = \left\{g \in \Omega\colon \int_\Delta |gf|d\mu < \infty \text{ for all } f \in \Lambda(\Gamma)\right\}$. Notice that our normed Köthe space L_ρ is also a Köthe space (since ρ is assumed to saturated).

Endow the space $M(\Delta, \mu)$ with the topology of convergence in measure on sets of finite measure. Then M becomes a linear Hausdorff space and the injection of L_ρ into M is continuous. Thus we have established the framework necessary to consider L_ρ as an intermediate space of L_1 and L_∞.

Let $\mu(\Delta) < \infty$. Then $L_\infty = L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty = L_1$ if and only if $\rho(\chi_\Delta) < \infty$ and $\rho'(\chi_\Delta) < \infty$. Furthermore, there is an equivalent norm which makes this embedding norm-reducing (Theorem 6.4). For this reason, *we will proceed under the assumption that* $\mu(\Delta) = \infty$.

Finally, we given a representation of the $L_1 + L_\infty$ norm which we will denote by $\|\cdot\|_+$.

THEOREM 2.1. *Let $f \in L_1 + L_\infty$ and let $s = \sup\{t\colon \mu\{|f| \geqq t\} \geqq 1\}$. Then*

$$\|f\|_+ = s + \int_{\{|f|>s\}} (|f| - s)d\mu\ .$$

A proof can be derived from Butzer and Berens [1, pp. 185–186].

3. Orlicz spaces as intermediate spaces. For basic Orlicz space theory, the reader is referred to [5], [8], or [15].

Let $\Phi\colon [0, \infty) \to [0, \infty)$ and $\Psi\colon [0, \infty) \to [0, \infty)$ be complementary Young's functions. Hence Φ and Ψ are increasing, absolutely continuous on the sets where they are finite, and convex. Let

$$\|f\|_{M\Phi} = \inf\left\{k > 0\colon \int_\Delta \Phi(|f|/k)d\mu \leqq 1\right\}\ .$$

The *Orlicz space* $L_{M\Phi}$ is the set of all complex-valued, μ-measurable functions satisfying $\|f\|_{M\Phi} < \infty$. Hence the Orlicz space $L_{M\Phi}$ is a normed Köthe space and, as such, it satisfies the properties stated in §2. In particular we can form the associate norm, denoted $\|\cdot\|_\Psi$,

$$\|f\|_\Psi = \sup\left\{\int_\Delta |fg|\, d\mu\colon \|g\|_{M\Phi} \leqq 1\right\},$$

and the associate space $L_\Psi = \{g\colon \|g\|_\Psi < \infty\}$.

We will denote the $L_1 \cap L_\infty$ norm by $\|\cdot\|_\cap$.

THEOREM 3.1. (a) *If Π is a (nontrivial) Young's function, then $L_1 \cap L_\infty \subset L_{M\Pi}$.* (b) *$L_1 \cap L_\infty$ is an Orlicz space. In particular there is a Young's function Ψ such that $\|f\|_\cap = \|f\|_{M\Psi}$ for all $f \in M$.*

Proof. Consider the Orlicz space given by $\Psi(u) = u$ for $0 \leqq u \leqq 1$ and $\Psi(u) = \infty$ for $1 < u$.

From Theorem 3.1 we see that $L_1 \cap L_\infty$ is the smallest Orlicz space.

Let Ψ be as defined in the proof of Theorem 3.1. Let Φ be the complementary Young's function of Ψ. One can check that $\Phi(u) = 0$ for $0 \leqq u \leqq 1$ and $\Phi(u) = u - 1$ for $1 \leqq u$.

LEMMA 3.2. *$L_{M\Phi}$, $(L_1 \cap L_\infty)'$, and $L_1 + L_\infty$ all consist of the same functions.*

It is not true that $\|\cdot\|_+ = \|\cdot\|_{M\Phi}$. For example let (Δ, Σ, μ) be $[0, \infty)$ with Lebesgue measure and let $f = 10\chi_{(0,1/2]} + 5\chi_{[1,3]}$. Then $\|f\|_{M\Phi} \leqq 5$ but $\|f\|_+ = 15/2$. However, the following is true.

THEOREM 3.3. (a) *For any $f \in L_1 + L_\infty$, we have $\|f\|_\Phi = \|f\|_+$.* (b) *$L_1 + L_\infty$ is an Orlicz space; in particular $(L_1 + L_\infty, \|\cdot\|_+) = (L_\Phi, \|\cdot\|_\Phi)$.*

Proof. Let $f \in L_1 + L_\infty$ and $g \in L_{M\Psi} = L_1 \cap L_\infty$. Then by Theorem 2.1 we get $\int |f|(g/\|g\|_\cap)d\mu \leqq \|f\|_+$. Hence

$$\|f\|_\Phi = \sup\left\{\int |f(g/\|g\|_\cap)|\,d\mu \colon g \in L_{M\Psi}\right\} \leqq \|f\|_+ .$$

To show the reverse inequality let $f \in L_1$ with $f \geqq 0$ and $s = \sup\{t \colon \mu\{f \geqq t\} \geqq 1\}$. Furthermore assume that f is a simple function (i.e., f is a linear combination of characteristic functions of sets of finite measure). Because f is simple, one can show that $\mu\{f > s\} \leqq 1$, $\mu\{f \geqq s\} \geqq 1$, and $\mu\{f = s\} \neq 0$. Now define $\alpha\colon \Delta \to [0, \infty)$ by $\alpha(x) = 1$ if $x \in \{f > s\}$, $\alpha(x) = (1 - \mu\{f > s\})/\mu\{f = s\}$ if $x \in \{f = s\}$ and $\alpha(x) = 0$ otherwise. Then $\|\alpha\|_\cap = 1$ and

$$\int |f\alpha|\,d\mu = s + \int_{\{f>s\}} (f - s)d\mu = \|f\|_+ .$$

Therefore, $\|f\|_+ = \int |f\alpha|\,d\mu \leqq \|f\|_\Phi$ by Hölders inequality [8, p. 7] and we have shown the equality for any simple function. Since both $\|\cdot\|_+$ and $\|\cdot\|_\Phi$ have the Fatou property, it is an easy matter to extend the result to an arbitrary $f \in L_1 + L_\infty$.

