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It is known that the fibre homotopy type of a spherical fibre space over a sphere is determined by its characteristic class. Our purpose is to describe the homotopy type of the total space of a spherical fibre space over a sphere in terms of its characteristic class, and to classify homotopy types of them by defining a kind of equivalence between characteristic classes.

I. M. James and J. H. C. Whitehead classified homotopy types of the total space of sphere bundles over spheres in [2] and [3]. Our results are a generalization of their theorems and also an answer to one of problems proposed by J. D. Stascheff in [7]. Let \mathcal{G}_k be the space of maps of a k -sphere into itself with degree 1 and let \mathcal{F}_k be the subspace of \mathcal{G}_k consisting of maps preserving the base point $(0, \dots, 0, 1)$. We denote by $\mathcal{E}_{k,n}(\chi)$ the total space of an orientable k -spherical fibre space over an n -sphere with $\chi \in \pi_{n-1}(\mathcal{G}_k)$ as its characteristic class. First we shall treat with the case where fibrations have cross-sections. Then we may suppose $\chi = i_{k*}(\xi)$ where $i_k: \mathcal{F}_k \rightarrow \mathcal{G}_k$ denotes the inclusion map.

Now let

$$\lambda: \pi_{n-1}(\mathcal{F}_k) \longrightarrow \pi_{k+n-1}(\mathcal{S}^k)$$

be the isomorphism defined by B. Steer in [5]. We are concerned with $\lambda(\xi)$ but not χ .

Then if $i_{k*}(\xi) = i_{k*}(\xi')$ we claim

$$(1) \quad \lambda(\xi') = \lambda(\xi) + [x, \iota_k]$$

for some $x \in \pi_n(\mathcal{S}^k)$ where $[,]$ denotes Whitehead product.

For, let i be the inclusion $\mathcal{R}_{k+1} \rightarrow \mathcal{G}_k$ where \mathcal{R}_{k+1} is the rotation group of \mathcal{S}^k . Clearly i induces a fibre map of the fibration $\mathcal{R}_{k+1} \rightarrow \mathcal{S}^k$ into the fibration $\mathcal{G}_k \rightarrow \mathcal{S}^k$. Since the restriction of λ on the image of $\pi_{n-1}(\mathcal{R}_k)$ is equal to (up to sign) ([5]), the homomorphism \mathcal{J} which is defined by G. W. Whitehead in [6], λ maps $\partial\pi_n(\mathcal{S}^k)$ onto the group $[\pi_n(\mathcal{S}^k), \iota_k]$ by the formula $\mathcal{J}\partial(x) = -[x, \iota_k]$ where ∂ denotes the boundary homomorphism taken from the homotopy sequences of fibrations. Thus, since $\xi' - \xi$ is contained in the $\partial\pi_n(\mathcal{S}^k)$, we obtain (1).

Let Σ be the natural projection

$$\pi_{k+n-1}(\mathcal{S}^k) \longrightarrow \pi_{k+n-1}(\mathcal{S}^k)/[\pi_n(\mathcal{S}^k), \iota_k].$$

A map of \mathcal{S}^k into itself with degree -1 canonically induces an endmorphism of $\pi_{k+n-1}(\mathcal{S}^k)/[\pi_n(\mathcal{S}^k), \iota_k]$. We denote by $\tilde{\Sigma}$ the composition of Σ and the endmorphism. The set

$$\mathcal{M}(\mathcal{E}_{k,n}(\chi)) = (\pm \Sigma \lambda(\xi), \pm \tilde{\Sigma} \lambda(\xi))$$

is independent from the choice of ξ by (1). Then we shall prove

THEOREM 1. *If the fibration $\chi_i (i = 1, 2)$ has a cross-section $(n, k \geq 2)$, $\mathcal{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathcal{E}_{k,n}(\chi_2)$ if and only if*

- (1) *if $n \neq k$, or $n = k = \text{even}$ $\mathcal{M}(\mathcal{E}_{k,n}(\chi_1)) = \mathcal{M}(\mathcal{E}_{k,n}(\chi_2))$*
- (2) *if $n = k = \text{odd}$ $d \cdot \lambda(\xi_1) \equiv \lambda(\xi_2) \pmod{[\pi_n(\mathcal{S}^k), \iota_k]}$ for some integer d , $(d, m) = 1$, where m is the order of $\lambda(\xi_2) \pmod{[\pi_n(\mathcal{S}^k), \iota_k]}$.*

If $\mathcal{E}_{k,n}(\chi)$ has the same homotopy type as $\mathcal{S}^k \times \mathcal{S}^n$ the fibration has a cross-section. Hence we have

COROLLARY 1.1. *$\mathcal{E}_{k,n}(\chi)$ has the same homotopy type as $\mathcal{S}^k \times \mathcal{S}^n$ if and only if the fibration χ is fibre homotopically trivial.*

Secondly we consider fibrations which do not necessarily have cross-sections. Therefore, we are mainly concerned in the case $n > k$. However, the case $n = k + 1$ is different from others, so we suppose $n \geq k + 2 \geq 4$.

Let $\bar{\rho}: \mathcal{S}^k \rightarrow \mathcal{S}^k$ be the homeomorphism defined by

$$\bar{\rho}(x_1, x_2, \dots, x_{k+1}) = (-x_1, x_2, \dots, x_{k+1}),$$

and let $\rho: \mathcal{E}_k \rightarrow \mathcal{E}_k$ be the homeomorphism induced by $\bar{\rho}(\rho(f)) = \bar{\rho} f \bar{\rho}$. For any $\alpha \in \pi_{n-1}(\mathcal{S}^k)$, from the diagram

$$\pi_{k+n-1}(\mathcal{S}^{n-1}) \xrightarrow{\alpha_*} \pi_{k+n-1}(\mathcal{S}^k) \xleftarrow{\lambda} \pi_{n-1}(\mathcal{S}^k) \xrightarrow{i_{k*}} \pi_{n-1}(\mathcal{E}_k),$$

we have the subgroup of $\pi_{k+n-1}(\mathcal{E}_k)$ defined by

$$\mathcal{E}(\alpha) = i_{k*} \lambda^{-1} \cdot \alpha_* \pi_{k+n-1}(\mathcal{S}^{n-1}).$$

