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Let X be a convergence space and $C(X)$ the R -algebra of all continuous real-valued functions on X , equipped with the continuous convergence structure. If the natural map from X into $C(C(X))$ is an embedding, then X is said to be a c -space. With each space X there is associated the c -modification cX which is a c -space with the property $C(X) = C(cX)$. This leads to the following theorems which are valid for any convergence space X : (1) $C(X)$ is a topological space iff cX is locally compact; (2) $C(X)$ is locally compact iff cX is finite.

1. Preliminaries. The continuous convergence structure on the function algebra $C(X)$ of continuous real-valued functions on a space X has been studied extensively by Binz and others during the past decade; see, for instance, [3], [4], [6], and [8]. This function space is typically called $C_c(X)$, but we will use instead the notation $C(X)$, assuming this space to be equipped with continuous convergence unless otherwise indicated. The term "space" will always mean "convergence space".

Let $\text{Hom } C(X)$ be the subspace of $C(C(X))$ consisting of all non-zero continuous homomorphisms on $C(X)$. X is c -embedded if the function $i_X: X \rightarrow \text{Hom } C(X)$ (defined by $i_X(x)(f) = f(x)$ for all f in $C(X)$) is a homeomorphism. We will use the term c -space in place of " c -embedded space"; this terminology is not only more concise, but avoids conflict with the different usage of the term " c -embedded" which is common in the literature.

Starting with a space X , let λX denote the *topological modification* and πX the *pretopological modification* of X . The symbol ωX will denote the *completely regular modification* of X , that is, the finest completely regular topological space on the same underlying set which is coarser than X . X will be called ω -Hausdorff if ωX is Hausdorff, and ω -regular if $\text{cl}_{\omega X} \mathcal{F} \rightarrow x$ whenever $\mathcal{F} \rightarrow x$. (Here, " $\mathcal{F} \rightarrow x$ in X " means "the filter \mathcal{F} converges to x in the space X "; cl_X designates the closure operator for X .) The following proposition will be useful later in the paper.

PROPOSITION 1.1. *A Hausdorff ω -regular space X is ω -Hausdorff.*

Proof. Let \mathcal{F} converge to x and y in ωX . Since $\dot{y} \geq \text{cl}_{\omega X} \mathcal{F}$, $\dot{y} \rightarrow x$ in ωX , and so $\dot{x} \geq \text{cl}_{\omega X} \dot{y}$. But $\dot{y} \rightarrow y$ in X , and, since X is ω -regular, $\dot{x} \rightarrow y$ in X . Therefore, $x = y$.

A space is said to be *pseudo-topological* if $\mathcal{F} \rightarrow x$ whenever each ultrafilter finer than \mathcal{F} converges to x . X is *locally compact* if every convergent filter contains a compact set, and *first countable* if every convergent filter contains a filter converging to the same point which has a countable filter base. The notation " $X \leq Y$ " for spaces X and Y will mean that X and Y have the same underlying set, and $\mathcal{F} \rightarrow x$ in Y implies $\mathcal{F} \rightarrow x$ in X .

A collection \mathcal{A} of subsets of a space X is called a *covering system* if each convergent filter contains a member of \mathcal{A} . If \mathcal{A} and \mathcal{A}_1 are covering systems such that each member of \mathcal{A}_1 is a subset of a member of \mathcal{A} , then \mathcal{A}_1 is said to be a *refinement* of \mathcal{A} .

PROPOSITION 1.2. *A subset A of a space X is compact iff, for each covering system \mathcal{A} of A , there is a refinement \mathcal{A}_1 of \mathcal{A} such that a finite subcollection of \mathcal{A}_1 covers A .*

A covering system \mathcal{B} for X will be called a *basic covering system* if, whenever $\mathcal{F} \rightarrow x$, there is a filter $\mathcal{G} \rightarrow x$ such that $\mathcal{G} \leq \mathcal{F}$ and \mathcal{G} has a filter base consisting of members of \mathcal{B} . Thus, if X is ω -regular, the set of all ωX -closed subsets of X forms a basic covering system for X .

The abbreviation "u.f." will be used for "ultrafilter". The symbol R will denote the real number system with its usual topology. The complement of a set A is written "Co A ", and the symbol \mathfrak{x} represents the fixed ultrafilter generated by $\{x\}$.

