

# Pacific Journal of Mathematics

**Chebyshev Centers in Spaces of Continuous  
Functions**

JOSEPH DINNEEN WARD

## CHEBYSHEV CENTERS IN SPACES OF CONTINUOUS FUNCTIONS

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**A bounded set  $F$  in a Banach space  $X$  has a Chebyshev center if there exists in  $X$  a "smallest" ball containing  $F$ . A Banach space  $X$  is said to admit centers if every bounded subset of  $X$  has a center. The purpose of this paper is to show that certain spaces of continuous functions admit centers.**

1. Introduction. Let  $X$  be a real normed linear space,  $G$  a subset of  $X$  and  $f$  an element of  $X$ . Then a best approximant,  $g_*$ , to  $f$  from  $G$  (if it exists) is a solution to

$$(1.1) \quad \inf \{ \|g - f\|, g \in G \} .$$

It may happen that  $f$  is not defined exactly but is known to lie in a bounded set  $F$ . It is reasonable then to approximate simultaneously all  $f \in F$  by solving

$$(1.2) \quad \inf \sup \{ \|g - f\|, f \in F \} \equiv R_g(F)$$

where the inf is taken over all  $g \in G$ . Thus we may view problem (1.2) as a natural generalization of the best approximation problem (1.1). If  $G = X$  then the solutions of (1.2) are called Chebyshev centers of  $F$ , following Garkavi [2]. In [3], Kadets and Zamyatin showed that  $C([a, b], R)$ , the space of real-valued continuous functions on  $[a, b]$ , admits centers. This means that (1.2) has a solution in  $C([a, b], R)$  for  $F$  an arbitrary bounded set in  $C([a, b], R)$ .

The purpose of this note is to show that the Kadets-Zamyatin result holds under much greater generality. Let

$\Omega =$  a paracompact Hausdorff space

$S =$  a normal space

$C(A, B) =$  space of continuous functions from  $A$  to  $B$ .

Our main theorems are:

**THEOREM 1.**  *$C(\Omega, X)$  admits centers if  $X$  is a finite dimensional, rotund space.*

**THEOREM 2.**  *$C(S, H)$  admits centers if  $H$  is an arbitrary Hilbert space.*

2. Proof of Theorem 1.

DEFINITION 2.1. Let  $X$  and  $E$  be Banach spaces and  $F: X \rightarrow 2^E$ .  $F$  is said to be *upper semi-continuous* (u.s.c) if the set  $\{x | F(x) \subset G\}$  is open in  $X$  for every open  $G \subset E$ .  $F$  is said to be *lower semi-continuous* (l.s.c) if the set  $\{x | F(x) \cap G\}$  is open in  $X$  for every open  $G \subset E$ .

*Proof of Theorem 1.* We use the following notation:

- $F$  = fixed but arbitrary bounded set in  $C(\Omega, X)$
- $\mathcal{N}(t)$  = the directed family of open  $t$ -neighborhoods,  $t \in \Omega$
- $\Omega(t, N) = \bigcup \{f(s)\}$  where  $f \in F, s \in N$
- $A_N(t)$  = convex closure of  $\Omega(t, N)$
- $B(x, K)$  = ball of radius  $K$  centered at  $x$
- $R(F)$  = Chebyshev radius of  $F$  with respect to  $C(\Omega, X)$  as defined in (1.2).

Suppose  $F$  is bounded by  $K$ . Now for each  $t \in \Omega$ , consider the net  $N \rightarrow A_N(t)$  defined on  $\mathcal{N}(t)$ . The range of this net lies in the metric space  $\mathcal{F}(B(0, K))$  whose elements are the compact, convex and nonempty subsets of  $B(0, K)$ . We put

$$(2.2) \quad A(t) = \lim A_N(t), N \in \mathcal{N}(t).$$

This limit exists in  $\mathcal{F}(B(0, K))$  by virtue of the compactness of this space and the monotonicity of the net  $\{A_N(t)\}, N \in \mathcal{N}(t)$ . It may be verified that

$$(2.3) \quad A(t) = \bigcap A_N(t), N \in \mathcal{N}(t).$$

We show that the map  $A: \Omega \rightarrow \mathcal{F}(B(0, K))$  is u.s.c.. This requires us to choose any nonempty open set  $G \subset X$  and then show that  $\{t \in \Omega: A(t) \subset G\}$  is open in  $\Omega$ . Let  $t_0$  belong to this set. Then by (2.2), there is an  $N \in \mathcal{N}(t_0)$  for which  $A_N(t_0) \subset G$ . Hence, if  $t \in N$ , we have by (2.3) that

$$A(t) \subset A_N(t) = A_N(t_0) \subset G$$

and so  $A$  is upper semi-continuous.

