

# Pacific Journal of Mathematics

**ON TWO CONGRUENCES FOR PRIMALITY**

M. V. SUBBA RAO

## ON TWO CONGRUENCES FOR PRIMALITY

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**In this paper we consider the congruences**

$$n\sigma(n) \equiv 2 \pmod{\varphi(n)}, \quad \varphi(n)t(n) + 2 \equiv 0 \pmod{n}.$$

1. **Introduction.** Apart from the classical Wilson's theorem (that a positive integer  $p > 1$  is a prime if and only if  $(p-1)! + 1 \equiv 0 \pmod{p}$ ) and its variants and corollaries, there is probably no other simple primality criterion in the literature in the form of a congruence. In this connection, we may recall Lehmer's congruence [1]:

$$(1.1) \quad n - 1 \equiv 0 \pmod{\phi(n)}.$$

This is satisfied by every prime. We do not yet know if it has any composite  $n$  as a solution. In 1932, Lehmer [1] showed that if there exists a composite number  $n$  satisfying (1.1), then  $n$  must be odd and square free and have at least seven distinct prime factors. This result was improved in 1944 by Fr. Schuh [4] who showed that such a  $n$  must have at least eleven prime factors. In 1970, E. Lieuwen [2] corrected an error in the proof of Schuh.

In the congruences we shall consider,

$$(1.2) \quad n\sigma(n) \equiv 2 \pmod{\phi(n)}$$

and

$$(1.3) \quad \phi(n)t(n) + 2 \equiv 0 \pmod{n},$$

where  $\phi(n)$  is Euler's totient, and  $t(n)$  and  $\sigma(n)$  are respectively the number and sum of the divisors of  $n$ . Each of these is satisfied whenever  $n$  is a prime. It is a simple matter to solve (1.2) completely (Theorem 1). However, the problem of solving (1.3) for all composite integers  $n$  seems to be a deep one, and we offer only a partial solution.

2. **THEOREM 1.** *The only composite numbers  $n$  satisfying (1.2) are  $n = 4, 6,$  and  $22$ .*

*Proof.* Let a solution of (1.2) be

$$n = 2^a p_1^{a_1} \cdots p_r^{a_r}$$

where  $p_1, \dots, p_r$  are the distinct odd prime divisors of  $n$ . If for some  $i$  ( $1 \leq i \leq r$ ),  $a_i > 1$ , then  $p_i \mid \phi(n)$  and  $p_i \mid n$ , so that  $p_i \mid 2$ , an absurdity. Hence

$$a_1 = a_2 = \cdots = a_r = 1.$$

An analogous argument shows that  $a = 0, 1$  or  $2$ . Hence  $n = 2^a p_1 p_2 \cdots p_r$ , where  $a = 0, 1$  or  $2$ . Next, when  $n$  is in this form,  $2^r \mid \sigma(n)$  and  $2^r \mid \phi(n)$ , so that we should have  $2^r \mid 2$ , on using the congruence. Hence  $r = 0$  or  $1$ , and we get  $n = 2, 4, p_1, 2p_1, 4p_1$  for the possible solutions of (1.2). However,  $n = 4p_1$  is impossible, for otherwise  $4 \mid \phi(n)$ , and this would imply, on using the congruence, that  $4 \mid 2$ .

In the next place, if  $n = 2p_1$ , we have

$$6p_1(p_1 + 1) \equiv 2 \pmod{(p_1 - 1)}.$$

This shows that  $(p_1 - 1) \mid 10$ , and this gives  $p_1 = 2, 3$ , and  $11$ . Hence all the possible composite solutions of (1.2) are  $n = 4, 6$ , and  $22$ , and these are indeed solutions of the congruence.

3. The solution of congruence (1.3). Up to 100,000, the only composite solution of (1.3) is  $n = 4$ , and the question naturally arises if there is any composite solution  $> 4$ . While this is still open, we devote the rest of the paper to obtain some information about such a solution if it exists.