Combining Theorem 3.1 and Theorem 3.3, we can say $L_{M\Pi} \subset (L_1 \cap L_\infty)' = L_1 + L_\infty$ for any Young's function Π. Hence $L_1 + L_\infty$ is the largest Orlicz space and we have

$$L_1 \cap L_\infty \subset L_{M\Pi} \subset L_1 + L_\infty .$$

An element $B \in \Sigma$ is called an *atom* if $A \in \Sigma$ and $A \subset B$ implies $\mu(A) = 0$ or $\mu(A) = \mu(B)$. If we restrict ourselves to the case that (Δ, Σ, μ) is nonatomic (i.e., has no atoms), then G. G. Gould [4] and Luxemburg and Zaanen [11] have obtained some results similar to ours. If μ has no atoms, then define the function norm $\|\cdot\|_G$ as

$$\|f\|_G = \sup\left\{\int_E |f|\, d\mu \colon \mu(E) = 1\right\} .$$

It was shown by Luxemburg and Zaanen and by Gould that for $f \in L_1 + L_\infty$, $\|f\|_G = \|f\|_+$. This is also mentioned by Butzer and Berens [1, p. 183]. Luxemburg and Zaanen have shown that the associate space of $(L_1 + L_\infty, \|\cdot\|_G)$ is the space $(L_1 \cap L_\infty, \|\cdot\|_\cap)$. One might hope that for each $f \in L_1 + L_\infty$ there exists a set E_f such that $\mu(E_f) = 1$ and $\|f\|_+ = \|f\|_G = \int_{E_f} |f|\, d\mu$. This is true for simple function, but it is not true for general functions as is shown by the following example.

Let (Δ, Σ, μ) be $[0, \infty)$ with Lebesgue measure and let $f(t) = (1 - 1/t)\chi_{[1,\infty)}$. Using Theorem 2.1 $\|f\|_G = \|f\|_+ = 1$. For any $E \subset [0, \infty)$ such that $\mu(E) = 1$ it follows that $\int_E |f|\, dt < 1 = \|f\|_+$.

Let us return to the question of whether all Orlicz spaces are intermediate spaces of L_1 and L_∞. It is easy to see that there are many spaces whose embeddings are not norm-reducing (e.g. $L_{M2\Psi}$, where $L_{M\Psi} = L_1 \cap L_\infty$). But we prove the following.

THEOREM 3.4. *Every Orlicz space $L_{M\Pi}$ has an equivalent Orlicz norm $\|\cdot\|_{M\Pi_1}$ for which it becomes an intermediate space of L_1 and L_∞.*

Proof. Let Ψ and Φ denote the Young's functions for $L_1 \cap L_\infty$ and $L_1 + L_\infty$, respectively. Let Π be a nontrivial Young's function. It may happen that there exists $u_0 (u < u_0 < \infty)$ such that $\Pi(u) = 0$ for $u \leqq u_0$ and $\Pi(u) = \infty$ for $u > u_0$. In this case $L_{M\Pi} = L_\infty$ as sets, so $\|\cdot\|_{M\Pi}$ is equivalent with the L_∞ norm. In all other cases, there is a $u_0 > 0$ such that $0 < \Pi(u_0) < \infty$. Now define Π_2 and Π_1 by $\Pi_2(u) = \Pi(u_0 u)/\Pi(u_0)$ for $u \geqq 0$ and $\Pi_1(u) = \Pi_2(u)$ for $0 \leqq u \leqq 1$ and $\Pi_1(u) = 2\Pi_2(u) - 1$ for $1 \leqq u$. Notice that Π_2 is continuous, convex, $\Pi_2(u) \geqq 0$ for all u, $\Pi_2(0) = 0$, and $\Pi_2(1) = 1$. This means that Π_1 is continuous, convex, $\Pi_1(u) \geqq 0$ for all u, $\Pi_1(0) = 0$ all and $\Pi_1(1) = 1$.

Thus Π_1 is a Young's function [8, p. 38, Remark (1)].

Because Π_2 is convex and $\Pi_2(1) = 1$, we have $\Pi_2(u) \geqq u$ for $u \geqq 1$; so $\Pi_1(u) \geqq 2u - 1$ for $u \geqq 1$. Therefore, $2\Phi(u) = 2u - 2 \leqq \Pi_1(u) \leqq \infty = \Psi(u)$ for $u \geqq 1$. Now for $0 \leqq u \leqq 1$, we have

$$2\Phi(u) = 0 \leqq \Pi_1(u) = \Pi(uu_0)/\Pi(u_0)$$
$$\leqq \frac{u\Pi(u_0)}{\Pi(u_0)} = u = \Psi(u) .$$

Hence for all $u \geqq 0$, $2\Phi(u) \leqq \Pi_1(u) \leqq \Psi(u)$. This means that

$$\|f\|_+ = \|f\|_\Phi \leqq 2\|f\|_{M\Phi} \leqq \|f\|_{M\Pi_1} \leqq \|f\|_{M\Psi} = \|f\|_\cap .$$

Next we will show that $L_{M\Pi}$ and $L_{M\Pi_1}$ consist of the same functions which means that $\|\cdot\|_{M\Pi}$ and $\|\cdot\|_{M\Pi_1}$ are equivalent. First notice that $\Pi_2(u) \leqq \Pi_1(u) \leqq 2\Pi_2(u)$ for all $u \geqq 0$. From which it follows that $\int \Pi(|f|/k)d\mu < \infty$ if and only if $\int \Pi_1(|f|/k)d\mu < \infty$. Therefore, $f \in L_{M\Pi}$ if and only if $f \in L_{M\Pi_1}$.

What about the space L_Π? Let Ω be the complementary Young's function for Π. Let Ω_1 be given by Theorem 3.4. Then the associate norm of $\|\cdot\|_{M\Omega_1}$ denoted by $\|\cdot\|_{\Pi_2}$ will make L_Π an intermediate space of L_1 and L_∞.