Then we claim

$$(2) \quad \mathcal{E}(\alpha) = \mathcal{E}(-\alpha) \text{ and } \rho_*(\mathcal{E}(\alpha)) = \mathcal{E}((-\iota_k)_* \alpha).$$

For, the former is clear and the latter follows from the following commutative diagram (see Lemma 2.2)

$$(3) \quad \begin{array}{ccccc} \pi_{k+n-1}(\mathcal{S}^k) & \xleftarrow{-\lambda} & \pi_{n-1}(\mathcal{S}^k) & \xrightarrow{i_{k*}} & \pi_{n-1}(\mathcal{E}_k) \\ (-\iota_k)_* \downarrow & & \downarrow \rho'_* & & \downarrow \rho_* \\ \pi_{k+n-1}(\mathcal{S}^k) & \xleftarrow{\lambda} & \pi_{n-1}(\mathcal{S}^k) & \xrightarrow{i_{k*}} & \pi_{n-1}(\mathcal{E}_k) \end{array}$$

where $\rho | \mathcal{F}_k = i_k \cdot \rho'$ is the natural factorization.

Now let $\mathcal{S}[\chi]$ ($\chi \in \pi_{n-1}(\mathcal{E}_k)$) be the set of elements

$$\{\chi, -\chi, \rho_*\chi, -\rho_*\chi\}$$

and let $\mathcal{P}_k: \mathcal{E}_k \rightarrow \mathcal{S}^k$ be the projection of the canonical fibration. We define a relation in $\pi_{n-1}(\mathcal{E}_k)$ as follows $\chi_1 \sim \chi_2$ if and only if $\theta_1 \equiv \theta_2 \pmod{\mathcal{S}(\mathcal{P}_k^*(\theta_1))}$ for some pair (θ_1, θ_2) , $\theta_i \in \mathcal{S}[\chi_i]$.

It can be easily checked by (2) that this is an equivalence relation.

THEOREM 2. *If $n \geq k + 2 \geq 4$, then $\mathcal{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathcal{E}_{k,n}(\chi_2)$ if and only if $\chi_1 \sim \chi_2$.*

If fibrations have cross-sections this is an alternative version of Theorem 1. For, since $\mathcal{P}_k^*(\chi_i) = 0$ we have $\chi_i = i_{k^*}(\xi_i)$. Then the condition $\chi_1 \sim \chi_2$ means that $\chi_1 = \pm\chi_2$ or $\chi_1 = \pm\rho_*\chi_2$, i.e.,

$$i_k^*(\xi_1) = \pm i_{k^*}(\xi_2) \quad \text{or} \quad i_{k^*}(\xi_1) = \pm i_{k^*}(\rho'_*)(\xi_2).$$

These are satisfied if and only if $\xi_1 = \pm\xi_2 + \partial\sigma$ or $\xi_1 = \pm\rho'_*\xi_2 + \partial\sigma$ where $\sigma \in \pi_n(\mathcal{S}^k)$. Now apply λ to the both side, then we have that

$$\lambda(\xi_1) \equiv \pm\lambda(\xi_2) \quad \text{or} \quad \pm(-\iota_k)_*\lambda(\xi_2) \pmod{[\pi_n(\mathcal{S}^k), \iota_k]}.$$

This is so if and only if $\mathcal{M}(\mathcal{E}_{k,n}(\chi_1)) = \mathcal{M}(\mathcal{E}_{k,n}(\chi_2))$.

From Theorem 2 the following is easily deduced.

COROLLARY 2.1. *Suppose that $\mathcal{J}\pi_{n-1}(\mathcal{P}_k) \supset \mathcal{P}_k(\chi)\pi_{k+n-1}(\mathcal{S}^{n-1})$. If $\mathcal{E}_{k,n}(\chi)$ ($n \geq k + 2 \geq 4$) has the same homotopy type as the total space of an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n , then the fibration itself is fibre homotopically equivalent to an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n .*

As special cases we have

COROLLARY 2.2. *Suppose that the fibration χ has a cross-section. If $\mathcal{E}_{k,n}(\chi)$ ($n \geq k + 2 \geq 4$) has the homotopy type of the total space of an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n , the fibration is fibre homotopically equivalent to an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n .*

COROLLARY 2.3. *A k -spherical fibring over \mathcal{S}^n is stable fibre homotopically equivalent to an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n if and only if the total space of the fibring has the same homotopy \mathcal{S} -type as the total space of an orthogonal \mathcal{S}^k -bundle over \mathcal{S}^n .*

2. $\mathcal{E}_{k,n}(\chi)$ as a CW-complex. Let $f: (\mathcal{S}^{n-1}, *) \rightarrow (\mathcal{E}_k, 1)$ be a representative of χ and let $\tilde{f}: \mathcal{S}^{n-1} \times \mathcal{S}^k \rightarrow \mathcal{S}^k$ be the adjoint map. We denote by $\mathcal{K}(f)$ the complex $\mathcal{S}^k \cup \mathcal{D}^n \times \mathcal{S}^k$ obtained from identifying (x, y) with $\tilde{f}(x, y)$ for $(x, y) \in \mathcal{S}^{n-1} \times \mathcal{S}^k$.

Then it is known that $\mathcal{E}_{k,n}(\chi)$ has the same homotopy type as $\mathcal{K}(f)$ (Prop. 1 of [4]). It may be considered that $\mathcal{K}(f)$ is given the natural CW-decomposition $\mathcal{S}^k \cup e^n \cup e^{k+n}$ in which attaching maps for cells are as follows

$$(4) \quad \begin{array}{l} \alpha: \mathcal{S}^{n-1} \longrightarrow \mathcal{S}^k, \alpha(x) = f(x, *) \\ \beta: \mathcal{S}^{k+n-1} = \mathcal{D}^n \times \mathcal{S}^{k-1} \cup \mathcal{S}^{n-1} \times \mathcal{D}^k \\ \longrightarrow \mathcal{D}^n \times * \cup \mathcal{S}^{n-1} \times \mathcal{S}^k \xrightarrow{\bar{\alpha} \cup \tilde{f}} \mathcal{S}^k \cup e^n \end{array}$$

where $\bar{\alpha}: (\mathcal{D}^n, \mathcal{S}^{n-1}) \rightarrow (\mathcal{S}^k \cup e^n, \mathcal{S}^k)$ denotes the characteristic map for e^n ($\alpha = \partial \bar{\alpha}$).