**THEOREM 2.** *Every composite solution  $n > 4$  of the congruence (1.3) satisfies the following conditions:*

(A)  *$n$  is square-free.*

(B) *If  $p$  is an odd prime divisor of  $n$ , then there is no prime divisor of the form  $px + 1$ .*

(C) *Let  $K$  be defined by the relation*

$$(3.1) \quad \phi(n)t(n) + 2 = Kn.$$

*Then  $K$  and  $n$  are of opposite parity and  $4 \nmid K$ .*

(D) *If  $n = m$  is a solution of (1.3), then  $n = 2m$  is not a solution.*

*Proof.* For an odd prime  $p$ , if  $p^2 \mid n$ , then  $p \mid \phi(n)$ ; hence on using (1.2),  $p \mid 2$ , which is absurd. Again if  $4 \mid n$  and  $n > 4$ , a simple argument shows that (1.3) is impossible. This establishes result (A). The proofs of (B), (C), and (D) are equally easy.

**LEMMA.** *For a given  $r$ , the number of solutions  $n$  of (2.11) having  $r$  prime divisors is finite. In fact, if  $p_1, p_2, \dots, p_r$  are the prime divisors of  $n$  in increasing order of magnitude, and if*

$$(3.2) \quad Q_r = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{p_r}\right)$$

where  $q_r$  is the  $r$ th prime in the sequence of primes  $2, 3, 5, \dots$  ( $q_1 = 2, q_2 = 3$  etc.), then

$$(3.3) \quad 2^r Q_r \leq K \leq 2^r,$$

$$(3.4) \quad p_1 < r \left(1 - \frac{K}{2^r}\right)^{-1},$$

and for  $i = 2, 3, \dots, r$ ,

$$p_{i-1} < p_i < (r - i + 1) \left(1 - \frac{K}{2^r} - \frac{1}{p_1} - \dots - \frac{1}{p_{i-1}}\right)^{-1}.$$

*Proof.* The relation (3.1) gives

$$\begin{aligned} K &= \frac{\phi(n)t(n)}{n} + \frac{2}{n} \\ &\leq t(n) + \frac{2}{n}, \end{aligned}$$

for  $n > 2$ . Hence  $K \leq t(n)$ . Since by Theorem 2,  $n$  is square free,  $n = p_1, p_2, \dots, p_r$ , so that  $t(n) = 2^r$ . Hence  $K \leq 2^r$ .

In the next place,

$$\begin{aligned} K &> 2^r \frac{\phi(n)}{n} \\ &= 2^r \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \geq 2^r Q_r. \end{aligned}$$

This completes the proof of (3.3). To prove (3.4), we note that

$$\begin{aligned} K &> 2^r \frac{\phi(n)}{n} = 2^r \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \\ &> 2^r \left(1 - \frac{1}{p_1} - \dots - \frac{1}{p_r}\right). \end{aligned}$$

Hence,

$$1 - \frac{K}{2^r} < \frac{1}{p_1} + \dots + \frac{1}{p_r} < \frac{r}{p_r},$$

and this gives

$$p_1 < r \left(1 - \frac{K}{2^r}\right)^{-1}.$$

Again, using

$$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r} < \frac{1}{p_1} + \frac{r-1}{p_2}$$

and proceeding as before, we get

$$(3.5) \quad p_1 < p_2 < (r-1) \left( 1 - \frac{K}{2^r} - \frac{1}{p_1} \right)^{-1}.$$

Continuing this process, we obtain

$$(3.6) \quad p_2 < p_3 < (r-2) \left( 1 - \frac{K}{2^r} - \frac{1}{p_1} - \frac{1}{p_2} \right)^{-1},$$

and finally,

$$(3.7) \quad p_{r-1} < p_r < \left( 1 - \frac{K}{2^r} - \frac{1}{p_1} - \dots - \frac{1}{p_{r-1}} \right)^{-1}.$$

This establishes (3.4).