4. Monotonic rearrangement.

Let $f \in M(\Delta, \mu)$, then the *monotonic rearrangement* of f is the function $f^*: [0, \infty) \to [0, \infty]$ defined by

$$f^*(t) = \inf\{y \geqq 0: \mu\{|f(x)| > y\} \leqq t\} .$$

Let f and g belong to $M(\Delta, \mu)$. Then f and g are called *equimeasurable* whenever $\mu\{|f(x)| > r\} = \mu\{|g(x)| > r\}$ for all $r \geqq 0$. If f and g are equimeasurable we write $f \sim g$. Notice that $f \sim g$ if and only if $f^* = g^*$. Since $\mu\{|f(x)| > r\} = m\{f^*(t) > r\}$ for all r, we will say that f and f^* are equimeasurable even though they are defined on different measure spaces. Hence f^* is the unique, nonnegative, monotonic nonincreasing, right-continuous function on $[0, \infty)$ which is equimeasurable with f. For properties of the montonic rearrangement refer to [9] and [14].

The following lemma, whose proof is straightforward, has several important consequences.

LEMMA 4.1. *Let Π be any Young's function and let f be μ-measurable. Then* $\int_\Delta \Pi(|f|)d\mu = \int_0^\infty \Pi(f^*)dt$.

COROLLARY 4.2. *Let Π be a Young's function and let f and g belong to $M(\mu)$.*

(i) $\|f\|_{M\Pi} = \|f^*\|_{M\Pi}$.
(ii) *If* $f \sim g$, *then* $\|f\|_{M\Pi} = \|g\|_{M\Pi}$.
(iii) *If* $f \in L_1 \cap L_\infty$ *and* $g \sim f$, *then* $g \in L_1 \cap L_\infty$.
(iv) $\|f\|_+ = \|f^*\|_{L_1([0,\infty)) \cap L_\infty([0,\infty))}$.

Now we are able to quickly prove a result which is stated by Butzer and Berens [1, p. 184, Prop. 3.3.7].

THEOREM 4.3. *Let* $f \in M(\mu)$, *then* $\|f\|_+ = \int_0^1 f^*(t)dt$.

Proof. From Corollary 4.2, we know that $\|f\|_+ = \|f^*\|_+$. So we will show that $\|f^*\|_+ = \int_0^1 f^*(t)dt$. Since f^* is a monotonic decreasing function, we know that $\{f^* > s_{f^*}\} \subset [0, 1) \subset \{f^* \geqq s_{f^*}\}$. So by Theorem 2.1

$$\|f\|_+ = s_{f^*} + \int_0^1 f^* dt - \int_0^1 s_{f^*} dt = \int_0^1 f^* dt \ .$$

This representation of $\|\cdot\|_+$ allows us to make the following statement about general Köthe spaces.

COROLLARY 4.4. *Let* Λ *be a Köthe space and let* Λ^* *be the set of all monotonic rearrangements of functions in* Λ *and let* Λ' *be the Köthe dual of* Λ. *Then the following are equivalent*:
(i) $L_1(\mu) \cap L_\infty(\mu) \subset \Lambda \subset L_1(\mu) + L_\infty(\mu)$.
(ii) $(\Lambda^* \cup \Lambda'^*) \subset L_1(m) + L_\infty(m)$.
(iii) $\int_0^1 f^*(t)dt < \infty$ *for all* $f \in (\Lambda \cup \Lambda')$.
(iv) $\int_0^r f^*(t)dt < \infty$ *for all* $f \in (\Lambda \cup \Lambda')$ *for any* $r > 0$.

5. Rearrangement invariant Köthe spaces.

DEFINITION 5.1. A Köthe space Λ is called *rearrangement invariant* if $f \in \Lambda$ and g equimeasurable with f implies $g \in \Lambda$.

(ii) A function norm ρ is called *rearrangement invariant* if $f \in L_\rho$ and g equimeasurable with f implies $\rho(f) = \rho(g)$.

Notice that if ρ is a rearrangement invariant function norm, then L_ρ is a rearrangement invariant Köthe space. However, a normed Köthe space may be rearrangement invariant but not norm rearrangement invariant. Most of the well-known examples of normed Köthe spaces are rearrangement invariant. Included are the L_p spaces ($1 \leqq p \leqq \infty$), Orlicz spaces and Lorentz spaces L_{pq}. Furthermore, given any Young's function Π and any $f \in M(\mu)$ we have that $\|f\|_{M\Pi} = \|f^*\|_{M\Pi}$ (Corollary 4.2).

DEFINITION 5.2. A function norm λ defined on $M([0, \infty), m)$ is called *universal* if for each totally σ-finite measure space (Δ, Σ, μ) the functional ρ defined on $M(\Delta, \mu)$ by $\rho(f) = \lambda(f^*)$ is a function norm. In this case we say that *ρ is induced by λ.*

Not every function norm on $M([0, \infty), m)$ is universal. Consider λ defined on $M([0, \infty), m)$ by $\lambda(f) = \|f\chi_{[0,1)}\|_1 + \|f\chi_{[1,\infty)}\|_\infty$. Let (S, ν) be a totally σ-finite measure space with sets A, B, and C such that $\nu(A) = 1/4$, $\nu(B) = 1/2$, and $\nu(C) = 3/4$. Let $f = 5\chi_B + 3\chi_A$ and $g = 4\chi_C$. Then $\rho(f) + \rho(g) = 25/4 < 17/2 = \rho(f + g)$ which means ρ is not a function norm. Therefore, λ is not universal.

Next we state a theorem that was proven by Silverman [14] and that has proven very useful for us.

LEMMA 5.3. (Silverman). *If (Δ, μ) has no atoms and if $f, g \in M(\mu)$, then $\int_0^\infty f^*g^*dt = \infty$ if and only if $\int_\Delta |f'g|d\mu = \infty$ for some $f' \sim f$.*

The theory of rearrangement invariant function norms has received some attention, most notably from Luxemburg [9]. However, each time the setting has been somewhat more restrictive than ours. Hence several cases of Lemma 5.4 and Theorem 5.5 are known. See [9] and [13].

LEMMA 5.4. *If (Δ, Σ, μ) is nonatomic, then for any $f, g \in M(\mu)$ we have* $\int_0^\infty f^*g^*dt = \sup\left\{\int_\Delta |fg'|d\mu\colon g' \sim g\right\}$.

