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SOME CONVERGENCE THEOREMS IN BANACH ALGEBRAS

J. J. KOLIHA

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This paper is concerned with finding necessary and sufficient conditions for the convergence of the sequence $\{f_n(a)\}$ of elements of Banach algebra, where $\{f_n\}$ is a sequence of analytic functions imitating the behavior of the sequence of integral powers. In particular, it is shown that the sequence $\{a^n\}$ converges iff the spectrum of a (with the possible exception of the point $\lambda = 1$) lies in the open unit disc and $\lambda = 1$ is a pole of $(\lambda - a)^{-1}$ of order ≤ 1 .

The spectral characterization of power convergent operators on Hilbert (or Banach) spaces given in [3] can be extended to elements of Banach algebras, however, the methods of [3], based on the direct decomposition of the underlying space are no longer applicable. The main purpose of this note is to prove certain convergence theorems in a complex unital Banach algebra \mathcal{A} , which will yield, as a special case, the following result (cf. [3] for operator formulation):

THEOREM 0. *Let $a \in \mathcal{A}$. The sequence $\{a^n\}$ converges iff*

- (i) $\text{Sp}(a) - \{1\}$ lies in the open unit disc, and
- (ii) 1 is a pole of $(\lambda - a)^{-1}$ of order ≤ 1 .

($\text{Sp}(a)$ denotes the spectrum of the element $a \in \mathcal{A}$.) Rephrasing the theorem slightly, we may say that the sequence $\{f_n(a)\}$ converges in \mathcal{A} iff $\{f_n(\lambda)\}$ converges uniformly to zero on $\text{Sp}(a) - \{1\}$ and 1 is a pole of $(\lambda - a)^{-1}$ of order ≤ 1 , where $f_n(\lambda) = \lambda^n$. In the sequel, we shall consider functions more general than $f_n(\lambda) = \lambda^n$, employing the operational calculus in a Banach algebra (cf. [2, Chapter V] or [1, Chapter VII]).

A complex function f of complex variable will be called (in this paper) *power-like* if the following two conditions are fulfilled:

- (1) f is analytic in a disc $\Delta(f) = \{\lambda: |\lambda| < \delta\}$, $\delta > 1$,
- (2) $(1 - f(\lambda))(1 - \lambda)^{-1}$ has a removable singularity at $\lambda = 1$.

A sequence $\{f_n\}$ of power-like functions will be called *admissible for \mathcal{A}* if

- (3) $(1 - x)f_n(x) \rightarrow 0$ for each $x \in \mathcal{A}$ with $\text{Sp}(x) \subset \bigcap_n \Delta(f_n)$ and with $\{f_n(x)\}$ convergent,

and

$$(4) \quad f_n(0) \longrightarrow 0 .$$

We offer some examples of sequences of power-like functions admissible for any algebra \mathcal{A} :

(i) The very prototype of such sequences, the sequence $\{\lambda^n\}$ of integral powers of λ .

(ii) The sequence of Cesàro means of the integral powers,

$$\frac{1}{n}(1 + \lambda + \dots + \lambda^{n-1}) .$$

(iii) Let $\{\gamma_n\}$ be any sequence of complex numbers convergent to 0. We may define f_n inductively by one of the following formulae [5, Proposition 2.1]:

$$\begin{aligned} f_{n+1}(\lambda) &= (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n, & f_1(\lambda) &\equiv 1, \\ f_{n+1}(\lambda) &= (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n\lambda, & f_1(\lambda) &\equiv 1, \\ f_{n+1}(\lambda) &= ((1 - \gamma_n)\lambda + \gamma_n)f_n(\lambda), & f_1(\lambda) &\equiv 1. \end{aligned}$$

In each of the three formulae, f_n is a polynomial of the form

$$(5) \quad f_n(\lambda) = 1 + (\lambda - 1)g_n(\lambda) ,$$

where g_n is a polynomial of degree $\leq n - 2$.

We observe that, by virtue of (2), each power-like function f_n can be written in the form (5) with g_n analytic in $\mathcal{A}(f_n)$.

THEOREM 1. *Let $\{f_n\}$ be an admissible sequence of power-like functions, and let $\text{Sp}(a) \subset \bigcap_n \mathcal{A}(f_n)$. Then $\{f_n(a)\}$ converges iff*

$$(6) \quad a = p + c ,$$

where

$$(7) \quad p^2 = p, \quad pc = cp = 0, \quad f_n(c) \longrightarrow 0 .$$

Proof. Suppose first that $f_n(a) \rightarrow p$. Then $(1 - a)p = p(1 - a) = 0$ in view of (3), and $ap = pa = p$. More generally,