Let  $R_x(A) = \sup \{R_x(A(t)): t \in \Omega\}$ . Following Olech [4], we introduce the map  $G: \Omega \rightarrow 2^X$  defined by

$$G(t) = \{\beta \in X: A(t) \subset B(\beta, R_x(A))\}.$$

Olech proved (under the assumption that  $X$  is uniformly rotund which is the same as rotund in finite dimensions) that the values  $G(t)$  are compact, convex and nonempty subsets of  $X$  and  $G$  is lower semi-continuous in  $t$ . Thus by appealing to the Michael selection theorem,

there is a continuous selection  $f$  for  $G$ .

It is clear that  $\|f - g\| \leq R_x(A)$  for all  $g \in F$ . It remains to show that  $R_x(A) \leq R(F)$ . Let  $\varepsilon$  be arbitrary and choose  $t \in \Omega$  so that  $R_x(A(t)) > R_x(A) - \varepsilon$ . Since  $f \in C(\Omega, X)$ , we may choose  $N \in \mathcal{N}(t)$  for which  $\text{osc}(f: N) < \varepsilon$ . Due to (2.2) and (2.3) we may assume that  $N$  has been chosen so "small" that there is a  $\gamma \in N$  and  $g \in F$  for which  $R_x(A) - 2\varepsilon < |g(\gamma) - f(\gamma)| \leq R(F)$ .

3. *Proof of Theorem 2.* The problem with  $X$  being infinite dimensional is that we have no right to expect  $\lim A_N(t), N \in \mathcal{N}(t)$ , to exist as in Theorem 1. Thus the method of proof of Theorem 1 must be abandoned. Nevertheless, Theorem 2 may still be proved.

*Proof of Theorem 2.* Let  $F \subset C(S, H)$  be bounded by  $K$ . There exist "approximate centers", call them  $f_n$ , such that  $f_n$  is within  $R(F) + 1/n$  of each element of  $F$ . We clearly have for any approximate center  $f_i$  and  $f_j$  the relationship  $\|f_i - f_j\| \leq 4K$ .

*Step 1.* We show that for arbitrary  $\delta > 0$ , there exists an  $\varepsilon_\delta > 0$  such that for any  $(R(F) + \varepsilon_\delta)$ -approximate center  $f_1$ , we may construct an  $(R(F) + \varepsilon_{\delta/2})$ -center  $f_2$  such that  $f_2 \in B(f_1, \delta)$ .

*Proof of Step 1.* Pick  $\varepsilon_\delta > 0$  so that  $\delta = (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2}$ . Pick  $g$  where  $g$  is an  $(R(F) + \varepsilon_{\delta/2})$ -approximate center for  $F$ . It is clear that  $\|g - f_1\| \leq 4K$ . Let  $F(t) = \{f(t): f \in F\}$ . By definition of approximate center, for all  $t \in S$ ,

$$B(f_1(t), R(F) + \varepsilon_\delta) \cap B(g(t), R(F) + \varepsilon_{\delta/2}) \supset F(t).$$

For convenience sake set

$$r_1 = R(F) + \varepsilon_\delta; r_2 = R(F) + \varepsilon_{\delta/2}$$

$$d(t) = \|f_1(t) - g(t)\|.$$

Define

$$f_2(t) = f_1(t) + \beta(t)(g(t) - f_1(t))$$

where

$$\beta(t) = \begin{cases} 1 & \text{if } (r_1^2 - r_2^2)/d^2 \geq 1 \\ ((r_1^2 - r_2^2)/d^2)^{1/2} & \text{if } (r_1^2 - r_2^2)/d^2 < 1. \end{cases}$$

Note that  $0 \leq \beta(t) \leq 1$  for all  $t \in S$ . We now make three claims about  $f_2$ .

(1)  $f_2$  is a continuous function, i.e.,  $f_2 \in C(S, H)$

- (2)  $\|f_2 - f_1\| \leq (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2} \leq \delta$   
 (3)  $f_2$  is an  $(R(F) + \varepsilon_{\delta/2})$ -approximate center of  $F$

*Proof of (1).* Since  $g$  and  $f_1$  are continuous functions and  $d(t) = \|f_1(t) - f_2(t)\|$  is continuous,  $\beta(t)$  is also continuous. This clearly implies the continuity of  $f_2$ .