2. c -spaces. It is shown in [8] that X is a c -space iff X is Hausdorff, ω -regular, and solid (see [8] for the definition of a *solid space*). We will begin by showing that X is a c -space iff X is Hausdorff, ω -regular, and pseudo-topological.

PROPOSITION 2.1. *A c -space is Hausdorff, ω -regular, and pseudo-topological.*

Proof. A c -space must be Hausdorff in order for the natural function from X into $\text{Hom } C(X)$ to be injective. That a c -space is ω -regular is established in [6].

To show that X must be pseudo-topological, it is sufficient to show that $C(X)$ is pseudo-topological, since this property is hereditary. Let Λ be a filter on $C(X)$, and assume $\Omega \rightarrow f$ for all u.f.'s $\Omega \geq \Lambda$. Let $\mathcal{F} \rightarrow x$ in X . To show $\Lambda \rightarrow f$, it is enough to show that $\Lambda(\mathcal{F}) \geq \bigcap \{\Omega(\mathcal{F}) : \Omega \text{ an u.f., } \Omega \geq \Lambda\}$. We will do this by showing that if \mathcal{A} is an u.f. on R and $\mathcal{A} \geq \Lambda(\mathcal{F})$, then $\mathcal{A} \geq \Omega(\mathcal{F})$ for some u.f. $\Omega \geq \Lambda$.

Let $\mathcal{F} = \{\Sigma : \Sigma \text{ a filter on } C(X), \Sigma \supseteq \mathcal{A}, \text{ and } \mathcal{A} \supseteq \Sigma(\mathcal{F})\}$. A standard Zorn's lemma argument establishes that \mathcal{F} contains a maximal element; call it Ω .

To show that Ω is an u.f., assume that $A \cup B \in \Omega$. If neither A nor B is in Ω , then the filter Ω_A generated by $\{A \cap M : M \in \Omega\}$, and Ω_B generated by $\{B \cap M : M \in \Omega\}$ must both be proper filters on $C(X)$ which are strictly finer than Ω . Thus Ω_A and Ω_B must both fail to be in \mathcal{F} , and so there are sets M_1, M_2 in Ω and F_1 and F_2 in \mathcal{F} such that $(M_1 \cap A)(F_1) \notin \mathcal{A}$ and $(M_2 \cap B)(F_2) \notin \mathcal{A}$. Let $M = M_1 \cap M_2$ and $F = F_1 \cap F_2$. But $A \cup B \in \Omega$, and so $(M \cap (A \cup B))(F) \in \mathcal{A}$. However, $(M \cap (A \cup B))(F) \subseteq (M_1 \cap A)(F_1) \cup (M_2 \cap B)(F_2)$, and this contradicts the fact that \mathcal{A} is an ultrafilter. Thus either A or B must be in Ω .

For any ω -Hausdorff space X , define \tilde{X} to be the space on the same underlying set with convergence defined as follows: $\mathcal{F} \rightarrow x$ in \tilde{X} iff, whenever $A \rightarrow f$ in $C(X)$, $A(\mathcal{F}) \rightarrow f(x)$ in R .

PROPOSITION 2.2. *The following statements are valid for any ω -Hausdorff space X : (a) $\omega X \subseteq \tilde{X} \subseteq X$; (b) $C(\tilde{X}) = C(X)$; (c) \tilde{X} is the finest c -space coarser than X .*

Proof. Assertions (a) and (b) are obvious. It is also clear from the definition that \tilde{X} is c -embedded. If X_1 is c -embedded and $X_1 \subseteq X$, then the identity map from $X \rightarrow X_1$ is continuous. Thus the induced map $C(X_1) \rightarrow C(X) = C(\tilde{X})$ is continuous, and so is the induced map from $\text{Hom } C(\tilde{X}) \rightarrow \text{Hom } C(X_1)$. But \tilde{X} is homeomorphic to $\text{Hom } C(\tilde{X})$ and $\text{Hom } C(X_1)$ is homeomorphic to X_1 , which establishes $X_1 \subseteq \tilde{X}$.

