Pacific Journal of Mathematics

BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS

BRYAN EDMUND CAIN AND RICHARD J. TONDRA

Vol. 52, No. 2 February 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 2, 1974

BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS

BRYAN E. CAIN AND RICHARD J. TONDRA

This paper establishes the existence of a domain (open connected subset) B of the complex plane C such that for every domain $\Omega \subset C$ and every compact set $K \subset \Omega$, there is a biholomorphic embedding $e: B \rightarrow \Omega$, such that $K \subset e(B) \subset \text{cl}\,[e(B)] \subset \Omega$.

1. Introduction. Let Ω_1 and Ω_2 be domains (i.e., open connected sets) in the complex plane C such that $\text{cl}\,\Omega_1 \subset \Omega_2$ (cl = closure). A domain Ω is a biholomorphic approximation of Ω_1 with respect to Ω_2 provided that there exists an invertible holomorphic function e defined on Ω such that

$$\text{cl}\,\Omega_1 \subset e(\Omega) \subset \text{cl}\,[e(\Omega)] \subset \Omega_2 .$$

The mapping e is a biholomorphic embedding (bh-embedding) of Ω into Ω_2. (Ω may also be considered a biholomorphic approximation of Ω_2 with respect to Ω_1.)

Homeomorphic domains may, of course, be biholomorphically inequivalent, and, moreover, may not even be close biholomorphic approximations of each other. For example, let $A(r, s) = \{z \in C: r < |z| < s\}$ when $0 < r < s < \infty$. Suppose that $0 < \varepsilon < 1 < t < \infty$ and that e is a bh-embedding of $A = A(r, s)$ such that

$$\text{cl}\,A(1, t) \subset e(A) \subset \text{cl}\,[e(A)] \subset A(1 - \varepsilon, t + \varepsilon) .$$

By taking the modules of these ring domains (cf. [1]) we obtain the inequality $t < s/r < (t + \varepsilon)/(1 - \varepsilon)$ which is precisely the condition r and s must satisfy for such an embedding e to exist.

Our main result establishes the existence of a domain $B \subset C$ which is a biholomorphic approximation of every bounded domain Ω_1 with respect to every domain Ω_2 containing $\text{cl}\,\Omega_1$.

2. The main theorem. Let $\hat{C}$ denote the Riemann sphere.

THEOREM 2.1. *There exists a domain $B \subset C$ such that for every domain $\Omega \subset \hat{C}$ and for every compact set $K \subset \Omega$ other than $\hat{C}$ there exists a biholomorphic embedding $e: B \rightarrow \Omega$ such that $K \subset e(B) \subset \text{cl}\,[e(B)] \subset \Omega$.*

REMARK. Actually such an embedding will exist if Ω is any connected Riemann surface (without boundary) and $K \subset \Omega$ is any planar compact surface other than $\hat{C}$. ("Planar" means homeomorphic to a subset of $\hat{C}$.) Indeed, by the trianguability of Ω there must exist a planar domain Ω_0 such that $K \subset \Omega_0 \subset \Omega$, and so it suffices to consider the planar case.

The following theorems are corollaries of Theorem 2.1.

COROLLARY 2.2. *Let $K \neq \hat{C}$ be a compact connected subset of a domain $\Omega \subset \hat{C}$. Then $K = \bigcap_{i=1}^{\infty} B_i$ where each B_i is bh-equivalent to B and* $\mathrm{cl}\, B_{i+1} \subset B_i$ *for $i = 1, 2, \cdots$.*

COROLLARY 2.3. *Let $\Omega \neq \phi$ be a domain in C. Then $\Omega = \bigcup_{i=1}^{\infty} B_i$ where each B_i is bh-equivalent to B and* $\mathrm{cl}\, B_i \subset B_{i+1}$ *for $i = 1, 2, \cdots$.*

3. *Proofs.* For each $a \in C$ and $r > 0$ set $D(a, r) = \{z: |z - a| < r\}$ and let $\bar{D}(a, r)$ denote $\mathrm{cl}\, D(a, r)$. Set $D = D(0, 1)$. A circle $\{z: |z - a| = r\}$ will be called "rational" provided that $\mathrm{Re}\, a$, $\mathrm{Im}\, a$, and $r > 0$ are rational numbers. The topological boundary of a domain Ω will be denoted $\partial\Omega$.