*Proof of (2).* It suffices to show that  $\|f_2(t) - f_1(t)\| \leq (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2} \leq \delta$  for all  $t \in S$ . Thus for fixed  $t_0$ ,  $\|f_2(t_0) - f_1(t_0)\| = \|\beta(t_0)(g(t_0) - f_1(t_0))\|$ . If  $\beta(t_0) = 1$ ,  $r_1^2 - r_2^2 \geq d^2$  so

$$\begin{aligned} \|\beta(t_0)(g(t_0) - f_1(t_0))\| &= \|g(t_0) - f_1(t_0)\| = d(t_0) \\ &\leq (r_1^2 - r_2^2)^{1/2} \leq \delta. \end{aligned}$$

If  $\beta(t_0) < 1$ ,

$$\begin{aligned} \|f_2(t_0) - f_1(t_0)\|^2 &= \|\beta(t_0)(f_2(t_0) - f_1(t_0))\|^2 \\ &= ((r_1^2 - r_2^2)/d^2)d^2 = r_1^2 - r_2^2 \end{aligned}$$

so  $\|f_2(t_0) - f_1(t_0)\| \leq (r_1^2 - r_2^2)^{1/2} \leq \delta$ . This proves (2).

*Proof of (3).* Since by (1)  $f_2 \in C(S, H)$ , it suffices to show that for each  $t_0 \in S$ ,

$$\begin{aligned} &B(f_2(t_0), R(F) + \varepsilon_{\delta/2}) \\ &\supset B(f_1(t_0), R(F) + \varepsilon_\delta) \cap B(g(t_0), R(F) + \varepsilon_{\delta/2}) \supset F(t_0). \end{aligned}$$

The above is equivalent to showing that for all  $x$  such that  $\|x - f_1\| \leq r_1$  and  $\|x - g\| \leq r_2$ , then  $\|x - f_2\| \leq r_2$ .

Without loss of generality assume  $f_1$  is 0. The above problem then simplifies to showing that the implication  $\|x\| \leq r_1$  and  $\|x - g\| \leq r_2$ , then  $\|x - f_2\| \leq r_2$  holds for all  $x \in V$  and for all  $V \subset H$  where  $V$  is a two dimensional subspace containing  $g$ . Hence we are reduced to a problem in two dimensional Hilbert space and a few simple applications of the Pythagorean theorem prove the assertion.

*Step 2.* Let  $f_1$  be any  $(R(F) + \varepsilon_{\delta_1})$ -approximate center of  $F$ . Having defined  $f_n$ , take  $f_{n+1}$  to be an  $(R(F) + \varepsilon_{\delta_{n+1}})$ -approximate center such that  $f_{n+1} \in B(f_n, \delta_n)$  and  $\delta_{n+1} = \delta_n/2$ , which we may do by Step 1. Evidently  $\varepsilon_{\delta_n} \rightarrow 0$  as  $n \rightarrow \infty$ .

Now consider  $\{f_n\}_{n=1}^\infty$ . For all  $i, j \geq K$ ,  $\|f_i - f_j\| \leq \|f_i - f_K\| + \|f_K - f_j\| \leq 2 \sum_{n=K}^\infty \delta_n/2^n = \delta_K/2^{K-1}$ . So  $\{f_n\}_{n=1}^\infty$  is a uniformly convergent sequence with limit point  $f'$ ,  $f' \in C(S, H)$ . Also for each  $g \in F$ ,

$$\begin{aligned} \sup \{\|g - f'\|, g \in F\} &\leq \sup \{\|g - f_n\| + \|f_n - f'\|\}, g \in F\} \\ &\leq R(F) + \varepsilon_{\delta_n} + \gamma_n \end{aligned}$$

where  $\gamma_n$  is a null sequence. Hence  $\sup \{\|g - f'\|, g \in F\} = R(F)$  and  $f'$  is a Chebyshev center of  $F$ .

REMARK 1. Since paracompact spaces are normal [1], Theorem 2 generalizes Theorem 1 in the case that the range space of the space of continuous functions is a finite dimensional Hilbert space.

REMARK 2. This author was unable to resolve the question whether Theorem 2 holds when the range space of  $C(S, H)$  is an arbitrary uniformly convex space.

The author thanks the referee whose suggestions simplified the proof of Theorem 2.

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Received October 9, 1973 and in revised form January 22, 1974. This research comprised a part of the author's Ph. D. thesis at Purdue University. The author wishes to express his gratitude to his advisor, Professor Richard Holmes, for numerous illuminating discussions.

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