Let j be the inclusion: $(\mathcal{S}^k \cup e^n, *) \rightarrow (\mathcal{S}^k \cup e^n, \mathcal{S}^k)$. Then we have

LEMMA 2.1. $\mathcal{P}_k(\chi) = \alpha$, and $j_*(\beta) = \pm[\bar{\alpha}, \iota_k]_r$ if $n > k + 1$ or $\alpha = 0$. Thus we can define the orientation of $\mathcal{K}(f)$ by $j_*(\beta) = [\bar{\alpha}, \iota_k]_r$.

Proof. The former follows from (4) and the definition of \mathcal{P}_k . Since the group $\pi_{k+n-1}(\mathcal{S}^k \cup e^n, \mathcal{S}^k)$ is isomorphic to the direct sum

$$\mathcal{K}[\bar{\alpha}, \iota_k]_r + \bar{\alpha}\pi_{k+n-1}(\mathcal{D}^n, \mathcal{S}^{n-1})$$

under the assumption, $j_*(\beta)$ is of the form

$$m[\bar{\alpha}, \iota_k]_r + \bar{\alpha}x$$

for some integer m and $x \in \pi_{k+n-1}(\mathcal{D}^n, \mathcal{S}^{n-1})$. Let \mathcal{H}_i ($i = k, n, k+n$) be generators of $\mathcal{H}^i(\mathcal{K}(f)) = \mathcal{K}$. Then, by the theorem in [1],

$$\mathcal{H}_k \cup \mathcal{H}_n = \pm m \mathcal{H}_{k+n}.$$

On the other hand, since $\mathcal{K}(f)$ has the homotopy type of $\mathcal{E}_{k,n}(\chi)$ we have

$$\mathcal{H}_k \cup \mathcal{H}_n = \pm \mathcal{H}_{k+n},$$

i.e., $m = \pm 1$. And moreover $\bar{\alpha}x = 0$ follows from the existence of the projection of the fibration.

Now we consider the special case where $0 = \alpha = \mathcal{P}_k(\chi)$. Then the map f may be considered as a map: $(\mathcal{S}^{n-1}, *) \rightarrow (\mathcal{F}_k, 1)$. Since $\tilde{f}|_{\mathcal{S}^{n-1} \times *} = *$, \mathcal{S}^n is naturally imbedded as the image of $\mathcal{D}^n \times *$. In this situation, after identifying $\pi_{k+n-1}(\mathcal{S}^k \vee \mathcal{S}^n)$ with $\pi_{k+n-1}(\mathcal{S}^k) +$

$\pi_{k+n-1}(\mathcal{S}^k \vee \mathcal{S}^n, \mathcal{S}^k)$, it follows from Lemma 2.1 that

$$(5) \quad \beta = \iota_{k*}(x) + [\iota_k, \iota_n] .$$

And also β may be considered as follows

$$(6) \quad \begin{array}{ccc} \mathcal{S}^{k+n-1} = \mathcal{D}^n \times \mathcal{S}^{k-1} \cup \mathcal{S}^{n-1} \times \mathcal{D}^k & \xrightarrow{1 \times \varphi_k} & \mathcal{D}^n \\ \times * \cup \mathcal{S}^{n-1} \times \mathcal{S}^k & \xrightarrow{\varphi_n \times * \cup * \times \tilde{f}} & \mathcal{S}^n \times * \cup * \times \mathcal{S}^k \end{array}$$

where φ_k denotes the identification map: $\mathcal{D}^k \rightarrow \mathcal{S}^k / \varphi^{k-1}$.

We make use of λ to determine x , so we recall the definition of λ . Let ε be the map: $\mathcal{S}^p \rightarrow \mathcal{F}_k$ defined by $\varepsilon(\) =$ the identity of \mathcal{S}^k and let h be a map: $(\mathcal{S}^p, *) \rightarrow (\mathcal{F}_k, 1)$. Since adjoint maps $\tilde{h}, \tilde{\varepsilon}: \mathcal{S}^p \times \mathcal{S}^k \rightarrow \mathcal{S}^k$ has the same restriction on $\mathcal{S}^p \vee \mathcal{S}^k$, the separation element $d(\tilde{h}, \tilde{\varepsilon}) \in \pi_{p+k}(\mathcal{S}^k)$ is defined. B. Steer defined $\lambda(h)$ by $d(\tilde{h}, \tilde{\varepsilon})$. For example we have (see the diagram (3))

$$\text{LEMMA 2.2.} \quad -\lambda \rho'_*(\tilde{\xi}) = (-\iota_k)_* \lambda(\tilde{\xi}) (\tilde{\xi} \in \pi_p(\mathcal{F}_k)) .$$

Proof. Let g be a representative of $\tilde{\xi}$. Then we have

$$\begin{aligned} (-\iota_k)_* \lambda(\tilde{\xi}) &= (-\iota_k)_* d(\tilde{g}, \tilde{\varepsilon}) = \bar{\rho}_* d(\tilde{g}, \tilde{\varepsilon}) = d(\bar{\rho}\tilde{g}, \bar{\rho}\tilde{\varepsilon}) \\ &= d(\bar{\rho}\tilde{g}, \tilde{\varepsilon}(id \times \bar{\rho})) = d(\bar{\rho}\tilde{g}(id \times \bar{\rho})(id \times \bar{\rho}), \tilde{\varepsilon}(id \times \bar{\rho})) \\ &= -d(\bar{\rho}\tilde{g}(id \times \bar{\rho}), \tilde{\varepsilon}) . \end{aligned}$$

Since $\tilde{\rho}' \cdot g(x, y) = \bar{\rho}(\tilde{g}(x, \bar{\rho}(y))) = \bar{\rho}\tilde{g}(id \times \bar{\rho})(x, y)$ we have $\tilde{\rho}'g = \bar{\rho}\tilde{g}(id \times \bar{\rho})$. Hence $d(\bar{\rho}\tilde{g}(id \times \bar{\rho}), \tilde{\varepsilon}) = d(\tilde{\rho}'g, \tilde{\varepsilon}) = \lambda(\rho'_*(\tilde{\xi}))$.