$$(8) \quad a^k p = p a^k = p, \quad k \geq 0 .$$

For each complex $\lambda \notin \text{Sp}(a) \cup \{1\}$,

$$(9) \quad (\lambda - a)^{-1} p = (\lambda - 1)^{-1} p .$$

This shows that $p = 0$ whenever $\lambda = 1$ is a regular point for $(\lambda - a)^{-1}$. Let C_n be a contour in $\mathcal{A}(f_n)$ enclosing $\text{Sp}(a) \cup \{1\}$. (C_n is a boundary of an open set $U_n(\supset \text{Sp}(a) \cup \{1\})$ consisting of a finite number

of closed rectifiable Jordan curves positively oriented with respect to U_n .) Then

$$\begin{aligned} pf_n(a) &= \frac{1}{2\pi i} \int_{c_n} f_n(\lambda)(\lambda - a)^{-1} p d\lambda \\ &= \frac{p}{2\pi i} \int_{c_n} f_n(\lambda)(\lambda - 1)^{-1} d\lambda = pf_n(1) = p ; \end{aligned}$$

we have used (9), and then (5) to get $f_n(1) = 1$. Consequently,

$$p^2 = p \lim_{n \rightarrow \infty} f_n(a) = \lim_{n \rightarrow \infty} pf_n(a) = p .$$

More generally, $p^k = p$ for each $k \geq 1$, and induction (utilizing (8)) yields

$$(10) \quad (a - p)^k = a^k - p , \quad k \geq 1 .$$

Let us write α_{nk} for $f_n^{(k)}(0)/k!$, and set $c = a - p$. Then

$$\begin{aligned} f_n(a) - (1 - f_n(0))p &= \sum_{k=0}^{\infty} \alpha_{nk} a^k - \left[\sum_{k=1}^{\infty} \alpha_{nk} \right] p \\ &= \sum_{k=0}^{\infty} \alpha_{nk} (a - p)^k = f_n(a - p) = f_n(c) , \end{aligned}$$

using the analyticity of f_n on $\Delta(f_n)(\supset \text{Sp}(a))$, and the identity (10). Therefore, $f_n(c)$ is defined, and

$$f_n(c) = (f_n(a) - p) + f_n(0)p \longrightarrow 0$$

by virtue of (4). Finally,

$$cp = pc = p(a - p) = pa - p^2 = 0 .$$

Assume, conversely, that (6) and (7) hold. Then

$$a^k = (p + c)^k = p + c^k ,$$

and

$$\begin{aligned} f_n(a) = f_n(p + c) &= \sum_{k=0}^{\infty} \alpha_{nk} (p + c)^k = f_n(0) + \sum_{k=1}^{\infty} \alpha_{nk} c^k + \left[\sum_{k=1}^{\infty} \alpha_{nk} \right] p \\ &= f_n(c) + (1 - f_n(0))p \longrightarrow p \quad \text{as } n \longrightarrow \infty . \end{aligned}$$

If $f_n(\lambda) = \lambda^n$ in the preceding theorem, we obtain the following result.

COROLLARY. $\{a^n\}$ converges iff $a = p + c$, where

$$p^2 = p , \quad pc = cp = 0 , \quad \lim_{n \rightarrow \infty} \|c^n\|^{1/n} < 1 .$$

The following theorem gives a sufficient condition for the con-

vergence of $\{f_n(a)\}$ if $\{f_n\}$ is an admissible sequence of power-like functions. A brief glance at Theorem 3 will tell the reader how far this condition is from being also necessary. The proof of the theorem could be based on our Theorem 1, on Theorem 5.5.1 [1, p. 174], and on Theorem VII.3.22 [2, p. 576]. We give a direct proof which appears to be fairly simple and straightforward.

THEOREM 2. *Let $\{f_n\}$ be an admissible sequence of power-like functions. If*

(i) *all f_n are analytic and uniformly convergent to zero on a fixed open neighborhood Ω of $\text{Sp}(a) - \{1\}$,*

and

(ii) *1 is a pole of $(\lambda - a)^{-1}$ of order ≤ 1 ,*

then $\{f_n(a)\}$ converges.

Proof. For a certain $\delta > 0$,

$$(11) \quad (\lambda - a)^{-1} = (\lambda - 1)^{-1}p + h(\lambda), \quad 0 < |\lambda - 1| < \delta,$$

where h is analytic in an open neighborhood of $\text{Sp}(a)$. We can select a contour C in Ω enclosing $\text{Sp}(a) - \{1\}$, and for each n we can find a positively oriented circle $C_n = \{\lambda: |\lambda - 1| = \varepsilon < \delta\}$ that misses C and such that f_n is analytic in an open neighborhood of C_n . Using (11), we get

$$\begin{aligned} f_n(a) - p &= \frac{1}{2\pi i} \int_{C+C_n} f_n(\lambda)(\lambda - a)^{-1} d\lambda - \frac{1}{2\pi i} \int_{C_n} (\lambda - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1} d\lambda + \frac{1}{2\pi i} \int_{C_n} (f_n(\lambda) - 1)(\lambda - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1} d\lambda + \frac{p}{2\pi i} \int_{C_n} g_n(\lambda) d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{C_n} (f_n(\lambda) - 1)h(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1} d\lambda, \end{aligned}$$

where g_n is specified in (5). Hence

$$\|f_n(a) - p\| \leq \frac{1}{2\pi} \sup_{\lambda \in C} \|f_n(\lambda)(\lambda - a)^{-1}\| \cdot l(C) \leq K \sup_{\lambda \in \Omega} |f_n(\lambda)|,$$

with

$$K = \frac{l(C)}{2\pi} \sup_{\lambda \in C} \|(\lambda - a)^{-1}\| < +\infty, \quad l(C) \text{ the length of } C.$$

This gives $f_n(a) \rightarrow p$, and completes the proof.