For a given  $r$ , (3.3) shows that  $K$  can take only a finite number of values, and (3.4)-(3.7) show that  $p_1, p_2, \dots, p_r$  can take only a finite number of values. Thus for a given  $r$ , the congruence (1.3) has got only a finite number of solutions, since for a given  $r$  the upper and lower bounds for  $K, p_1, p_2, \dots, p_r$  are fixed by the relations (3.3) and (3.4). The actual solutions corresponding to any given  $r$  can be obtained after a finite number of trials. Following this method, we have obtained the following results. (The details of the numerous computations involved in the proofs of Theorems 3 and 4 below are available with the authors.)

**THEOREM 3.** *Any composite solution  $n > 4$  of (1.3) must have at least 4 distinct odd prime factors.*

**THEOREM 4.** *For the congruence (1.3) we have the following:*

(3.8) *If  $K = 1$  or  $3 \leq K \leq 14$ , there are no solutions.*

(3.9) *If  $K = 2$ , the only solutions are all the primes and 4.*

(3.10) *If  $K = 15$ , then  $r = 4$  or 5.*

(3.11) *If  $17 \leq K \leq 29$ , then  $r = 5$ .*

(3.12) *If  $K = 30$  or 31, then  $r = 5$  or 6.*

(3.13) *If  $33 \leq K \leq 58$ , then  $r = 6$ .*

(3.14) *If  $59 \leq K \leq 63$ , then  $r = 6$  or 7.*

(3.15) *If  $65 \leq K \leq 116$ , then  $r = 7$ .*

(3.16) *If  $117 \leq K \leq 127$ , then  $r = 7$  or 8.*

(3.17) *If  $129 \leq K \leq 230$ , then  $r = 8$ .*

(3.18) *If  $231 \leq K \leq 255$ , then  $r = 8$  or 9.*

(3.19) *If  $257 \leq K \leq 457$ , then  $r = 9$ .*

(3.20) *If  $458 \leq K \leq 551$ , then  $r = 9$  or 10.*

(3.21) *If  $513 \leq K \leq 909$ , then  $r = 10$ .*

(3.22) *If  $910 \leq K \leq 1023$ , then  $r = 10$  or 11.*

*Proof.* We illustrate the proof for the case when  $n$  is odd. Using the lemma, we have

$$2^r \geq K > 2^r \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) > 2^r \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{17}\right) \left(1 - \frac{1}{23}\right) \cdots \left(1 - \frac{1}{p_r}\right),$$

on using part (B) of Theorem 2 and Theorem 3. Giving  $K$  successive integral values and examining the consistency of the resulting inequalities while keeping in view the restrictions of Theorem 2, we get the results of the theorem.

REMARK. Any solution  $n$  of (3.1) satisfies the relation

$$2^r < \frac{6480}{19019} K e^\gamma \log x (1 + \log^{-2} x)$$

where  $\gamma$  is Euler's constant,  $r$  is the number of distinct prime factors of  $n$  and  $x = q_{r+5}$ . To show this, we note that

$$2^r = t(n) < K \frac{n}{\phi(n)} < K \left(1 - \frac{1}{3}\right)^{-1} \left(1 - \frac{1}{5}\right)^{-1} \left(1 - \frac{1}{17}\right)^{-1} \left(1 - \frac{1}{23}\right)^{-1} \prod_{10 \leq i \leq r+5} \left(1 - \frac{1}{q_i}\right)^{-1},$$

on using Theorems 2 and 3. Hence

$$2^r < K \cdot \frac{1}{2} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{18}{19} \cdot Q_{r+5}^{-1}$$

where  $Q_{r+5}$  is defined as in (3.2). We now use the estimate given by Rosser and Schoenfeld [3, Theorem 8, Corollary 1] for  $Q_{r+5}^{-1}$ , namely  $Q_{r+5}^{-1} < e^\gamma \log x (1 + \log^{-2} x)$ , where  $x = q_{r+5}$ ; and obtain the stated result.

In the next theorem,  $q_u$  denotes, as already noted, the  $u$ th prime in the sequence of primes  $q_1 = 2, q_2 = 3, \dots$ .