LEMMA 2.3. *In the expression in (4) we have $x = \lambda(\tilde{\xi})$, up to sign, where $\tilde{\xi}$ denotes the homotopy class of f .*

For the proof of Lemma 2.3 we prepare the following general

LEMMA 2.4. *Let \mathcal{L} be a 1-connected CW-complex and let \mathcal{K} be a complex $\mathcal{L} \cup e^N (\alpha \sim 0)$. Let f, g be maps: $\mathcal{K} \rightarrow \mathcal{L}$ such that $f|_{\mathcal{L}} = g|_{\mathcal{L}}$ and let ζ be a map: $\mathcal{S}^N \rightarrow \mathcal{K}$ which induces the isomorphism: $\mathcal{H}_N(\mathcal{S}^N, *) \rightarrow \mathcal{H}_N(\mathcal{K}, \mathcal{L})$. Then we have $d(f, g) = f_*(\zeta) - g_*(\zeta)$ (up to sign).*

Proof. Since $\alpha \sim 0$ there exists a homotopy equivalence $\varphi: (\mathcal{L} \vee \mathcal{S}^N, \mathcal{L}) \rightarrow (\mathcal{K}, \mathcal{L})$ relative to \mathcal{L} . Let δ be the inclusion $\mathcal{S}^N \rightarrow \mathcal{L} \vee \mathcal{S}^N$. Then

$$d(f, g) = \pm d(f\varphi, g\varphi) = \pm ((f\varphi)_* \delta - (g\varphi)_* \delta) .$$

From $\varphi^{-1}\zeta \in \pi_N(\mathcal{L} \vee \mathcal{S}^N)$ and the assumption on ζ we have

$$\varphi_*^{-1}\zeta = \pm\delta + \eta(\eta \in \pi^N(\mathcal{L})), \text{ i.e., } \zeta = \pm\varphi_*(\delta) + \varphi_*(\eta).$$

Hence

$$\begin{aligned} f_*(\zeta) - g_*(\zeta) &= f_*(\pm\varphi_*(\delta) + \varphi_*(\eta)) - g_*(\pm\varphi_*(\delta) + \varphi_*(\eta)) \\ &= \pm(f_*\varphi_*(\delta) - g_*\varphi_*(\delta)) = \pm d(f, g). \end{aligned}$$

Proof of Lemma 2.3. Let \mathcal{Q} be the identification map:

$$\mathcal{S}^{n-1} \times \mathcal{S}^k \rightarrow \mathcal{S}^{n-1} \times \mathcal{S}^k / \mathcal{S}^{n-1} \times *.$$

The maps

$$\tilde{f}\mathcal{Q}^{-1}, \tilde{\varepsilon}\mathcal{Q}^{-1}: \mathcal{S}^{n-1} \times \mathcal{S}^k / \mathcal{S}^{n-1} \times * \rightarrow \mathcal{S}^k$$

are well-defined and has the same restriction on $* \times \mathcal{S}^k / \mathcal{S}^{n-1} \times *$. The complex $\mathcal{S}^{n-1} \times \mathcal{S}^k / \mathcal{S}^{n-1} \times *$ has a form $\mathcal{S}^k \cup e^{k+n-1}(\alpha \sim 0)$. Then we apply Lemma 2.4 to the case where

$$\begin{aligned} \mathcal{H} &= \mathcal{S}^{n-1} \times \mathcal{S}^k / \mathcal{S}^{n-1} \times *, \quad \mathcal{L} = * \times \mathcal{S}^k / \mathcal{S}^{n-1} \times *, \\ N &= n + k - 1, \quad f = \tilde{f}\mathcal{Q}^{-1}, \quad g = \tilde{\varepsilon}\mathcal{Q}^{-1} \text{ and } \mathcal{R} = \mathcal{S}^k. \end{aligned}$$

Thus we have

$$\lambda(f) = d(\tilde{f}, \tilde{\varepsilon}) = d(\tilde{f}\mathcal{Q}^{-1}, \tilde{\varepsilon}\mathcal{Q}^{-1}) = \pm((f\mathcal{Q}^{-1})_*(\zeta) - (\tilde{\varepsilon}\mathcal{Q}^{-1})_*(\zeta))$$

for any $\zeta: (\mathcal{S}^{k+n-1}, *) \rightarrow (\mathcal{H}, \mathcal{L})$ which induces an isomorphism

$$\zeta_*: \mathcal{H}_{k+n-1}(\mathcal{S}^{k+n-1}, *) \rightarrow \mathcal{H}_{k+n-1}(\mathcal{H}, \mathcal{L}).$$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}^{k+n-1} = \mathcal{D}^n \times \mathcal{S}^{k-1} \cup \mathcal{S}^{n-1} \times \mathcal{D}^n & \rightarrow & \mathcal{D}^n \\ \times * \cup \mathcal{S}^{n-1} \times \mathcal{S}^k & \xrightarrow[* \cup \mathcal{Q}]{} & \mathcal{S}^{n-1} \times \mathcal{S}^k / \mathcal{S}^{n-1} \times * \\ \downarrow \varphi_n \cup \tilde{f} & & \downarrow \tilde{f}\mathcal{Q}^{-1} \\ \mathcal{S}^n \vee \mathcal{S}^k & \longrightarrow & \mathcal{S}^k. \end{array}$$

Since we can take ζ with the composition of two maps in the upper row it follows from $(\tilde{\varepsilon}\mathcal{Q}^{-1})_*(\zeta) = 0$ that $\lambda(f) = \pm(\tilde{f}\mathcal{Q}^{-1})_*(\zeta)$. From the diagram (6) the proof is completed.