Theorem 2 has a partial converse which will be proved after the following two auxiliary results.

LEMMA 1. *If $x_n x \rightarrow 1$ and $xx_n \rightarrow 1$, then x is invertible, and $x_n \rightarrow x^{-1}$.*

Proof. Let N be a fixed positive integer such that

$$\|1 - x_N x\| < \frac{1}{2}.$$

For each $\varepsilon > 0$ we can find a positive integer n_0 such that

$$\|xx_n - xx_m\| < \varepsilon/(2\|x_N\|) \quad \text{whenever } n, m > n_0.$$

Since

$$x_n - x_m = (1 - x_N x)(x_n - x_m) + x_N(xx_n - xx_m),$$

we get

$$\|x_n - x_m\| < \frac{1}{2}\|x_n - x_m\| + \frac{1}{2}\varepsilon,$$

and

$$\|x_n - x_m\| < \varepsilon \quad \text{whenever } n, m > n_0.$$

Hence $x_n \rightarrow y$ for some $y \in \mathcal{A}$, and $yx = xy = 1$.

LEMMA 2. *Let $\{f_n\}$ be an arbitrary sequence of power-like functions with $\bigcap_n \Delta(f_n) \supset \text{Sp}(c)$. If $f_n(c) \rightarrow 0$, then 1 is a regular point for $(\lambda - c)^{-1}$, and*

$$g_n(c) \longrightarrow (1 - c)^{-1},$$

with g_n defined in (5).

Proof. If $f_n(c) \rightarrow 0$, then

$$g_n(c)(1 - c) = (1 - c)g_n(c) \longrightarrow 1.$$

The result follows on taking $x_n = g_n(c)$ and $x = 1 - c$ in Lemma 1.

A special case of Lemma 1 for the algebra of bounded linear operators on a Banach space and with f_n polynomials of a certain form has been proved in [4, Proposition 5]. A particularly simple form of Lemma 2 is the following well known result: If $c^n \rightarrow 0$, then the series $\sum_n c^n$ converges to $(1 - c)^{-1}$. Also:

$$n^{-1}c^n + 1 \longrightarrow 0 \implies n^{-1}(1 + c + \cdots + c^{n-1}) \longrightarrow (1 - c)^{-1},$$

$$n^{-1}(1 + c + \cdots + c^{n-1}) \longrightarrow 0 \implies \sum_{k=0}^{n-2} \frac{n-k-1}{n} c^k \longrightarrow (1-c)^{-1},$$

etc.

THEOREM 3. *Let $\{f_n\}$ be an admissible sequence of power-like functions with $\bigcap_n \Delta(f_n) \supset \text{Sp}(a)$. If $\{f_n(a)\}$ converges, then*

(i) $f_n(\lambda) \rightarrow 0$ uniformly on $\text{Sp}(a) - \{1\}$,

and

(ii) 1 is a pole of $(\lambda - a)^{-1}$ of order ≤ 1 .

Proof. Suppose $f_n(a) \rightarrow p$. The elements p and $c = a - p$ satisfy the conditions (6) and (7), in particular, $f_n(c) \rightarrow 0$. By Lemma 2, 1 is a regular point for $(\lambda - c)^{-1}$, and hence the function

$$h(\lambda) = (\lambda - c)^{-1}(1 - p)$$

is analytic in a certain open neighborhood of 1. The function

$$u(\lambda) = h(\lambda) + (\lambda - 1)^{-1}p$$

has a pole of order ≤ 1 at $\lambda = 1$. The elements $\lambda - a$ and $u(\lambda)$ commute (whenever the latter is defined). Moreover,

$$\begin{aligned} (\lambda - a)u(\lambda) &= (\lambda - a)(\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}(\lambda - a)p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c) + p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c + (\lambda - c)p) \\ &= (\lambda - c)^{-1}(\lambda - c) \\ &= 1. \end{aligned}$$

Hence, $u(\lambda) = (\lambda - a)^{-1}$, and

$$(12) \quad (\lambda - a)^{-1} = (\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}p.$$

The identity (12) shows that

$$\text{Sp}(a) - \{1\} = \text{Sp}(c).$$

Finally, $f_n(c) \rightarrow 0$ implies $f_n(\lambda) \rightarrow 0$ uniformly on $\text{Sp}(c)$ [1, p. 584], and the proof is complete.

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