Proof. Because of Lemma 5.3 we can assume that $\int_0^\infty f^*g^*\,dt < \infty$. Further, without loss of generality we may assume that $f, g \in M^+(\mu)$. Let $\varphi = \sum_{i=1}^{m+1} a_i\chi_{A_i}$ be a simple function in $M^+(\mu)$ where $a_1 > a_2 > \cdots > a_m > a_{m+1} = 0$ and $A_{m+1} = \Delta\backslash(\bigcup_{i=1}^m A_i)$. Let $g \in M^+(\mu)$ be arbitrary. Then $g^* \in M^+([0, \infty))$, so for each pair of integers $\langle n, k\rangle$ such that $0 \leqq k \leqq 2^{2n}$ let

$$E_{n,k} = \{t \in [0, \infty)\colon k/2^n < g^*(t) \leqq (k + 1)/2^n\}$$

and

$$E_{n,2^{2n}+1} = [0, \infty)\Big\backslash\left(\bigcup_{k=0}^{2^{2n}} E_{n,k}\right).$$

Set

$$\psi_n = \sum_{k=0}^{2^{2n}} (k/2^n)\chi_{E_{n,k}}.$$

Then $\{\psi_n\}_{n=1}^\infty$ is as a sequence of simple functions such that $\psi_n^* \uparrow g^*$. Notice that for a fixed n_0 the sets $\{E_{n_0, k}\}_{k=0}^{2^{2n_0}}$ are disjoint sets and each $E_{n_0,k}$ is the disjoint union of a finite number of sets $\{E_{n_0+1,j}\}_{j \in F_{n_0,k}}$. Hence, since (Δ, μ) has no atoms, by induction we can define the sets $\tilde{E}_{n,k}$ in Δ such that

(1) $\tilde{E}_{n_0,k_1} \cap \tilde{E}_{n_0,k_2}$ is empty for $k_1 \neq k_2$.

(2) $\mu(\tilde{E}_{n,k}) = m(E_{n,k})$.

(3) $\mu(A_i \cap \tilde{E}_{n,k}) = m(A_i^* \cap E_{n,k})$.

(4) $\mu(\tilde{E}_{n_1,k_1} \cap \tilde{E}_{n_2,k_2}) = m(E_{n_1,k_1} \cap E_{n_2,k_2})$.

Next we define the simple functions $\tilde{\psi}_n: \Delta \to [0, \infty)$ by

$$\tilde{\psi}_n = \sum_{k=0}^{2^{2n}} (k/2^n)\chi_{\tilde{E}_{n,k}} .$$

Because of the properties of the sets $\{\tilde{E}_{n,k}\}$, one can show that ψ_n and $\tilde{\psi}_n$ are equimeasurable for all n and that $\{\tilde{\psi}_n(x)\}_{n=1}^\infty$ is an increasing sequence for each $x \in \Delta$. Also $\int_\Delta \varphi\tilde{\psi}_n d\mu = \int_0^\infty \varphi^*\psi_n dt$ since $\mu(A_i \cap \tilde{E}_{n,k}) = m(A_i^* \cap E_{n,k})$. Let $\tilde{g}(x) = \lim_{n\to\infty} \tilde{\psi}_n(x)$. Then $\tilde{g}^* = \lim_n \tilde{\psi}_n^* = \lim_n \psi_n^* = g^*$, so $\tilde{g}$ and g are equimeasurable and $\int_\Delta \varphi\tilde{g}d\mu = \int_0^\infty \varphi^* g^* dt$.

Hence the equation is true for arbitrary g and simple functions φ. The extension to arbitrary functions follows easily.

The next result was also stated by Luxemburg [9]. A proof follows from Lemma 5.4.

THEOREM 5.5. *Let (Δ, μ) be a nonatomic measure space and let ρ be a function norm defined on $M(\mu)$.*

(i) *If ρ is rearrangement invariant, then ρ' is rearrangement invariant.*

(ii) *ρ is rearrangement invariant if and only if*

$$\rho(f) = \sup\left\{\int_0^\infty f^*g^*dt: \rho'(g) \leqq 1\right\} .$$

A partition $P = \{E_j\}_{j=1}^n$ in Δ is defined to a finite disjoint collection of sets of positive measure. Define the average function of $f \in M(\mu)$ with respect to P to be

$$f_P = \sum_{j=1}^n \left(\int_{E_j} f d\mu/\mu(E_j)\right)\chi_{E_j} .$$

A function norm ρ defined on $M(\mu)$ is said to satisfy *Property* (J) if for each partition P and any $f \in L_\rho$, we have $\rho(f_P) \leqq \rho(f)$. This is similar to the levelling length property introduced by Ellis and Halperin [3].

Let R be the set of all nonnegative, monotonic nonincreasing, right-continuous functions defined on $[0, \infty)$. Then the monotonic rearrangement of any measurable function belonging to $M(\mu)$ is contained in R. Also $g^* = g$ for any $g \in R$.

The next result is stated in terms of the levelling length property by Luxemburg ([9, p. 132]).

THEOREM 5.6. *Let (Δ, μ) be non-atomic and let ρ be a rearrangement invariant function norm on $M(\mu)$. Then ρ has property (J).*

Proof. Let $f \in M^+(\mu)$ and let $P = \{E_j\}_{j=1}^n$ be a partition in Δ. Let $b_j = \left(\int_{E_j} f d\mu/\mu(E_j)\right)$. Renumber the E_j, if necessary, so that $b_1 \geqq b_2 \geqq \cdots \geqq b_n$. Set $E_{n+1} = \Delta \backslash \bigcup_{j=1}^n E_j$ and $b_{n+1} = 0$; hence

$$f_P^* = \sum_{j=1}^{n+1} b_j \chi E_j^*$$

where

$$E_j^* = [y_{j-1}, y_j) = \left[\sum_{l=1}^{j-1} \mu(E_l), \sum_{l=1}^{j} \mu(E_l)\right)$$

with the understanding that $y_0 = 0$ and $y_{n+1} = \infty$.