LEMMA 2.3. *Let \mathcal{B} be a basic covering system for X , and let t be the topology on $C(X)$ with subbase $\{(B, W) : B \in \mathcal{B}, W \text{ open in } R\}$, where $(B, W) = \{g \in C(X) : g(B) \subseteq W\}$. Then $C(X) \subseteq C_t(X)$.*

Proof. Let $\Phi \rightarrow h$ in $C_t(X)$ and $\mathcal{G} \rightarrow y$ in X ; let W be an open neighborhood of $h(y)$ in R . Let $\mathcal{H} \rightarrow y$ in X such that $\mathcal{G} \supseteq \mathcal{H}$, and \mathcal{H} has a filter base in \mathcal{B} . Let $B \in \mathcal{B}$ be a basic set in \mathcal{H} such that $h(B) \subseteq W$. Then (B, W) is t -open, and hence $(B, W) \in \Phi$. But $(B, W)(B) \subseteq W$, and it follows that $\Phi(\mathcal{H}) \rightarrow h(y)$ in R . Thus $\Phi(\mathcal{G}) \rightarrow h(y)$ in R , and so $\Phi \rightarrow h$ in $C(X)$.

THEOREM 2.4. *X is a c -space iff X is Hausdorff, ω -regular, and pseudo-topological.*

Proof. Assume the three conditions. By Proposition 1.1, X is ω -Hausdorff. Thus \tilde{X} exists, and it is sufficient to show that $X = \tilde{X}$. Since X and \tilde{X} are both pseudo-topological, and $\tilde{X} \leq X$, we can complete the proof by showing that each u.f. which \tilde{X} -converges to x also X -converges to x .

Suppose \mathcal{G} is an u.f., $\mathcal{G} \rightarrow x$ in \tilde{X} , and $\mathcal{G} \rightarrow x$ in X . If \mathcal{H} is an \tilde{X} -convergent filter, then $\mathcal{G} \not\leq \text{cl}_{\omega_X} \mathcal{H}$, and so there is an ω_X -closed set H in \mathcal{H} such that $\text{Co } H \in \mathcal{G}$. The set of all such H 's forms a covering system \mathcal{A} for X , and the set \mathcal{B} of all ω_X -closed subsets of members of \mathcal{A} is a basic covering system for X consisting entirely of sets whose complements are in \mathcal{G} . Let $C_i(X)$ be the topological space derived from \mathcal{B} as in Lemma 2.3. Then, by the same lemma and Proposition 2.2, $C_i(X) \geq C(X) = C(\tilde{X})$.

Let f in $C(X)$ be defined by $f(x) = 0$, all x in X . Let \mathcal{A} be the t -neighborhood filter at f . Then $\mathcal{A} \rightarrow f$ in $C_i(X)$, which implies $\mathcal{A} \rightarrow f$ in $C(\tilde{X})$, and so $\mathcal{A}(\mathcal{G}) \rightarrow 0$ in R . Let W be a neighborhood of 0 not containing 1. Then there are sets G in \mathcal{G} and L in \mathcal{A} such that $L(G) \subseteq W$. L contains a set of the form $(A_1, V_1) \cap \cdots \cap (A_n, V_n)$, where each A_i is in \mathcal{B} and each V_i is an open neighborhood of 0 in R . Since $\text{Co } A_i \in \mathcal{G}$ for $i = 1, \dots, n$, we can choose $G_1 \subseteq G$ such that G_1 is in \mathcal{G} and $G_1 \cap (\bigcup A_i)$ is the empty set. Let z be any element of G_1 , and let g in $C(X)$ be constructed such that $g(z) = 1$ and $g(\cup A_i) = 0$. Then g is in (A_i, V_i) for all i , but $g(G)$ is not a subset of W . This contradiction establishes that $\mathcal{G} \rightarrow x$ in X , and the proof is complete.

For any space X , let $cX = \text{Hom } C(X)$. Then cX is a c -space, and we will refer to it as the c -modification of X . The next result can be easily verified.

- PROPOSITION 2.5. (a) $C(X)$ and $C(cX)$ are homeomorphic.
 (b) If X is ω -Hausdorff, then \tilde{X} and cX are homeomorphic.

When X is ω -Hausdorff, it is convenient to think of cX as coinciding with \tilde{X} . In general, the underlying set for cX can be thought of as consisting of equivalence classes relative to the following equivalence relation on X : $x \sim y$ iff $f(x) = f(y)$, for all f in $C(X)$.