To construct B consider the domains Ω satisfying: (1) $\partial\Omega$ has finitely many components, (2) each component of $\partial\Omega$ is a rational circle, (3) $\mathrm{cl}\, \Omega \subset D$ and its outer boundary is centered at the origin. Let $E_1, E_2, \cdots$ be an enumeration of these domains. Let s_j be the radius of the outer boundary of E_j and let ϕ_j be the linear fractional transformation of D onto $H = \{z: \mathrm{Re}\, z > 0\}$ which carries -1 to 0, $+1$ to ∞, and $-s_j$ to 1 if $j = 1$ and to $\phi_{j-1}(s_{j-1})$ if $j > 1$. Let $B = H \backslash \bigcup_{j=1}^{\infty} \phi_j[D(0, s_j) \backslash E_j]$.

To show that B has the desired properties, we prove the following lemma using the "small mesh grid" technique (often employed in texts on function theory), rather than the theory of trianguability. A bounded domain $\Omega \subset C$ will be called a Jordan domain if $\partial\Omega$ consists of finitely many disjoint Jordan curves.

LEMMA 3.1. *Let K be a compact subset of a domain $\Omega \subset C$. Then there exists a Jordan domain Ω_0 such that $K \subset \Omega_0 \subset \mathrm{cl}\, \Omega_0 \subset \Omega$.*

Sketch of proof. Since Ω is connected, there exists a connected compact set K_0 such that $K \subset K_0 \subset \Omega$. Thus we may assume that K is connected. With r picked so small that $[K + \bar{D}(0, \sqrt{2}r)] \subset \Omega$ let L be the union of those squares of a grid of squares with edge length r which intersect K. If $a \in L$ is a vertex of precisely two squares of L select the positive number $s_a < r/2$ to be so small that $\bar{D}(a, s_a) \subset \Omega$. Let L_0 denote the union of all the $\bar{D}(a, s_a)$'s. Then straightforward arguments show that $\Omega_0 = \mathrm{int}\,(L \cup L_0)$ is the desired Jordan domain.

Now let Ω and K be as described in Theorem 2.1. Lemma 3.1

provides a Jordan domain Ω_0 such that $K \subset \Omega_0 \subset \text{cl}\,\Omega_0 \subset \Omega$. According to Theorem 2 page 237 of [2] there is a bh-embedding h of Ω_0 into D such that (1) the outer boundary of $h(\Omega_0)$ is ∂D and (2) $\partial[h(\Omega_0)]$ has finitely many components and each is a circle. Each of the circles bounding $h(\Omega_0)$ can be "approximated" arbitrarily closely by a rational circle which lies in $h(\Omega_0)$. We require that the approximation to the unit circle be centered at 0. Since $h(K)$ is a compact subset of $h(\Omega_0)$, when the approximations are close enough, the approximating circles will bound a domain which contains $h(K)$. This region, by its definition, is one of the E_j's, say E_k. Then

$$h(K) \subset E_k \subset \phi_k^{-1}(B) \subset h(\Omega_0)$$

and so applying h^{-1} will establish Theorem 2.1.

To prove Corollary 2.2 we let $B_1 = e_1(B)$ where e_1 is the bh-embedding of B such that $K \subset B_1 \subset \text{cl}\,B_1 \subset \Omega$. For $i > 1$ we let G_i be the component of $[K + D(0, 1/(i-1))] \cap B_{i-1}$ which contains K, and we set $B_i = e_i(B)$ where e_i is the bh-embedding of B, given by Theorem 2.1, such that $K \subset B_i \subset \text{cl}\,B_i \subset G_i$.

To prove Corollary 2.3 we pick $a \in \Omega$ and for large n we can let G_n be the component of $\{z: \text{dist}\,(z, \boldsymbol{C}\backslash\Omega) > 1/n \text{ and } |z| < n\}$ which contains a. Since $\text{cl}\,G_n$ is a compact subset of G_{n+1} there exists a bh-embedding $e_n: B \to G_{n+1}$ such that $B_n = e_n(B) \supset \text{cl}\,G_n$. That $\Omega = \bigcup G_n$ (and hence $\Omega = \bigcup B_n$) follows from the arc connectedness of Ω. These B_n's are the required domains (except for re-indexing).