3. Proof of Theorem 1. Let \mathcal{H} be a complex of the form

$$\mathcal{S}^k \vee \mathcal{S}^n \cup e^{k+n}$$

where $\beta = \iota_k\alpha + [\iota_k, \iota_n]$ under the decomposition

$$\pi_{k+n-1}(\mathcal{S}^k \vee \mathcal{S}^n) = \pi_{k+n-1}(\mathcal{S}^k) + \pi_{k+n-1}(\mathcal{S}^n) + \mathcal{R}[\iota_k, \iota_n].$$

By the cellular homotopy theorem \mathcal{K}_1 has the same homotopy type as \mathcal{K}_2 if and only if there exists a homotopy equivalence ($n, k \geq 2$)

$$\Phi: \mathcal{S}^k \vee \mathcal{S}^n \rightarrow \mathcal{S}^k \vee \mathcal{S}^n$$

such that $\Phi_*(\beta_1) = \pm\beta_2$. Now consider the case $n \neq k$. It is obvious that a map Φ is homotopy equivalence if and only if $\Phi|_{\mathcal{S}^k} = \pm\iota_k + \iota_n \circ \tau (\tau \in \pi_k(\mathcal{S}^n))$, and $\Phi|_{\mathcal{S}^n} = \pm\iota_n$ if $n < k = \pm\iota_k$, and $\Phi|_{\mathcal{S}^n} = \iota_k \circ \sigma + \pm\iota_n (\sigma \in \pi_n(\mathcal{S}^k))$ if $n > k$. From easy computation of $\Phi_*(\beta_1)$ we can obtain

LEMMA 3.1. *If $n \neq k$, \mathcal{K}_1 has the same homotopy type as \mathcal{K}_2 if and only if the set $\{\pm\alpha_1, \pm(-\iota_k)_*\alpha_1\}$ is equal to the set*

$$\{\pm\alpha_2, \pm(\iota_k)_*\alpha_2\} \bmod [\pi_n(\mathcal{S}^k), \iota_k].$$

Next we consider the case $n = k$. By the same way as in [2] we have

LEMMA 3.2. (James and Whitehead). *If $n = k = \text{even}$, \mathcal{K}_1 and \mathcal{K}_2 have the same homotopy type if and only if*

$$\{\pm\alpha_1\} \equiv \{\alpha_2\} \bmod [\pi_n(\mathcal{S}^k), \iota_k].$$

LEMMA 3.3. (James and Whitehead). *If $n = k = \text{odd}$, \mathcal{K}_1 and \mathcal{K}_2 have the same homotopy type if and only if there exists an integer d which is prime to m_2 and $d\alpha_1 \equiv \alpha_2 \bmod [\pi_n(\mathcal{S}^k), \iota_k]$ where m_2 is the order of $\alpha_2 \bmod [\pi_n(\mathcal{S}^k), \iota_k]$.*

Thus Theorem 1 follows from Lemmas 3.1, 3.2, 3.3, and 2.3.

4. Some Lemmas. Let \mathcal{L} be a complex of the form $\mathcal{S}^k \cup e^n$ with the characteristic map $\bar{\alpha}: (\mathcal{D}^n, \mathcal{S}^{n-1}) \rightarrow (\mathcal{L}, \mathcal{S}^k)$ for the n -cell. Let \mathcal{L} be the complex obtained from identifying \mathcal{S}^k of two copies of \mathcal{L} , i.e., $\bar{\mathcal{L}} = e^n \cup \mathcal{L} \cup \mathcal{L} \cup e^n$. It may be considered that two maps $\mu_i (i = 1, 2): \mathcal{L} \rightarrow \bar{\mathcal{L}}$ and a map $\nu: \bar{\mathcal{L}} \rightarrow \mathcal{L}$ are naturally defined and satisfy $\nu\mu_i = \text{the identity}$. Since $\mu_1|_{\mathcal{S}^k} = \mu_2|_{\mathcal{S}^k}$ the separation element $d(\mu_1, \mu_2)$ is defined. Then we have

LEMMA 4.1. *If $\beta \in \pi_{k+n-1}(\mathcal{L})$ and $j_*(\beta) = m[\bar{\alpha}, \iota_k]_*$, then $\mu_{1*}(\beta) - \mu_{2*}(\beta) = m[d(\mu_1, \mu_2), \iota_k]$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
 \pi_*(\mathcal{S}^k) & \xrightarrow{\quad} & \pi_*(\overline{\mathcal{L}}) & \xrightarrow{\quad} & \pi_*(\overline{\mathcal{L}}, \mathcal{S}^k) \\
 \uparrow id & & \uparrow \uparrow \uparrow & & \uparrow \mu_{1*} \downarrow \quad \nu_* \uparrow \mu_{2*} \\
 \pi_*(\mathcal{S}^k) & \xrightarrow{i_*} & \pi_*(\mathcal{L}) & \xrightarrow{j_*} & \pi_*(\mathcal{L}, \mathcal{S}^k)
 \end{array}$$

which is taken from the homotopy sequence of the pair and $* = k + n - 1$.

From the commutativity it follows that

$$j_+(\mu_{1*}(\beta) - \mu_{2*}(\beta)) = m[\mu_{1*}\bar{\alpha} - \mu_{2*}\bar{\alpha}, \iota_k]_r .$$

On the other hand, we have

$$j_+[d(\mu_1, \mu_2), \iota_k] = [j_+d(\mu_1, \mu_2), \iota_k]_r = [\mu_{1*}\bar{\alpha} - \mu_{2*}\bar{\alpha}, \iota_k]_r .$$

Thus, for some element $\gamma \in \pi_*(\mathcal{S}^k)$, it holds

$$m[d(\mu_1, \mu_2), \iota_k] = \mu_{1*}(\beta) - \mu_{2*}(\beta) + i_+(\gamma) .$$

Applying ν_* to the both side, then, from

$$\nu_*d(\mu_1, \mu_2) = d(\nu\mu_1, \nu\mu_2) = d(id, id) = 0 \text{ and } \nu\mu_i(\beta) = \beta ,$$

we have $\nu_*i_+(\gamma) = 0$. Hence $i_+(\gamma) = 0$ from the commutativity of the diagram.

As an application of Lemma 4.1 we have

LEMMA 4.2. *Let f, g be maps: $\mathcal{L} \rightarrow \mathcal{L}$ such that $f|_{\mathcal{L}} = g|_{\mathcal{L}}$. For any $\beta, j_*(\beta) = m[\bar{\alpha}, \iota_k]_r$, we have*

$$f_*(\beta) - g_*(\beta) = m[d(f, g), f|_{\mathcal{S}^k}] .$$

Proof. Define a map $f \cup g: \overline{\mathcal{L}} \rightarrow \mathcal{L}$ by

$$(f \cup g)\mu_1 = f, \text{ and } (f \cup g)\mu_2 = g .$$

Since $d(f, g) = d((f \cup g)\mu_1, (f \cup g)\mu_2) = (f \cup g)_*d(\mu_1, \mu_2)$ the proof is completed by applying $(f \cup g)_*$ to the both side of the equality in Lemma 4.1.