Define the function $h: [0, \infty) \rightarrow [0, \infty)$ by

$$h(t) = \sum_{j=1}^{n} (f\chi_{E_j})^*(t - y_{j-1})\chi_{E_j^*}(t) .$$

The collection $P' = \{E_j^*\}_{j=1}^n$ is a partition in $[0, \infty)$, and

$$h_{P'} = \sum_{j=1}^{n} \frac{\int_0^{\mu(E_j)} (f\chi_{E_j})^*(t)dt}{m(E_j^*)} \chi_{E_j^*} = \sum_{j=1}^{n} \frac{\int_{E_j} f d\mu}{\mu(E_j)} \chi_{E_j^*} = f_P^* .$$

For each x such that $y_{j-1} \leqq x \leqq y_j$ we know that

$$\int_{y_{j-1}}^{x} h(t)dt \geqq \int_{y_{j-1}}^{x} h_{P'}(t)dt = \int_{y_{j-1}}^{x} f_P^*(t)dt \tag{1}$$

since h is nondecreasing on E_j^*. Let $\varphi = \sum_{i=1}^{m+1} a_i \chi_{A_i} (a_1 > a_2 > \cdots > a_m > a_{m+1} = 0,\ A_{m+1} = [0, \infty) \backslash \bigcup_{i=1}^m A_i)$ be a simple function in R (the set of monotonic rearrangements). Then by Hardy's theorem (Luxemburg [9, p. 34]) we have

$$\int_{E_j^*} h\varphi dt = \int_{E_j^*} f_P^* \varphi dt .$$

For $1 \leqq j \leqq n + 1$, set $\varphi_j = \varphi \chi_{E_j^*}$. Since h and φ are nonincreasing on E_j^* we know that $(h\chi_{E_j^*})^*(t) = h(t + y_{j-1})$ and $\varphi_j^*(t) = \varphi(t + y_{j-1})$. Hence

$$\int_0^\infty (f\chi_{E_j})^* \varphi_j^* dt = \int_0^\infty (h\chi_{E_j^*})^* \varphi_j^* dt = \int_{E_j^*} h\varphi dt .$$

Because (Δ, μ) is nonatomic, for each $j = 1, 2, \cdots, n+1$ we can define a function $\tilde{\varphi}_j: E_j \to [0, \infty)$ which is equimeasurable with φ_j. Since φ_j is simple, we have seen in the proof of Lemma 5.4 that there exist functions $\tilde{f}_j: E_j \to [0, \infty) (1 \leqq j \leqq n+1)$ such that $\tilde{f}_j$ is equimeasurable with $f\chi_{E_j}$ and $\int_{E_j} \tilde{f}_j \tilde{\varphi}_j d\mu = \int_0^\infty (f\chi_{E_j})^* (\varphi_j)^* dt$. Let

$$\tilde{\varphi} = \sum_{j=1}^{n+1} \tilde{\varphi}_j \chi_{E_j} \quad \text{and} \quad f_1 = \sum_{j=1}^{n+1} \tilde{f}_j \chi_{E_j} .$$

Then f_1 is equimeasurable with f and

$$\int_\Delta f_1 \tilde{\varphi} d\mu \geqq \sum_{j=1}^{n} \int_0^\infty (f\chi_{E_j})^* \varphi_j^* dt \geqq \sum_{j=1}^{n} \int_{E_j^*} f_P^* \varphi dt = \int_0^\infty f_P^* \varphi dt .$$

Hence

$$\int_0^\infty f^* \varphi dt = \sup \left\{ \int_\Delta |f_1 \varphi'| d\mu : \varphi' \sim \varphi \right\} \geqq \int_\Delta f_1 \tilde{\varphi} d\mu \geqq \int_0^\infty f_P^* \varphi dt .$$

Now let $g \in R$ be arbitrary, then there exists a sequence of simple functions φ_k such that $\varphi_k \uparrow g$ a.e. on $[0, \infty)$. Then φ_k can be chosen to lie in R for each k. Since ρ is rearrangement invariant

$$\begin{aligned} \rho(f_P) &= \sup \left\{ \lim \int_0^\infty f_P^* \varphi_n dt : \varphi_n \uparrow g \quad \text{and} \quad \rho'(g) \leqq 1 \right\} \\ &\leqq \sup \left\{ \lim \int_0^\infty f^* \varphi_n dt : \varphi_n \uparrow g \quad \text{and} \quad \rho'(g) \leqq 1 \right\} = \rho(f) . \end{aligned}$$

Therefore ρ has property (J).

We will give an example at the end of this section to show that a universal function norm does not necessarily have property (J).

Let Γ be any nontrivial subset of R. Define the functional $F = F_\Gamma$ on $M(\Delta, \mu)$ by $F(f) = \sup \left\{ \int_0^\infty f^* h dt : h \in \Gamma \right\}$. Then F is a function norm with the Fatou property.

THEOREM 5.7. (a) *If λ is a rearrangement function norm on $M([0, \infty))$, then λ is universal.*

(b) *Let ρ be a function norm defined on $M(\Delta, \mu)$ which is induced by a universal function norm λ. Then for each $f \in M(\Delta, \mu)$ we have $\rho'(f) = \sup \left\{ \int_0^\infty f^* h dt : h \in R \text{ and } \lambda(h) \leqq 1 \right\}$.*

(c) *If λ is rearrangement invariant on $M([0, \infty))$, then λ' is universal; moreover, if $\rho(f) = \lambda(f^*)$, then $\rho'(f) = \lambda'(f^*)$.*

Proof. To prove (a) let $\Gamma = \{g^* : \lambda'(g) \leqq 1\}$. Then for $f \in M([0,$

∞)) we have $F_\Gamma(f) = \lambda(f)$ which means λ is universal.

In the proof of (b) we may assume that λ is rearrangement invariant and by Theorem 5.6 λ has property (J).