For the purpose of studying $C(X)$, X can be replaced by the c -space cX . If one wishes to determine what properties of $C(X)$ are induced by given properties of X , it would naturally be of interest to know when a given property of X extends to cX . Two such properties are "Lindelof" and "second countable"; for definitions of these concepts in a convergence space setting, the reader is referred to [6].

PROPOSITION 2.6. *If X is Lindelof (second countable), then cX is Lindelof (second countable).*

Proof. In Theorem 1, [6], Feldman shows that $C(X)$ is first countable whenever X is Lindelof, and that X is Lindelof whenever X is a c -space and $C(X)$ is first countable. The assertion for "Lindelof" follows immediately from these results. The assertion for "second countable" can be proved similarly with the help of Theorem 2 of [6].

In the next section, we will show that cX is locally compact whenever X is locally compact. We conclude this section with a simple example which shows that X can be first countable when cX is not first countable.

EXAMPLE 2.7. Let Y be the interval $[0, 1]$ with the usual topology. Let X be the space with the same underlying set whose convergence to nonzero points is discrete, and with convergence to 0 defined as follows: $\mathcal{F} \rightarrow 0$ in X iff there is a free filter \mathcal{G} and a point y in Y such that: (1) \mathcal{G} is finer than the Y -neighborhood filter at y ; (2) $\mathcal{F} \supseteq \mathcal{G} \cap \dot{0}$. In other words, $\mathcal{F} \rightarrow 0$ in X means that \mathcal{F} is finer than the Y -neighborhood filter at 0, or else \mathcal{F} is finer than $\mathcal{N}(y)$ for some y in X , where $\mathcal{N}(y)$ is generated by sets of the form $(V - \{y\}) \cup \{0\}$ and V is a Y -neighborhood of y .

The space X is clearly compact and first countable. But cX (which turns out to be finest pseudo-topological space coarser than X) is homeomorphic to the one-point compactification of the interval $(0, 1]$ with the discrete topology, and so is not first countable.

3. Local compactness. In this section, we examine the consequences of assuming that either X or $C(X)$ is locally compact. Arens, [1], proved for a completely regular topological space X that $C(X)$ is a topological space iff X is locally compact. In Theorem 3.6, we show that Arens' theorem is valid in the larger class of ω -regular convergence spaces. We also show that when $C(X)$ is a topology, then $C(X)$ has the compact-open topology relative to cX , but not, in general, relative to X .

LEMMA 3.1. *Let X be a space, $\mathcal{F} \rightarrow x$ in X , and $\Phi \rightarrow f$ in $C(X)$. Let W be an open neighborhood of $f(x)$ in R .*

(a) *If F is a compact set in \mathcal{F} such that $f(F) \subseteq W$, then there is a set A in Φ such that $A(F) \subseteq W$.*

(b) *If A is a compact set in Φ and $A(x) \subseteq W$, then there is a set F in \mathcal{F} such that $A(F) \subseteq W$.*

Proof. The proofs of (a) and (b) are essentially the same, so we will prove only (b).

Let $\{A_i: i \in I\}$ be the collection of all filters on $C(X)$ which converge in $C(X)$ to a point in A . Let $A_i \rightarrow g$ in A ; then there are sets L_i in A_i and F_i in \mathcal{F} such that $L_i(F_i) \subseteq W$. The set $\{L_i: i \in I\}$ is a covering system for A which, by Proposition 1.2, reduces to a finite subcover L_1, \dots, L_n of A . Let F_1, \dots, F_n be the corresponding members of \mathcal{F} , and let $F = \bigcap F_k$. Then $(\bigcup L_k)(F) \subseteq W$, and so $A(F) \subseteq W$.

THEOREM 3.2. *If X is a locally compact space, then $C(X)$ is a topological space.*

Proof. $C(X)$ is known to be a convergence group, and it is also known (see [7], Theorem 5, §3) that a pretopological convergence group is topological. Thus we can complete the proof by showing that, whenever f is in $C(X)$, $\mathcal{F} \rightarrow x$ in X , and W is an open neighborhood of $f(x)$ in R , there is a neighborhood V of f in $C(X)$ and F in \mathcal{F} such that $V(F) \subseteq W$. Assume that W_1 is a closed neighborhood of $f(x)$ contained in W and F_0 a compact set in \mathcal{F} . Let $F_1 = F_0 \cap f^{-1}(W_1)$. Then F_1 is compact, F_1 is in \mathcal{F} , and $f(F_1) \subseteq W$, so that Lemma 3.1(a) can be applied to obtain, for each filter $A \rightarrow f$ in $C(X)$, a set $L_A \in A$ such that $L_A(F_1) \subseteq W_1$. If V is the union of these L_A 's, over all A 's converging to f in $C(X)$, then V is a neighborhood of f , and $V(F_1) \subseteq W$.