Let id be the identity map of $\mathcal{L} (n \geq k + 2 \geq 4)$ and let $w: \mathcal{L} \rightarrow \mathcal{L}$ be a map with $w|_{\mathcal{S}^k} = id|_{\mathcal{S}^k}$. In general, $d(id, w)$ is belonging to $\pi_n(\mathcal{L})$. However, we have

LEMMA 4.3. *w is a homotopy equivalence preserving the orientation of the n -cell if and only if $d(id, w)$ is contained in $i_*\pi_n(\mathcal{S}^k)$.*

Proof. Let x_n, y_n be the orientation generators of $\mathcal{H}_n(\mathcal{L})$, and $\mathcal{H}_n(\mathcal{S}^n)$ respectively, and let δ be $d(id, w)$. Since $x_n - w_*(x_n) = \delta_*(y_n)$, $x_n = w_*(x_n)$ holds if and only if $\delta_*(y_n) = 0$. On the other hand, the diagram

$$\pi_n(\mathcal{S}^k) \xrightarrow{i_*} \pi_n(\mathcal{L}) \longrightarrow \pi_n(\mathcal{L}, \mathcal{S}^k) = \mathcal{H}_n(\mathcal{L}, \mathcal{S}^k) = \mathcal{H}_n(\mathcal{L})$$

shows that $\delta_*(x_n) = 0$ is equivalent to $\delta \in i_*\pi_n(\mathcal{S}^k)$.

Now we prepare lemmas for the proof of Theorem 2. In what follows, we use the notations in § 2 and suppose $n \geq k + 2 \geq 4$.

LEMMA 4.4. *Let i be the inclusion: $\mathcal{S}^k \rightarrow \mathcal{S}^k \cup e^n \subset \mathcal{H}(f)$. Then we have*

$$i_*^{-1}(0) = \alpha_*\pi_{k+n-1}(\mathcal{S}^{n-1}).$$

Proof. Since the pair $(\mathcal{H}(f), \mathcal{S}^k)$ is homotopy equivalent to $(\mathcal{E}_{k,n}(\chi), \mathcal{S}^k)$

$$\pi_{k+n}(\mathcal{H}(f), \mathcal{S}^k) = \pi_{k+n}(\mathcal{S}^n) = E\pi_{k+n-1}(\mathcal{S}^{n-1}).$$

Hence from the homotopy sequence of the triple $(\mathcal{H}(f), \mathcal{S}^k \cup e^n, \mathcal{S}^k)$ we obtain

$$\pi_{k+n}(\mathcal{S}^k \cup e^n, \mathcal{S}^k) = \partial\pi_{k+n+1}(\mathcal{H}(f), \mathcal{S}^k \cup e^n) \cup \bar{\alpha}_*\pi_{k+n}(\mathcal{D}^n, \mathcal{S}^{n-1}).$$

Thus we have that

$$i_*^{-1}(0) = \partial\pi_{k+n}(\mathcal{S}^k \cup e^n, \mathcal{S}^k) = \alpha_*\pi_{k+n-1}(\mathcal{S}^{n-1}).$$

Let $\chi_i (i = 1, 2)$ be elements such that $\mathcal{P}_{k^*}(\chi_1) = \mathcal{P}_{k^*}(\chi_2) = \alpha$. Then $\beta_i \in \pi_{k+n-1}(\mathcal{S}^k \cup e^n)$ and there exists an element $\xi \in \pi_{n-1}(\mathcal{F}_k)$ which satisfies $i_{k^*}(\xi) = \chi_1 - \chi_2$.

LEMMA 4.5. *There exists a homotopy equivalence $\varphi: \mathcal{S}^k \cup e^n \rightarrow \mathcal{S}^k \cup e^n$ which satisfies*

- (1) $\varphi_*(e^k) = e^k, \varphi_*(e^n) = e^n$
- (2) $\beta_1 - \varphi_*(\beta_2) = i_*\lambda(\xi)$ (up to sign).

Proof. Let $\kappa: \mathcal{S}^n \rightarrow \mathcal{S}_1^n \vee \mathcal{S}_2^n$ be a map of type $(1, -1)$ and let χ be the fibration induced from $\chi_1 \vee \chi_2$ by κ , i.e., $\chi = \chi_1 - \chi_2$. Since $i_{k^*}(\xi) = \chi\mathcal{H}(f)$ has the form $\mathcal{S}^k \vee \mathcal{S}^n \cup e^{k+n}$ by (5). It may be considered that κ induces a map $\bar{\kappa}$:

$$\begin{aligned} \mathcal{H}(f) &= \mathcal{S}^k \vee \mathcal{S}^n \cup e^{k+n} \longrightarrow \mathcal{H}(f_1) \cup \mathcal{H}(f_2) \\ &= \mathcal{D}_1^n \times \mathcal{S}^k \cup \mathcal{S}^k \cup \mathcal{D}_2^n \times \mathcal{S}^k \end{aligned}$$

which satisfies

$$\bar{\kappa}_*(e^{k+n}) = e_1^{k+n} - e_2^{k+n}, \quad \bar{\kappa}_*(e^n) = e_1^n - e_2^n \text{ and } \bar{\kappa}_*(e^k) = e^k.$$

Let $\bar{\kappa}: \mathcal{S}^k \vee \mathcal{S}^n \rightarrow e_1^n \cup \mathcal{S}^k \cup e_2^n$ be the map obtained from the restriction of $\bar{\kappa}$ on $\mathcal{S}^k \vee \mathcal{S}^n$ and let i_j be the inclusion: $e_j^n \vee \mathcal{S}^k \rightarrow e_1^n \cup \mathcal{S}^k \cup e_2^n$. Then we have

$$(*) \quad \bar{\kappa}_*(\beta) = i_{1*}(\beta_1) - i_{2*}(\beta_2).$$

Define the map $r: e_1^n \cup \mathcal{S}^k \cup e_2^n \rightarrow \mathcal{S}^k \cup e^n$ by

$$r|_{e_1^n \cup \mathcal{S}^k} = \text{identity} = r|_{\mathcal{S}^k \cup e_2^n}.$$

We claim that

(**) $r_*(\omega)$ is contained in \tilde{i}_* -image where $\omega = \bar{\kappa}|_{\mathcal{S}^n}$ and \tilde{i} denotes the inclusion: $\mathcal{S}^k \rightarrow e_1^n \cup \mathcal{S}^k \cup e_2^n$.