It is not hard to see that

$$\rho'(f) \leqq \sup\left\{\int_0^\infty f^* h dt \colon h \in R \quad \text{and} \quad \lambda(h) \leqq 1\right\}.$$

Now we will show the reverse inequality for simple functions. Assume $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ is a simple function in $M^+(\Delta, \mu)$ where $a_1 > a_2 > \cdots > a_n > 0$ and the A_i are mutually disjoint. Then $\varphi^* = \sum_{i=1}^n a_i \chi_{A_i^*}$ where $m(A_i^*) = \mu(A_i)$. Let $g \in R$ and define $\tilde{g} \colon \Delta \to [0, \infty)$ by

$$\tilde{g} = \sum_{i=1}^n \left(\int_{A_i^*} g dt / m(A_i^*)\right)\chi_{A_i}.$$

Then $\tilde{g}^* = g_P$ where P is the partition $\{A_i^*\}_{i=1}^n$ in $[0, \infty)$. So if $\lambda(g) \leqq 1$, by property (J), $\rho(\tilde{g}) = \lambda(\tilde{g}^*) = \lambda(g_P) \leqq \lambda(g) \leqq 1$. Also

$$\int_\Delta \varphi \tilde{g} d\mu = \int_0^\infty \varphi^* g dt$$

which means

$$\sup\left\{\int_0^\infty \varphi^* g dt \colon g \in R,\ \lambda(h) \leqq 1\right\} \leqq \sup\left\{\int_\Delta \varphi h d\mu \colon h \in M(\Delta, \mu),\ \rho(h) \leqq 1\right\}$$
$$= \rho'(\varphi).$$

Therefore, (b) is true for every simple function in $M(\Delta, \mu)$ and the extension to arbitrary functions follows from the Fatou property.

We conclude this section with the following example. Let $\mathscr{I} = \{I_i\}_{i=1}^\infty$ be the partition of $[0, \infty)$ with $I_i = [i-1, i)$. For any $f \in M^+([0, \infty))$ define $f_\mathscr{I}$ to be the average function $f_\mathscr{I} = \sum_{i=1}^\infty \left(\int_{I_i} f dt\right)\chi_{I_i}$. Some of the properties of $f_\mathscr{I}$ are

(i) $f_\mathscr{I} = 0$ if and only if $f = 0$ a.e. on $[0, \infty)$.

(ii) $(af_\mathscr{I}) = a(f_\mathscr{I})$.

(iii) $(f + g)_\mathscr{I} = f_\mathscr{I} + g_\mathscr{I}$.

(iv) If $f_n \uparrow f$, then $(f_n)_\mathscr{I} \uparrow f_\mathscr{I}$.

Define the functional λ_0 on $M^+([0, \infty))$ by $\lambda_0(f) = \|f_\mathscr{I}\|_\infty$. Then λ_0 is a function norm with the Fatou property.

λ_0 *is universal.* Notice that λ_0 is universal if and only if $(\lambda_0)_m(f) = \lambda_0(f^*)$ is a function norm. For any $f \in M([0, \infty))$, $f^* \in R$ which means that $\int_{I_1} f^* dt \geqq \int_{I_i} f^* dt$ for all $i = 1, 2, \cdots$. Hence $(\lambda_0)_m(f) = \int_{I_1} f^* dt = \int_0^1 f^* dt = \|f\|_{L_1 + L_\infty}$. Therefore, $(\lambda_0)_m$ is a function norm which makes λ_0 universal.

λ_0 *is not rearrangement invariant* and in fact L_{λ_0} is not even rearrangement invariant. Let $f = \sum_{i=1}^{\infty} i\chi_{[i,i+1/i)}$. Then

$$\lambda_0(f) = \sup\left\{\int_{I_i} f\,dt\right\}_{i=1}^{\infty} = 1\ .$$

Let $\{A_i\}_{i=1}^{\infty}$ be the subsets of $[0, \infty)$ defined by $A_i = [\sum_{k=1}^{i-1} 1/k, \sum_{k=1}^{i} 1/k)$. Define $f_1 = \sum_{i=1}^{\infty} i\chi_{A_i}$. Then f and f_1 are equimeasurable but $\lambda_0(f_1) = \infty$. Hence L_{λ_0} is not rearrangement invariant.

λ_0 *does not have property* (J). Let $P = \{[1/2, 2)\}$ and let $\varphi = 6\chi_{[1/2,1)} + 4\chi_{[1,2]}$. Then $\varphi_P = (14/3)\chi_{[1/2,2)}$ and $\lambda_0(\varphi_P) = 14/3$. But $\lambda_0(\varphi) = 4$. Thus $\lambda_0(\varphi) < \lambda_0(\varphi_P)$ which means λ_0 does not have property (J).

λ_0' *is not universal*. One can show that $\lambda_0'(g) = \sum_{i=1}^{\infty} \|g\chi_{I_i}\|_\infty$. Let $f = 3\chi_{[0,1)}$ and $g = 2\chi_{[1/2,3/2)}$. Then $(\lambda_0')_m(f) + (\lambda_0')_m(g) = 5 < 7 = (\lambda_0')_m(f + g)$ which means λ_0' is not universal.

6. Universally rearrangement invariant function norms. If (Δ, Σ, μ) is a σ-finite measure space, then Δ can be written as the union of a sequence of disjoint sets $\Delta_0, e_1, e_2, \cdots$ belonging to Σ such that Δ_0 is atom free and each e_i is an atom of finite measure. Let $\{B_i\}_{i=1}^{\infty}$ be a collection of disjoint intervals on the positive real axis such that $B_i = [a_i, b_i]$ and $b_i - a_i = \mu(e_i)(i = 1, 2, \cdots)$. Set $\Delta_1 = \Delta_0 \cup (\bigcup_{i=1}^{\infty} B_i)$ and let $(\Delta_1, \Sigma_1, \mu_1)$ be the direct sum of the measure space $(\Delta_0, \Sigma \cap \Delta_0, \mu)$ and the spaces $(B_i, m)(i = 1, 2\cdots)$. Then $(\Delta_1, \Sigma_1, \mu_1)$ is a nonatomic σ-finite measure space with $\mu_1(\Delta_1) = \mu(\Delta) = \infty$. Furthermore, $M(\Delta, \Sigma, \mu)$ can be identified with a subset of $M(\Delta_1, \Sigma_1, \mu_1)$, in particular the set of all functions which are constant on the intervals B_i. We will say that (Δ, Σ, μ) is *embedded* in $(\Delta_1, \Sigma_1, \mu_1)$.