The preceding proof made use of Lemma 3.1(a) to show that $C(X) = \pi C(X)$ when X is locally compact. An analogous argument, based on Lemma 3.1(b), establishes the following result.

LEMMA 3.3. *Let X be any space such that $C(X)$ is locally compact. Then $C(X) = C(\pi X)$.*

PROPOSITION 3.4. *If $C(X)$ is a locally compact space, then $C(X) = C(\lambda X)$.*

Proof. In view of Lemma 3.3, it is sufficient to assume that X is pretopological and show that $C(X) = C(\lambda X)$. Let $\Phi \rightarrow f$ in $C(X)$ and $x \in X$. Let W be an open neighborhood of $f(x)$ in R . Select a compact set A in Φ and a neighborhood B of x such that $A(B) \subseteq W$. Given z in B , we can use Lemma 3.1 to find a neighborhood B_z of z such that $A(B_z) \subseteq W$. Let $B_1 = \bigcup \{B_z: z \text{ in } B\}$; then $A(B_1) \subseteq W$. Next, given z in B_1 , use Lemma 3.1 again to find a neighborhood C_z of z such that $A(C_z) \subseteq W$. Let $B_2 = \bigcup \{C_z: z \text{ in } B_1\}$; then $A(B_2) \subseteq W$.

Continue in this way to obtain B_n such that $A(B_n) \subseteq W$ for all natural numbers n . Let $U = \bigcup \{B_n: n \text{ a natural number}\}$. Then U is a λX -neighborhood of x and $A(U) \subseteq W$. It follows that $\Phi \rightarrow f$ in $C(\lambda X)$, and the proof is complete.

COROLLARY 3.5. (a) *If X is locally compact, then $C(X) = \lambda C(X)$.*
 (b) *If $C(X)$ is locally compact, then $C(X) = C(\lambda X)$.*

THEOREM 3.6. *If X is an ω -regular space, then $C(X)$ is a topological space iff X is locally compact.*

Proof. Assume that $C(X)$ is a topological space, and let f in $C(X)$ be the constant map $f(x) = 0$, all x in X . Let W be any open neighborhood of 0 in R not containing 1. Let $\mathcal{F} \rightarrow x$ in X ; since X is ω -regular, there is an ωX -closed set F_0 in \mathcal{F} such that, for some neighborhood U_0 of f in $C(X)$, $U_0(F_0) \subseteq W$. We will complete the proof by showing that F_0 is compact.

Let \mathcal{A} be a covering system for F_0 . Let $\mathcal{A}_1 = \mathcal{A} \cup \{\text{Co } F_0\} \cup \{B \cup \text{Co } F_0: B \in \mathcal{A}\}$; then \mathcal{A}_1 is a covering system for X . By Lemma 2, [6], we can replace \mathcal{A}_1 by a refinement \mathcal{A}_2 composed of ωX -closed sets. Let \mathcal{B} be the basic covering system for X obtained by adding to the collection \mathcal{A}_2 all ωX -closed subsets of members of \mathcal{A} . If t is the topology on $C(X)$ defined from \mathcal{B} as in Lemma 2.3, then $C(X) \subseteq C_t(X)$ follows from the same lemma. Thus there is a t -neighborhood U_1 of f , with $U_1 = \bigcap \{(F_k, W_k): k = 1, \dots, n\}$, such that $U_1 \subseteq U_0$, where the sets F_1, \dots, F_n are ωX -closed members of \mathcal{B} . To show that $F_0 \subseteq \bigcup F_k$, assume the contrary, and let $z \in F_0 - (\bigcup F_k)$. Then there is h in $C(X)$ such that $h(z) = 1$ and $h(\bigcup F_k) = 0$. This yields a contradiction, since h is in U_1 , a subset of U_0 , but $h(z) = 1$ implies $U_0(F_0) \not\subseteq W$.