For, consider the commutative diagram

$$\begin{array}{ccccc} \pi_n(\mathcal{S}^k \vee \mathcal{S}^n, \mathcal{S}^k) & \xrightarrow{\bar{\kappa}_*} & \pi_n(e_1^n \cup \mathcal{S}^k \cup e_2^n, \mathcal{S}^k) & \xrightarrow{r_*} & \pi_n(\mathcal{S}^k \vee e^n, \mathcal{S}^k) \\ \uparrow j_{1*} & & \uparrow j_{2*} & & \uparrow j_{3*} \\ \pi_n(\mathcal{S}^k \vee \mathcal{S}^n) & \xrightarrow{\bar{\kappa}_*} & \pi_n(e_1^n \cup \mathcal{S}^k \cup e_2^n) & \xrightarrow{r_*} & \pi_n(\mathcal{S}^k \cup e^n). \end{array}$$

Let z_n be the element of $\pi_n(\mathcal{S}^k \vee \mathcal{S}^n)$ which is represented by \mathcal{S}^n . Then we have

$$\begin{aligned} j_{3*}r_*(\omega) &= j_{3*}r_*\bar{\kappa}(z_n) = r_*j_{2*}(\bar{\kappa}(z_n)) = r_*\bar{\kappa}_*j_{1*}(z_n) \\ &= r_*(i_{1*}(\bar{\alpha}_1) - i_{2*}(\bar{\alpha}_2)) = \bar{\alpha} - \bar{\alpha} = 0. \end{aligned}$$

Thus (**) is proved.

Now, by applying r_* to the both side of (*) we have

$$r_*\bar{\kappa}_*(\beta) = \beta_1 - \beta_2.$$

On the other hand, by using (5), we have

$$\begin{aligned} r_*\bar{\kappa}_*(\beta) &= r_*\bar{\kappa}_*(\iota_k(\pm\lambda(\xi)) + [\iota_k, z_n]), (\iota_n = z_n) \\ &= i_*(\pm\lambda(\xi)) + [\iota_k, r_*(\omega)] \\ &= i_*(\pm\lambda(\xi) + [\iota_k, \omega']), (\omega' \in \pi_n(\mathcal{S}^k), \tilde{i}_*\omega' = r_*(\omega) \text{ by (**)}) \\ &= i_*(\pm\lambda(\xi) \pm [\omega', \iota_k]) \end{aligned}$$

$$\text{i.e., } \beta_1 - \beta_2 = i_*(\pm\lambda(\xi) \pm [\omega', \iota_k]).$$

If we take a map $\varphi: \mathcal{S}^k \cup e^n \rightarrow \mathcal{S}^k \cup e^n$ such that $d(id, \varphi) = \mp\omega'$, it follows from Lemma 4.2 and Lemma 2.1 that

$$\beta_2 - \varphi_*(\beta_2) = i_*(\mp[\omega', \iota_k]) \quad \text{i.e., } \beta_1 - \varphi_*(\beta_2) = i_*(\pm\lambda(\xi)).$$

Since $d(id, \varphi) \in i_*\pi_n(\mathcal{S}^k)$ φ satisfies (1) by Lemma 4.3.

LEMMA 4.6. *There exist homotopy equivalences $u': \mathcal{H}(-f) \rightarrow \mathcal{H}(f)$ and $u'': \mathcal{H}(\rho\rho) \rightarrow \mathcal{H}(f)$ which satisfy*

- (1) $u'_*(e^k) = e^k$ and $u'_*(e^n) = -e^n$,
- (2) $u''_*(e^k) = -e^k$ and $u''_*(e^n) = e^n$.

Proof. Let \mathcal{U} be the identification map: $\mathcal{S}^k + \mathcal{D}^n \times \mathcal{S}^k \rightarrow \mathcal{H}(f)$ and define u', u'' as follows

$$\rho_n(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n), ((x_1, x_2, \dots, x_n) \in \mathcal{D}^n)$$

$$u'(x) = x, \quad u''(x) = \bar{\rho}x \text{ if } x \in \mathcal{S}^k \text{ and}$$

$$u'(y, z) = \mathcal{U}(\rho_n y, z), \quad u''(y, z) = \mathcal{U}(y, \bar{\rho}z) \text{ if } (y, z) \in \mathcal{D}^n \times \mathcal{S}^k.$$

u' and u'' are well-defined by the formulas

$$-\bar{f} = \bar{f}((\rho_n | \mathcal{S}^{n-1}) \times id) \text{ and } \tilde{\rho}f = \bar{\rho}\bar{f}(id \times \bar{\rho}).$$

5. Proof of Theorem 2. First of all we prove

LEMMA 5.1. *If $\mathcal{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathcal{E}_{k,n}(\chi_2)$ there exists a pair $(\theta_1, \theta_2), \theta_i \in \mathcal{S}[\chi_i]$ which satisfies*

(A) $\mathcal{P}_{k^*}(\theta_1) = \mathcal{P}_{k^*}(\theta_2)$

(B) *there exists a homotopy equivalence $\psi: \mathcal{H}(g_1) \rightarrow \mathcal{H}(g_2)$ with $\psi_*(e_i^i) = e_i^i$ ($i = k, n$).*

Proof. Let $h: \mathcal{H}(f_1) \rightarrow \mathcal{H}(f_2)$ be a homotopy equivalence which may be considered as a cellular map. Then we have

$$\mathcal{P}_{k^*}(\chi_1) = \pm(\mathcal{P}_{k^*}(\chi_2)) \text{ or } \pm(-\iota_k)_* \mathcal{P}_{k^*}(\chi_2).$$

Since it is clear that each element on the right hand side can be obtained as $\mathcal{P}_{k^*}(\theta_2)$ of a suitable $\theta_2 \in \mathcal{S}[\chi_2]$, there exists a pair (χ_1, θ_2) which satisfies (A), and a homotopy equivalence $u: \mathcal{H}(f_1) \rightarrow \mathcal{H}(g_2)$ by Lemma 4.6.

