

# Pacific Journal of Mathematics

**BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS**

BRYAN EDMUND CAIN AND RICHARD J. TONDRA

## BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS

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**This paper establishes the existence of a domain (open connected subset)  $B$  of the complex plane  $C$  such that for every domain  $\Omega \subset C$  and every compact set  $K \subset \Omega$ , there is a biholomorphic embedding  $e: B \rightarrow \Omega$ , such that  $K \subset e(B) \subset \text{cl}[e(B)] \subset \Omega$ .**

1. Introduction. Let  $\Omega_1$  and  $\Omega_2$  be domains (i.e., open connected sets) in the complex plane  $C$  such that  $\text{cl } \Omega_1 \subset \Omega_2$  ( $\text{cl}$  = closure). A domain  $\Omega$  is a biholomorphic approximation of  $\Omega_1$  with respect to  $\Omega_2$  provided that there exists an invertible holomorphic function  $e$  defined on  $\Omega$  such that

$$\text{cl } \Omega_1 \subset e(\Omega) \subset \text{cl}[e(\Omega)] \subset \Omega_2 .$$

The mapping  $e$  is a biholomorphic embedding (*bh*-embedding) of  $\Omega$  into  $\Omega_2$ . ( $\Omega$  may also be considered a biholomorphic approximation of  $\Omega_2$  with respect to  $\Omega_1$ .)

Homeomorphic domains may, of course, be biholomorphically inequivalent, and, moreover, may not even be close biholomorphic approximations of each other. For example, let  $A(r, s) = \{z \in C: r < |z| < s\}$  when  $0 < r < s < \infty$ . Suppose that  $0 < \varepsilon < 1 < t < \infty$  and that  $e$  is a *bh*-embedding of  $A = A(r, s)$  such that

$$\text{cl } A(1, t) \subset e(A) \subset \text{cl}[e(A)] \subset A(1 - \varepsilon, t + \varepsilon) .$$

By taking the modules of these ring domains (cf. [1]) we obtain the inequality  $t < s/r < (t + \varepsilon)/(1 - \varepsilon)$  which is precisely the condition  $r$  and  $s$  must satisfy for such an embedding  $e$  to exist.

Our main result establishes the existence of a domain  $B \subset C$  which is a biholomorphic approximation of every bounded domain  $\Omega_1$  with respect to every domain  $\Omega_2$  containing  $\text{cl } \Omega_1$ .

2. The main theorem. Let  $\hat{C}$  denote the Riemann sphere.

**THEOREM 2.1.** *There exists a domain  $B \subset C$  such that for every domain  $\Omega \subset \hat{C}$  and for every compact set  $K \subset \Omega$  other than  $\hat{C}$  there exists a biholomorphic embedding  $e: B \rightarrow \Omega$  such that  $K \subset e(B) \subset \text{cl}[e(B)] \subset \Omega$ .*

**REMARK.** Actually such an embedding will exist if  $\Omega$  is any connected Riemann surface (without boundary) and  $K \subset \Omega$  is any planar compact surface other than  $\hat{C}$ . ("Planar" means homeomorphic

to a subset of  $\hat{C}$ .) Indeed, by the trianguability of  $\Omega$  there must exist a planar domain  $\Omega_0$  such that  $K \subset \Omega_0 \subset \Omega$ , and so it suffices to consider the planar case.

The following theorems are corollaries of Theorem 2.1.

**COROLLARY 2.2.** *Let  $K \neq \hat{C}$  be a compact connected subset of a domain  $\Omega \subset \hat{C}$ . Then  $K = \bigcap_{i=1}^{\infty} B_i$  where each  $B_i$  is bh-equivalent to  $B$  and  $\text{cl } B_{i+1} \subset B_i$  for  $i = 1, 2, \dots$ .*

**COROLLARY 2.3.** *Let  $\Omega \neq \phi$  be a domain in  $C$ . Then  $\Omega = \bigcup_{i=1}^{\infty} B_i$  where each  $B_i$  is bh-equivalent to  $B$  and  $\text{cl } B_i \subset B_{i+1}$  for  $i = 1, 2, \dots$ .*

3. *Proofs.* For each  $a \in C$  and  $r > 0$  set  $D(a, r) = \{z: |z - a| < r\}$  and let  $\bar{D}(a, r)$  denote  $\text{cl } D(a, r)$ . Set  $D = D(0, 1)$ . A circle  $\{z: |z - a| = r\}$  will be called "rational" provided that  $\text{Re } a, \text{Im } a$ , and  $r > 0$  are rational numbers. The topological boundary of a domain  $\Omega$  will be denoted  $\partial\Omega$ .