The next definition is due to Luxemburg [9, p. 98].

DEFINITION 6.1. Let (Δ, Σ, μ) be embedded in $(\Delta_1, \Sigma_1, \mu_1)$. Define the transformation $T_\mu\colon M(\Delta_1, \mu_1) \to M(\Delta, \mu)$ by

$$T_\mu(f) = f\chi_{\Delta_0} + \sum_{i=1}^{\infty}\left(\int_{B_i} f\,dt/m(B_i)\right)\chi_{e_i}\ .$$

A function norm ρ on $M(\Delta, \Sigma, \mu)$ is said to be *universally rearrange-invariant* whenever $\rho(T_\mu f_1) \geqq \rho(f)$ for all $f \in M^+(\Delta, \mu)$ and all $f_1 \in M(\Delta_1, \mu_1)$ satisfying $f_1 \sim f$.

Notice that if (Δ, μ) is non-atomic, then ρ is universally rearrangement invariant if and only if ρ is rearrangement invariant.

Lemma 6.2 relates the subjects of the previous section to the concept of universally rearrangement invariant (compare [9, p. 121, Theorem 12.2]).

LEMMA 6.2. (a) *Let ρ be a function norm defined on $M(\Delta, \mu)$.*

Then the following are equivalent:

(i) ρ *is induced by a universal function norm.*

(ii) ρ *is universally rearrangement invariant.*

(iii) $\rho(f) = \sup\left\{\int_0^\infty f^*g^*dt : \rho'(g) \leqq 1\right\}$ *for all* $f \in M^+(\Delta, \mu)$.

(b) *If* ρ *is universally rearrangement invariant, then* ρ' *is universally rearrangement invariant.*

We are now able to show that the function norms induced by a universal function norm behave very much like the Orlicz norms with respect to $L_1 \cap L_\infty$ and $L_1 + L_\infty$. We will need to use a result of Silverman [14, p. 230].

THEOREM 6.3. (Silverman). *Let* (Δ, μ) *be nonatomic and let* Λ *be a Köthe space in* $M(\Delta, \mu)$. *If* Λ *is rearrangement invariant then* $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.

THEOREM 6.4. *Let* ρ *be a universally rearrangement invariant function norm defined on* $M(\Delta, \mu)$. *Then*

(a) $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$.

(b) *there is an equivalent universally rearrangement invariant function norm* ρ_1 *such that* L_{ρ_1} *is an intermediate space of* L_1 *and* L_∞.

Proof. To prove (a) notice that since ρ is universally rearrangement invariant, there exists a rearrangement invariant function norm λ defined on $M([0, \infty))$ such that $\rho(f) = \lambda(f^*)$. λ' is rearrangement invariant so by Theorem 6.3 we have $L_1 \cap L_\infty \subset L_\lambda, L_{\lambda'} \subset L_1 + L_\infty$. Hence $\|f\|_{L_1+L_\infty} = \int_0^1 f^*dt < \infty$ for all $f \in (L_\rho \cup L_{\rho'})$. So by Corollary 4.4 we know $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$.

To prove (b) let $\Gamma = \{g: \rho'(g) \leqq 1\}$ be the unit ball for $L_{\rho'}$ and let $B_\cap = \{g: \|g\|_\cap \leqq 1\}$ and $B_+ = \{g: \|g\|_+ \leqq 1\}$ be the unit balls for $L_1 \cap L_\infty$ and $L_1 + L_\infty$, respectively. ρ' is universally rearrangement invariant which means $L_1 \cap L_\infty \subset L_{\rho'} \subset L_1 + L_\infty$. Hence there is a constant a such that $(1/a)\rho' \leqq \|\cdot\|_\cap$, i.e., $B_\cap \subset a\Gamma$. Now set $\Gamma_1 = a\Gamma \cap B_+$ and define ρ_1 by $\rho_1(f) = \sup\left\{\int_0^\infty f^*g^*dt : g \in \Gamma_1\right\}$. Lemma 6.2 says that ρ_1 is universally rearrangement invariant. Because $B_\cap \subset \Gamma_1 \subset B_+$ we have $\|\cdot\|_+ \leqq \rho_1 \leqq \|\cdot\|_\cap$.

Now we will show that ρ_1 and ρ are equivalent. Notice that $a\rho(f) = \sup\left\{\int_0^\infty f^*g^*dt : g \in a\Gamma\right\}$. Hence $\rho_1 \leqq a\rho$ because $\Gamma_1 \subset a\Gamma$. Since $L_{\rho'} \subset L_1 + L_\infty$, there is a constant b_1 such that $1/b_1\|\cdot\|_+ \leqq \rho'$ (we may choose b_1, such that $b_1 > 1/a$). So $\Gamma \subset b_1B_+$ and thus $a\Gamma \subset ab_1B_+$. Let $b = ab_1$, then $b\Gamma_1 = b(a\Gamma \cap B_+) = ba\Gamma \cap bB_+$. Notice that $a\Gamma \subset b\Gamma_1$ which means that $(a/b)\Gamma \subset \Gamma_1$ or $(a/b)\rho \leqq \rho_1$. Hence ρ and ρ_1 are equivalent.

7. Universal and universally rearrangement invariant Köthe spaces. The concepts of the previous sections of this paper can be generalized to the general Köthe spaces.

DEFINITION 7.1. A Köthe space $\Lambda(\Gamma)$ is called *universal* if

$$\Lambda = \left\{ f \in M(\Delta, \mu) : \int_0^\infty f^* g^* dt < \infty \quad \text{for all} \quad g \in \Gamma \right\}.$$

Hence the functions in a universal Köthe space are characterized by the action of their monotonic rearrangements as was the case of a normed Köthe space induced by a universal function norm.

The following concept is due to Luxemburg [9].

DEFINITION 7.2. A Köthe space $\Lambda = \Lambda(\Gamma)$ defined on $M(\Delta, \mu)$ is said to be *universally rearrangement invariant* whenever $f \in \Lambda$ implies $T_\mu f_1 \in \Lambda$ for all $f_1 \in M(\Delta_1, \mu_1)$ satisfying $f_1 \sim f$.