We suppose that $u_*(e^k) = \varepsilon_k e_2^k$ and $u_*(e_1^n) = \varepsilon_n e_2^n \cdot (\varepsilon_k, \varepsilon_n = \pm 1)$. Then we have the equation

(C) $(\varepsilon_k \iota_k)(\mathcal{P}_{k^*}(\chi_1) = \varepsilon_n \mathcal{P}_{k^*}(\theta_2)$.

Hence, by (A), we have

(D) $\varepsilon_n (\varepsilon_k \iota_k) \mathcal{P}_{k^*}(\chi_1) = \mathcal{P}_{k^*}(\theta_2) = \mathcal{P}_{k^*}(\chi_1)$.

The case of $\varepsilon_k = 1$. Since, by Lemma 4.6, there exists a homotopy equivalence $u': \mathcal{H}(\varepsilon_n f_1) \rightarrow \mathcal{H}(f_1)$ with $u'_*(e^k) = e^k$ and $u'_*(e^n) = \varepsilon_n e^n$, the set

$$\{\theta_1 = \varepsilon_n \chi_1, \theta_2, \psi = u \cdot u'\}$$

satisfies (A) and (B).

The case of $\varepsilon_k = -1$. Similarly, by Lemma 4.6, there exists a

homotopy equivalence $u'': \mathcal{K}(\varepsilon_n \rho_* f_1) \rightarrow \mathcal{K}(f_1)$ with $u''_*(e^k) = -e^k$ and $u''_*(e^n) = e^n$. The set

$$\{\theta_1 = \varepsilon_n \rho_* \chi_1, \theta_2, \psi = u \cdot u''\}$$

satisfies (\mathcal{A}) and (\mathcal{B}) by $\mathcal{P}_{k^*}(\varepsilon_n \rho_* \chi_1) = \varepsilon_n(-\iota_k) \mathcal{P}_{k^*}(\chi_1)$.

Proof of Theorem 2. First we suppose that $\mathcal{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathcal{E}_{k,n}(\chi_2)$. We choose $(\theta_1, \theta_2, \psi)$ as stated in Lemma 5.1. Let g_i be a representative of θ_i and let γ_i be the attaching class for the $(k+n)$ -cell of $\mathcal{K}(g_i)$. Let $\varphi: \mathcal{S}^k \cup e^n \rightarrow \mathcal{S}^k \cup e^n$ be a map as stated in Lemma 4.5 ($\chi_i = \theta_i$) and let $\bar{\psi}$ be the map obtained from the restriction of ψ on $\mathcal{S}^k \cup e^n$. Since $\psi_*(\gamma) = \gamma_2$ we have

$$\begin{aligned} 0 &= \gamma_2 - \bar{\psi}_*(\gamma_1) \\ &= \gamma_2 - \varphi_*(\gamma_1) + \varphi_*(\gamma_1) - \bar{\psi}_*(\gamma_1) \\ &= (\gamma_2 - \varphi_*(\gamma_1)) + [d(\varphi, \bar{\psi}), \iota_k] && \text{by Lemma 4.2 and Lemma 2.1} \\ &= i_*(\pm\lambda(\gamma)) + [d(\varphi, \bar{\psi}), \iota_k] && \text{by Lemma 4.5 and} \\ & && \theta_2 - \theta_1 = i_{k^*}(\gamma). \end{aligned}$$

On the other hand, since $d(\varphi, \psi) = d(\varphi, id) + d(id, \bar{\psi}), d(\varphi \bar{\psi})$ is contained in $i_*\pi_n(\mathcal{S}^k)$ by Lemma 4.3. Hence we obtain that

$$\begin{aligned} \lambda(\gamma) &= [\delta, \iota_k] + i_*^{-1}(0) \text{ for some } \delta \in \pi_n(\mathcal{S}^k) \text{ i.e.,} \\ \eta &\equiv \lambda^{-1}[\delta, \iota_k] \text{ mod } \lambda^{-1}i_*^{-1}(0) = \lambda^{-1}\mathcal{P}_{k^*}(\theta_1)\pi_{k+n-1}(\mathcal{S}^{n-1}) \end{aligned}$$

by Lemma 4.4. By applying i_{k^*} to the both side we have

$$\theta_2 - \theta_1 \equiv 0 \text{ mod } \mathcal{E}(\mathcal{P}_{k^*}(\theta_1)), \text{ i.e., } \chi_1 \sim \chi_2.$$

Secondly we assume that $\chi_1 \sim \chi_2$. Hence there exists a pair (θ_1, θ_2) such that $\theta_1 \equiv \theta_2 \text{ mod } \mathcal{E}(\mathcal{P}_{k^*}(\theta_1))$ which means

$$\theta_1 - \theta_2 = i_{k^*}(\eta), \eta \in \pi_{n-1}(\mathcal{S}^k), \lambda(\eta) \in \mathcal{P}_{k^*}(\theta_1)\pi_{k+n-1}(\mathcal{S}^{n-1}).$$

Since $\mathcal{P}_{k^*}(\theta_1) = \mathcal{P}_{k^*}(\theta_2)$ there exists a homotopy equivalence $\varphi: \mathcal{S}^k \cup e^n \rightarrow \mathcal{S}^k \cup e^n$ which satisfies (see Lemma 4.5)

$$\gamma_1 - \varphi_*(\gamma_2) = i_*(\pm\lambda(\eta)).$$

Since $i_*(\pm\lambda(\eta)) \in i_*\mathcal{P}_{k^*}(\theta_1) \cdot \pi_{k+n-1}(\mathcal{S}^{n-1}) = 0$ by Lemma 4.4, we have $\gamma_1 = \varphi_*(\gamma_2)$, i.e., φ is extendable over $\mathcal{K}(g_1)$ to $\mathcal{K}(g_2)$. Then, by Lemma 4.6, $\mathcal{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathcal{E}_{k,n}(\chi_2)$.

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