To construct  $B$  consider the domains  $\Omega$  satisfying: (1)  $\partial\Omega$  has finitely many components, (2) each component of  $\partial\Omega$  is a rational circle, (3)  $\text{cl } \Omega \subset D$  and its outer boundary is centered at the origin. Let  $E_1, E_2, \dots$  be an enumeration of these domains. Let  $s_j$  be the radius of the outer boundary of  $E_j$  and let  $\phi_j$  be the linear fractional transformation of  $D$  onto  $H = \{z: \text{Re } z > 0\}$  which carries  $-1$  to  $0$ ,  $+1$  to  $\infty$ , and  $-s_j$  to  $1$  if  $j = 1$  and to  $\phi_{j-1}(s_{j-1})$  if  $j > 1$ . Let  $B = H \setminus \bigcup_{j=1}^{\infty} \phi_j[D(0, s_j) \setminus E_j]$ .

To show that  $B$  has the desired properties, we prove the following lemma using the "small mesh grid" technique (often employed in texts on function theory), rather than the theory of trianguability. A bounded domain  $\Omega \subset C$  will be called a Jordan domain if  $\partial\Omega$  consists of finitely many disjoint Jordan curves.

**LEMMA 3.1.** *Let  $K$  be a compact subset of a domain  $\Omega \subset C$ . Then there exists a Jordan domain  $\Omega_0$  such that  $K \subset \Omega_0 \subset \text{cl } \Omega_0 \subset \Omega$ .*

*Sketch of proof.* Since  $\Omega$  is connected, there exists a connected compact set  $K_0$  such that  $K \subset K_0 \subset \Omega$ . Thus we may assume that  $K$  is connected. With  $r$  picked so small that  $[K + \bar{D}(0, \sqrt{2}r)] \subset \Omega$  let  $L$  be the union of those squares of a grid of squares with edge length  $r$  which intersect  $K$ . If  $a \in L$  is a vertex of precisely two squares of  $L$  select the positive number  $s_a < r/2$  to be so small that  $\bar{D}(a, s_a) \subset \Omega$ . Let  $L_0$  denote the union of all the  $\bar{D}(a, s_a)$ 's. Then straightforward arguments show that  $\Omega_0 = \text{int}(L \cup L_0)$  is the desired Jordan domain.

Now let  $\Omega$  and  $K$  be as described in Theorem 2.1. Lemma 3.1

provides a Jordan domain  $\Omega_0$  such that  $K \subset \Omega_0 \subset \text{cl } \Omega_0 \subset \Omega$ . According to Theorem 2 page 237 of [2] there is a  $bh$ -embedding  $h$  of  $\Omega_0$  into  $D$  such that (1) the outer boundary of  $h(\Omega_0)$  is  $\partial D$  and (2)  $\partial[h(\Omega_0)]$  has finitely many components and each is a circle. Each of the circles bounding  $h(\Omega_0)$  can be "approximated" arbitrarily closely by a rational circle which lies in  $h(\Omega_0)$ . We require that the approximation to the unit circle be centered at 0. Since  $h(K)$  is a compact subset of  $h(\Omega_0)$ , when the approximations are close enough, the approximating circles will bound a domain which contains  $h(K)$ . This region, by its definition, is one of the  $E_j$ 's, say  $E_k$ . Then

$$h(K) \subset E_k \subset \phi_k^{-1}(B) \subset h(\Omega_0)$$

and so applying  $h^{-1}$  will establish Theorem 2.1.

To prove Corollary 2.2 we let  $B_1 = e_1(B)$  where  $e_1$  is the  $bh$ -embedding of  $B$  such that  $K \subset B_1 \subset \text{cl } B_1 \subset \Omega$ . For  $i > 1$  we let  $G_i$  be the component of  $[K + D(0, 1/(i-1))] \cap B_{i-1}$  which contains  $K$ , and we set  $B_i = e_i(B)$  where  $e_i$  is the  $bh$ -embedding of  $B$ , given by Theorem 2.1, such that  $K \subset B_i \subset \text{cl } B_i \subset G_i$ .

To prove Corollary 2.3 we pick  $a \in \Omega$  and for large  $n$  we can let  $G_n$  be the component of  $\{z: \text{dist}(z, C \setminus \Omega) > 1/n \text{ and } |z| < n\}$  which contains  $a$ . Since  $\text{cl } G_n$  is a compact subset of  $G_{n+1}$  there exists a  $bh$ -embedding  $e_n: B \rightarrow G_{n+1}$  such that  $B_n = e_n(B) \supset \text{cl } G_n$ . That  $\Omega = \bigcup G_n$  (and hence  $\Omega = \bigcup B_n$ ) follows from the arc connectedness of  $\Omega$ . These  $B_n$ 's are the required domains (except for re-indexing).