To conclude that F_0 is compact, let $G_i = F_i \cap F_0$, $i = 1, \dots, n$, and let $\mathcal{A}^* = \{B \cap F_0: B \in \mathcal{A}_2\}$. Then \mathcal{A}^* is an ωX -closed refinement of \mathcal{A} , and each G_i is a subset of some member of \mathcal{A}^* . Since $F_0 \subseteq \bigcup G_k$, the compactness of F_0 follows from Proposition 1.2, and the proof is complete.

Even in the class of topological spaces, there are ω -regular spaces which are not completely regular; an example of such a space can be found in [5], page 85, Ex. 4.

COROLLARY 3.7. *If X is locally compact, then cX is locally compact.*

Proof. This follows because $C(X)$ is a topology (Theorem 3.2),

cX is ω -regular (Proposition 2.1), and $C(cX)$ is homeomorphic to $C(X)$ (Proposition 2.5).

An immediate consequence of Theorem 2.3 and Corollary 3.7 is

COROLLARY 3.8. *For any space X , $C(X)$ is a topological space iff cX is locally compact.*

$C(X)$ is said to have the compact-open topology relative to X if $C_i(X) = C(X)$, where $C_i(X)$ is the topological space derived, as in Lemma 2.3, from the collection \mathcal{B} of all compact subsets of X . When X is a completely regular topological space, then it is known that $C(X)$ has the compact-open topology relative to X whenever $C(X)$ is a topology. The situation for convergence spaces can be summarized as follows.

THEOREM 3.9. *Let $C(X)$ be a topological space.*

- (a) *If X is ω -regular, then $C(X)$ has the compact-open topology relative to X .*
- (b) *$C(X)$ always has the compact-open topology relative to cX .*

Proof. Both assertions follow from the fact that $C(X)$ has the compact-open topology relative to X whenever X is locally compact. Assume that X is locally compact, and let \mathcal{B} be the collection of all compact subsets of X ; let t be the compact-open topology on $C(X)$. $C(X) \subseteq C_i(X)$ follows from Lemma 2.3. If $\Phi \rightarrow f$ in $C(X)$ and $f \in (K, W)$, where $K \in \mathcal{B}$ and W is open in R , then the argument used in proving Lemma 3.1 can be applied to obtain a set A in Φ such that $A(K) \subseteq W$. Thus $A \subseteq (K, W)$, and $C(X) = C_i(X)$ is established.

It is not generally true that $C(X)$ has the compact-open topology relative to X whenever $C(X)$ is topological. One can obtain a counter-example by taking X to be the space of Theorem 6.21, [2].

THEOREM 3.10. *$C(X)$ is locally compact iff cX is finite.*

Proof. If cX is finite, then $C(X) = C(cX)$ is a finite dimensional topological linear space, and hence locally compact.

Conversely, assume that $C(X)$ is locally compact; for convenience, let $Y = cX$. Then Y is a completely regular topological space, since Y is a subspace of $C(C(X))$, which has the compact-open topology by Theorem 3.9.

Let $A = \{f \in C(X): |f(x)| \leq 1, \text{ for all } x \text{ in } X\}$. A is evidently

closed in $C(X)$. Note that the filter on $C(X)$ generated by $\{(1/n)A: n = 1, 2, \dots\}$ converges to the zero function; since $C(X)$ is locally compact, some set of the form $(1/n)A$ is compact, and it follows that A is compact in $C(X)$. We can also regard A as a subset of the product R^Y ; A will then be compact relative to the product topology on R^Y .

Assume that Y is infinite. If Y were discrete, then $C(Y) = R^Y$, and $C(Y)$ would not be locally compact. Thus some element y in Y has a neighborhood filter distinct from \dot{y} . For each open set V in the neighborhood filter at y , choose y_V in $V - \{y\}$ and f_V in A such that $f_V(y) = 1$ and $f_V(Y - V) = 0$. Since A is compact in R^Y , the net (f_V) has a pointwise-convergent subnet $(f_{V_\alpha})_{\alpha \in \mathcal{A}}$. Let f be the pointwise-convergent limit of this subnet. Then $(f_{V_\alpha}(y))_{\alpha \in \mathcal{A}} \rightarrow f(y) = 1$, and $(f_{V_\alpha}(y_{V_\beta}))_{\alpha \in \mathcal{A}} \rightarrow f(y_{V_\beta}) = 0$ for each $\beta \in \mathcal{A}$. But the net $(y_\beta)_{\beta \in \mathcal{A}} \rightarrow y$, and so f is not in $C(Y)$. But $f \in A \subseteq C(Y)$, since A is compact, a contradiction. It follows that $Y = cX$ is finite.