**THEOREM 5.** *Let  $K$  and  $m$  be given and let  $q_u$  be the smallest prime factor of  $n$  which is a solution of the simultaneous equations*

(3.8) 
$$\phi(n)t(n) + 2 = Kn$$

(3.9) 
$$t(n) = mK.$$

*Then  $n$  has a prime factor at least as large as*

$$q_u^m + O(u^m \exp - \log^b u)$$

*where  $b$  is any number  $< 3/5$ .*

*Proof.* By Theorem 2,  $n$  is square free. Let it have  $r$  distinct prime divisors.

Then A. Walfisz [5, Satz 4, p. 187] has shown that if  $\pi(x)$  denotes, as usual, the number of primes  $\leq x$ , and

$$li\ x = \int_2^x \frac{dt}{\log t},$$

then

$$\pi(x) = li(x) + O(x \exp - A \log^{3/5} x (\log \log x)^{-1/5}),$$

where  $A$  is a positive constant. It follows that

$$\pi(x) = li(x) + O(x \exp - \log^a x)$$

for all  $a < 3/5$ . By using a standard argument, we can show that

$$\sum_{q \leq x} \frac{1}{q} = \log \log x + c + O(\exp - \log^a x),$$

$q$  varying over primes.

It follows that

$$\begin{aligned} \sum_{q \leq x} -\log \left(1 - \frac{1}{q}\right) &= \sum_{q \leq x} \frac{1}{q} + \sum_q \left\{ -\log \left(1 - \frac{1}{q}\right) - \frac{1}{q} \right\} + O\left(\frac{1}{x}\right) \\ &= \log \log x + c + O(\exp - \log^a x) \end{aligned}$$

for all  $a < 3/5$ , where  $c$  is an absolute constant (not necessarily the same as the  $c$  used before).

Hence for any given  $h$  for which  $h = O(x^m)$ , we have

$$\begin{aligned} (3.10) \quad \sum_{x \leq q \leq x^m+h} -\log \left(1 - \frac{1}{q}\right) \\ = \log \log (x^m + h) - \log \log x + O(\exp - \log^a x) \end{aligned}$$

for all  $a < 3/5$ . If we choose  $h = x^m \exp(-\log^b x)$ , where  $b < a < 3/5$ , we get

$$\begin{aligned} \sum_{x \leq q \leq x^m+h} -\log \left(1 - \frac{1}{q}\right) &= \log m + \frac{\exp - \log^b x}{m \log x} \\ &+ O\left\{ \frac{\exp - 2 \log^b x}{\log x} + O(\exp - \log^a x) \right\}, \end{aligned}$$

and this is greater than  $\log m$  for all sufficiently large  $x$ . Again, if we take  $h = -x^m \exp(-\log^b x)$  where  $b < a < 3/5$ , then

$$\sum_{x \leq q \leq x^{m+h}} -\log\left(1 - \frac{1}{q}\right) = \log m - \frac{\exp(-\log^b x)}{m \log x} + O\left(\frac{\exp(-2 \log^b x)}{\log x}\right) + O(\exp(-\log^a x)),$$

which is less than  $\log m$  for all sufficiently large  $x$ . Hence, if  $g(x)$  is the smallest number such that

$$\sum_{x \leq q \leq g(x)} -\log\left(1 - \frac{1}{q}\right) \geq \log m,$$

then  $g(x) = x^m + O(x^m \exp(-\log^a x))$  for all  $a < 3/5$ . Now going back to the relation

$$2^r \phi(n) + 2 = Kn.$$

This gives, with  $m = 2^r/K$ , the result

$$m + 2/\phi(n) = n/\phi(n).$$

Taking  $q_u$  to be the smallest prime divisor of  $n$ , let the integer  $v$  be defined to be the smallest integer with the property

$$m < \prod_{i=u}^v \frac{q_i}{q_i - 1}$$

that is,

$$\sum_{q_u \leq q \leq q_v} -\log\left(1 - \frac{1}{q}\right) > \log m.$$

Then it follows that  $n$  must have a prime factor other than  $q_u$  and at least as large as  $q_v$ . The previous investigation shows that

$$q_v = q_u^m + O(q_u^m \exp(-\log^a(q_u^m))),$$

that is,

$$q_v = q_u^m + O(u^m \exp(-\log^b u)) \text{ for any } b < a < 3/5.$$

Hence, we have proved the theorem.