4. Some applications to holomorphic extension problems. Let $K \subset \boldsymbol{C}$ be compact and let $f: K \to \boldsymbol{C}$. It is easy to extend f to a holomorphic function F defined on a domain containing K (caution: domains are connected) if there exist: (1) a domain Ω, (2) a biholomorphic function e on Ω such that $K \subset e(\Omega)$, and (3) a holomorphic extension G of $g = f \circ e\,|_{e^{-1}(K)}$ to all of Ω. Indeed $F = G \circ e^{-1}$ is the required extension. Conversely if f has such an extension F the existence of Ω, e, and G is trivial. For let the domain Ω be the domain of F, set $e(z) = z$, and take $G = F$. Thus we have an equivalent formulation of the problem of holomorphically extending a function $f: K \to \boldsymbol{C}$ to a domain containing K. Theorem 4.2 shows that another equivalent formulation is obtained when in the discussion above the variable domain Ω is replaced by the fixed domain B. We first show that for a more restricted class of sets K this extension question is very naturally formulated with D in the role of Ω.

THEOREM 4.1. *Let $K \subset \boldsymbol{C}$ be compact and let $f: K \to \boldsymbol{C}$. Suppose* *that K and $C\backslash K$ are connected. Then there exists a holomorphic extension F of f to a domain containing K if and only if there exist* (a) *a bh-embedding e of D such that $K \subset e(D)$ and* (b) *a holomorphic extension G of $g = f \circ e|_{e^{-1}(K)}$ to all of D.*

Proof. Since the "if" part of this theorem is treated in the discussion above we confine our remarks to the "only if" part. Assume that the extension F exists, and let $\Omega \supset K$ be its domain. It suffices to find a bh-mapping e of D such that $K \subset e(D) \subset \Omega$. This is trivial if K is a singleton: so we assume K is not a singleton. Then the Riemann Mapping theorem shows that $\hat{C}\backslash K$ is bh-equivalent to D (it is simply connected because K is connected). Let $h: \hat{C}\backslash K \rightarrow D$ be the Riemann mapping. Since $h^{-1}(\bar{D}(0, r))$ is simply connected for $0 < r < 1$ we know that $V_r = \hat{C}\backslash h^{-1}(\bar{D}(0, r))$ is nonempty, open, and simply connected for $0 < r < 1$. Thus each V_r with $0 < r < 1$ is bh-equivalent to D. Since $h(\hat{C}\backslash\Omega)$ is a compact subset of D it lies in $D(0, s)$ for some $s < 1$, and the Riemann mapping e of D onto V_s is the required map.

If in Theorem 4.1 D is replaced by B the assumption that K and $C\backslash K$ are connected may be dropped.

THEOREM 4.2. *Let $K \subset C$ be compact and let $f: K \rightarrow C$. There exists a holomorphic extension F of f to a domain containing K if and only if there exist* (a) *a bh-mapping e of B such that $K \subset e(B)$ and* (b) *a holomorphic extension G of $g = f \circ e|_{e^{-1}(K)}$ to all of B.*

Proof. As in the proof of Theorem 4.1 the "if" part has already been settled and we begin the "only if" part by letting $\Omega \supset K$ be the domain of F. An application of Theorem 2.1 gives a bh-embedding e of B such that $K \subset e(B) \subset \Omega$. This is the required mapping.

REMARK. Comparing Theorems 4.1 and 4.2 tempts one to conjecture the existence of a sequence of domains $D = \Omega_1, \Omega_2, \cdots, \Omega_\infty = B$ such that $\hat{C}\backslash\Omega_n$ has n components and for which Theorem 4.1 will remain true when it is modified by: (1) Replacing its second sentence with "Suppose K is connected and $C\backslash K$ has n components", and (2) Replacing D with Ω_n. The discussion in the introduction shows that this conjecture fails, since for $n = 2$, Ω_2 must be bh-equivalent to $A(r, s)$ for some r, s with $0 \leqq r < s \leqq \infty$ and so Ω_2 cannot be embedded between $A(1, t)$ (the domain of f) and $A(1 - \varepsilon, t + \varepsilon)$ (the domain of the extension F) unless $t < s/r < (t + \varepsilon)/(1 + \varepsilon)$.

References

1. W. H. J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Van Nostrand Mathematical Studies #12, D. Van Nostrand, Princeton, N. J., 1967.
2. G. M. Goluzin, *Geometric Theory of Function of a Complex Variable*, Translations of Math. Mono., vol. **26**, Providence, Rhode Island, 1969.

Received July 19, 1973.

IOWA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid Additional copies may be obtained at cost in multiples of 50.

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 52, No. 2 February, 1974