The preceding theorem and Corollary 3.8 imply that $C(X)$ is topological whenever $C(X)$ is locally compact. Combining this result with Proposition 3.4, we obtain the following.

COROLLARY 3.11. *If $C(X)$ is locally compact, then $C(X) = \lambda C(X) = C(\lambda X)$.*

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| | |
|---|-----|
| Harm Bart, <i>Spectral properties of locally holomorphic vector-valued functions</i> | 321 |
| J. Adrian (John) Bondy and Robert Louis Hemminger, <i>Reconstructing infinite graphs</i> | 331 |
| Bryan Edmund Cain and Richard J. Tondra, <i>Biholomorphic approximation of planar domains</i> | 341 |
| Richard Carey and Joel David Pincus, <i>Eigenvalues of seminormal operators, examples</i> | 347 |
| Tyrone Duncan, <i>Absolute continuity for abstract Wiener spaces</i> | 359 |
| Joe Wayne Fisher and Louis Halle Rowen, <i>An embedding of semiprime P.I.-rings</i> | 369 |
| Andrew S. Geue, <i>Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators</i> | 377 |
| Charles Lemuel Hagopian, <i>Locally homeomorphic λ connected plane continua</i> | 403 |
| Darald Joe Hartfiel, <i>A study of convex sets of stochastic matrices induced by probability vectors</i> | 405 |
| Yasunori Ishibashi, <i>Some remarks on high order derivations</i> | 419 |
| Donald Gordon James, <i>Orthogonal groups of dyadic unimodular quadratic forms. II</i> | 425 |
| Geoffrey Thomas Jones, <i>Projective pseudo-complemented semilattices</i> | 443 |
| Darrell Conley Kent, Kelly Denis McKennon, G. Richardson and M. Schroder, <i>Continuous convergence in $C(X)$</i> | 457 |
| J. J. Koliha, <i>Some convergence theorems in Banach algebras</i> | 467 |
| Tsang Hai Kuo, <i>Projections in the spaces of bounded linear operations</i> | 475 |
| George Berry Leeman, Jr., <i>A local estimate for typically real functions</i> | 481 |
| Andrew Guy Markoe, <i>A characterization of normal analytic spaces by the homological codimension of the structure sheaf</i> | 485 |
| Kunio Murasugi, <i>On the divisibility of knot groups</i> | 491 |
| John Phillips, <i>Perturbations of type I von Neumann algebras</i> | 505 |
| Billy E. Rhoades, <i>Commutants of some quasi-Hausdorff matrices</i> | 513 |
| David W. Roeder, <i>Category theory applied to Pontryagin duality</i> | 519 |
| Maxwell Alexander Rosenlicht, <i>The nonminimality of the differential closure</i> | 529 |
| Peter Michael Rosenthal, <i>On an inversion theorem for the general Mehler-Fock transform pair</i> | 539 |
| Alan Saleski, <i>Stopping times for Bernoulli automorphisms</i> | 547 |
| John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces. II</i> | 553 |
| Vashishtha Narayan Singh, <i>Reproducing kernels and operators with a cyclic vector. I</i> | 567 |
| Peggy Strait, <i>On the maximum and minimum of partial sums of random variables</i> | 585 |
| J. L. Brenner, <i>Maximal ideals in the near ring of polynomials modulo 2</i> | 595 |
| Ernst Gabor Straus, <i>Remark on the preceding paper: "Ideals in near rings of polynomials over a field"</i> | 601 |
| Masamichi Takesaki, <i>Faithful states on a C^*-algebra</i> | 605 |
| R. Michael Tanner, <i>Some content maximizing properties of the regular simplex</i> | 611 |
| Andrew Bao-hwa Wang, <i>An analogue of the Paley-Wiener theorem for certain function spaces on $SL(2, \mathbb{C})$</i> | 617 |
| James Juei-Chin Yeh, <i>Inversion of conditional expectations</i> | 631 |