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# Pacific Journal of Mathematics

Vol. 52, No. 1

January, 1974

David R. Adams, <i>On the exceptional sets for spaces of potentials</i> . . . . .	1
Philip Bacon, <i>Axioms for the Čech cohomology of paracompacta</i> . . . . .	7
Selwyn Ross Caradus, <i>Perturbation theory for generalized Fredholm operators</i> . . . . .	11
Kuang-Ho Chen, <i>Phragmén-Lindelöf type theorems for a system of nonhomogeneous equations</i> . . . . .	17
Frederick Knowles Dashiell, Jr., <i>Isomorphism problems for the Baire classes</i> . . . . .	29
M. G. Deshpande and V. K. Deshpande, <i>Rings whose proper homomorphic images are right subdirectly irreducible</i> . . . . .	45
Mary Rodriguez Embry, <i>Self adjoint strictly cyclic operator algebras</i> . . . . .	53
Paul Erdős, <i>On the distribution of numbers of the form <math>\sigma(n)/n</math> and on some related questions</i> . . . . .	59
Richard Joseph Fleming and James E. Jamison, <i>Hermitian and adjoint abelian operators on certain Banach spaces</i> . . . . .	67
Stanley P. Gudder and L. Haskins, <i>The center of a poset</i> . . . . .	85
Richard Howard Herman, <i>Automorphism groups of operator algebras</i> . . . . .	91
Worthen N. Hunsacker and Somashekhar Amrith Naimpally, <i>Local compactness of families of continuous point-compact relations</i> . . . . .	101
Donald Gordon James, <i>On the normal subgroups of integral orthogonal groups</i> . . . . .	107
Eugene Carlyle Johnsen and Thomas Frederick Storer, <i>Combinatorial structures in loops. II. Commutative inverse property cyclic neofields of prime-power order</i> . . . . .	115
Ka-Sing Lau, <i>Extreme operators on Choquet simplexes</i> . . . . .	129
Philip A. Leonard and Kenneth S. Williams, <i>The septic character of 2, 3, 5 and 7</i> . . . . .	143
Dennis McGavran and Jingyal Pak, <i>On the Nielsen number of a fiber map</i> . . . . .	149
Stuart Edward Mills, <i>Normed Köthe spaces as intermediate spaces of <math>L_1</math> and <math>L_\infty</math></i> . . . . .	157
Philip Olin, <i>Free products and elementary equivalence</i> . . . . .	175
Louis Jackson Ratliff, Jr., <i>Locally quasi-unmixed Noetherian rings and ideals of the principal class</i> . . . . .	185
Seiya Sasao, <i>Homotopy types of spherical fibre spaces over spheres</i> . . . . .	207
Helga Schirmer, <i>Fixed point sets of polyhedra</i> . . . . .	221
Kevin James Sharpe, <i>Compatible topologies and continuous irreducible representations</i> . . . . .	227
Frank Siwiec, <i>On defining a space by a weak base</i> . . . . .	233
James McLean Sloss, <i>Global reflection for a class of simple closed curves</i> . . . . .	247
M. V. Subba Rao, <i>On two congruences for primality</i> . . . . .	261
Raymond D. Terry, <i>Oscillatory properties of a delay differential equation of even order</i> . . . . .	269
Joseph Dinneen Ward, <i>Chebyshev centers in spaces of continuous functions</i> . . . . .	283
Robert Breckenridge Warfield, Jr., <i>The uniqueness of elongations of Abelian groups</i> . . . . .	289
V. M. Warfield, <i>Existence and adjoint theorems for linear stochastic differential equations</i> . . . . .	305