Observe that if (Δ, μ) is nonatomic then Λ is universally rearrangement invariant if and only if Λ is rearrangement invariant.

LEMMA 7.3. *Let $\Lambda(\Gamma)$ be a Köthe space.*

(a) *Λ is universal if and only if Λ is universally rearrangement invariant.*

(b) *If Λ is universal, then Λ' is also universal.*

Proof. Assume $\Lambda(\Gamma)$ is universal. Let $f \in \Lambda$, $f_1 \in (\Delta_1)$, and $f_1 \sim f$. Then for any $g \in \Gamma$ we have $\int_\Delta T_\mu f_1 g d\mu = \int_\Delta f_1 g d\mu \leqq \int_0^\infty f^* g^* dt < \infty$. Therefore, Λ is universally rearrangement invariant.

Next assume that Λ is universally rearrangement invariant. Let $\Pi = \left\{ f : \int_0^\infty f^* g^* dt < \infty \text{ for all } g \in \Gamma \right\}$. Easily $\Pi \subset \Lambda$. Suppose $f \in \Lambda$ but $f \notin \Pi$. This means that $\int_0^\infty f^* g_0^* dt = \infty$ for some $g_0 \in \Gamma$. By Lemma 5.3 we know that there exists an $f_1 \in M(\Delta_1)$ such that $\int_{\Delta_1} f_1 g_0 d\mu_1 = \infty$ and $f_1 \sim f$. But $\int_\Delta T_\mu f_1 g_0 d\mu = \int_{\Delta_1} f_1 g_0 d\mu_1 = \infty$ which contradicts the fact that Λ is universally rearrangement invariant. Therefore, $\Pi = \Lambda$ and Λ is universal.

The next result is an extension of Theorem 6.3.

THEOREM 7.4. *If $\Lambda(\Gamma)$ is a universal Köthe space in $M(\Delta, \mu)$, then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.*

Proof. In $[0, \infty)$ let $I_n = [0, n)$ and let $\Omega([0, \infty))$ be the locally integrable functions in $M([0, \infty))$ with respect to $\{I_n\}_{n=1}^{\infty}$. Let $\Gamma^* = \{g^*: g \in \Gamma\}$ and $\Gamma_1 = \{h \in \Omega([0, \infty)): h^* \in \Gamma^*\}$. Form the Köthe space $\Lambda_1 = \Lambda(\Gamma_1)$ in $M([0, \infty))$. If $f \in \Lambda_1$ and $g \in \Gamma_1$, then $\int_0^\infty fg'dt < \infty$ for all $g' \sim g_1$. Hence $\int_0^\infty f^*g^*dt < \infty$ and therefore

$$\Lambda_1 = \left\{f \in \Omega([0, \infty)): \int_0^\infty f^*h^*dt < \infty \quad \text{for all} \quad h \in \Gamma_1\right\}$$

which means Λ_1 is rearrangement invariant. So $L_1([0, \infty)) \cap L_\infty([0, \infty)) \subset \Lambda_1 \subset L_1([0, \infty)) + L_\infty([0, \infty))$. This means that $(\Lambda^* \cup \Lambda'^*) \subset L_1([0, \infty)) + L_\infty([0, \infty))$. Hence by Corollary 4.4 $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.

Returning to normed Köthe spaces we are now able to prove

THEOREM 7.5. *If L_ρ is a universal Köthe space, then there is a norm ρ_1 such that ρ and ρ_1 are equivalent and ρ_1 is universally rearrangement invariant.*

Proof. Define ρ_1 by $\rho_1(f) = \sup\left\{\int_0^\infty f^*g^*dt: \rho'(g) \leqq 1\right\}$. Easily ρ_1 is universally rearrangement invariant. In order to show that ρ_1 and ρ are equivalent, we will show that $L_{\rho_1} = L_\rho$. It is easy to show that $L_{\rho_1} \subset L_\rho$. On the other hand, suppose $f \in L_\rho$ and $f \notin L_{\rho_1}$. There is a sequence of functions $\{g_n\} \subset L_{\rho'}$ such that $g_n \geqq 0$, $\rho'(g_n) \leqq 1$, and $\int_0^\infty f^*g^*dt > n^3$. Let $h_k = \sum_{n=1}^k g_n/n^2$ and $h = \sum_{n=1}^\infty g_n/n^2$. Then $\rho'(h) \leqq \liminf \sum_{n=1}^k 1/n^2\rho'(g_n) \leqq \pi^2/6$. Since all the g_n are nonnegative we know that $h_k \geqq g_k$ for each k, which means $\int_0^\infty f^*h^*dt \geqq \int_0^\infty f^*g_k^*dt > k^3$ for all $k = 1, 2, \cdots$. Therefore $\int_0^\infty f^*h^*dt = \infty$. But as before this contradicts the fact that L_ρ is universal. Therefore, $L_{\rho_1} = L_\rho$ and we have completed the proof.

Theorem 7.5 was also given by Luxemburg [9] for his restricted case.

Combining Theorem 7.4, Theorem 7.5, and Theorem 6.4(b) we have

THEOREM 7.6. *If Λ is a universal Köthe space, then*

$$L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty .$$

Furthermore, if Λ is normed, i.e., *$\Lambda = L_\rho$, then there exists an equivalent universally rearrangement invariant norm ρ_1 such that $\|\cdot\|_+ \leqq \rho_1 \leqq \|\cdot\|_\cap$.*

We conclude with an example that shows that $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$ does not necessarily imply that L_ρ is universal. Let (Δ, μ) be $(-\infty, \infty)$ with Lebesgue measure and let

$$\rho(f) = \|f\chi_{(-\infty,0)}\|_\infty + \|f\chi_{[0,\infty)}\|_1 .$$

Clearly $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$ but L_ρ is not universal.

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Received January 18, 1973 and in revised form November 28, 1973. This research was partially supported by N. S. F. Grant GP-30284. This paper constitutes the major portion of the author's doctoral dissertation written at Louisiana State University under the direction of Professor James R. Dorroh. The author wishes to express his sincere gratitude to Dr. Dorroh.

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Printed in Japan by Intarnational Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 52, No. 1 January, 1974