4. Some applications to holomorphic extension problems. Let  $K \subset C$  be compact and let  $f: K \rightarrow C$ . It is easy to extend  $f$  to a holomorphic function  $F$  defined on a domain containing  $K$  (caution: domains are connected) if there exist: (1) a domain  $\Omega$ , (2) a biholomorphic function  $e$  on  $\Omega$  such that  $K \subset e(\Omega)$ , and (3) a holomorphic extension  $G$  of  $g = f \circ e|_{e^{-1}(K)}$  to all of  $\Omega$ . Indeed  $F = G \circ e^{-1}$  is the required extension. Conversely if  $f$  has such an extension  $F$  the existence of  $\Omega$ ,  $e$ , and  $G$  is trivial. For let the domain  $\Omega$  be the domain of  $F$ , set  $e(z) = z$ , and take  $G = F$ . Thus we have an equivalent formulation of the problem of holomorphically extending a function  $f: K \rightarrow C$  to a domain containing  $K$ . Theorem 4.2 shows that another equivalent formulation is obtained when in the discussion above the variable domain  $\Omega$  is replaced by the fixed domain  $B$ . We first show that for a more restricted class of sets  $K$  this extension question is very naturally formulated with  $D$  in the role of  $\Omega$ .

**THEOREM 4.1.** *Let  $K \subset C$  be compact and let  $f: K \rightarrow C$ . Suppose*

that  $K$  and  $C \setminus K$  are connected. Then there exists a holomorphic extension  $F$  of  $f$  to a domain containing  $K$  if and only if there exist (a) a  $bh$ -embedding  $e$  of  $D$  such that  $K \subset e(D)$  and (b) a holomorphic extension  $G$  of  $g = f \circ e|_{e^{-1}(K)}$  to all of  $D$ .

*Proof.* Since the “if” part of this theorem is treated in the discussion above we confine our remarks to the “only if” part. Assume that the extension  $F$  exists, and let  $\Omega \supset K$  be its domain. It suffices to find a  $bh$ -mapping  $e$  of  $D$  such that  $K \subset e(D) \subset \Omega$ . This is trivial if  $K$  is a singleton: so we assume  $K$  is not a singleton. Then the Riemann Mapping theorem shows that  $\hat{C} \setminus K$  is  $bh$ -equivalent to  $D$  (it is simply connected because  $K$  is connected). Let  $h: \hat{C} \setminus K \rightarrow D$  be the Riemann mapping. Since  $h^{-1}(\bar{D}(0, r))$  is simply connected for  $0 < r < 1$  we know that  $V_r = \hat{C} \setminus h^{-1}(\bar{D}(0, r))$  is nonempty, open, and simply connected for  $0 < r < 1$ . Thus each  $V_r$  with  $0 < r < 1$  is  $bh$ -equivalent to  $D$ . Since  $h(\hat{C} \setminus \Omega)$  is a compact subset of  $D$  it lies in  $D(0, s)$  for some  $s < 1$ , and the Riemann mapping  $e$  of  $D$  onto  $V_s$  is the required map.

If in Theorem 4.1  $D$  is replaced by  $B$  the assumption that  $K$  and  $C \setminus K$  are connected may be dropped.

**THEOREM 4.2.** *Let  $K \subset C$  be compact and let  $f: K \rightarrow C$ . There exists a holomorphic extension  $F$  of  $f$  to a domain containing  $K$  if and only if there exist (a) a  $bh$ -mapping  $e$  of  $B$  such that  $K \subset e(B)$  and (b) a holomorphic extension  $G$  of  $g = f \circ e|_{e^{-1}(K)}$  to all of  $B$ .*

*Proof.* As in the proof of Theorem 4.1 the “if” part has already been settled and we begin the “only if” part by letting  $\Omega \supset K$  be the domain of  $F$ . An application of Theorem 2.1 gives a  $bh$ -embedding  $e$  of  $B$  such that  $K \subset e(B) \subset \Omega$ . This is the required mapping.

**REMARK.** Comparing Theorems 4.1 and 4.2 tempts one to conjecture the existence of a sequence of domains  $D = \Omega_1, \Omega_2, \dots, \Omega_\infty = B$  such that  $\hat{C} \setminus \Omega_n$  has  $n$  components and for which Theorem 4.1 will remain true when it is modified by: (1) Replacing its second sentence with “Suppose  $K$  is connected and  $C \setminus K$  has  $n$  components”, and (2) Replacing  $D$  with  $\Omega_n$ . The discussion in the introduction shows that this conjecture fails, since for  $n = 2$ ,  $\Omega_2$  must be  $bh$ -equivalent to  $A(r, s)$  for some  $r, s$  with  $0 \leq r < s \leq \infty$  and so  $\Omega_2$  cannot be embedded between  $A(1, t)$  (the domain of  $f$ ) and  $A(1 - \varepsilon, t + \varepsilon)$  (the domain of the extension  $F$ ) unless  $t < s/r < (t + \varepsilon)/(1 + \varepsilon)$ .

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