

# Pacific Journal of Mathematics

**SOME CONVERGENCE THEOREMS IN BANACH ALGEBRAS**

J. J. KOLIHA

## SOME CONVERGENCE THEOREMS IN BANACH ALGEBRAS

J. J. KOLIHA

**This paper is concerned with finding necessary and sufficient conditions for the convergence of the sequence  $\{f_n(a)\}$  of elements of Banach algebra, where  $\{f_n\}$  is a sequence of analytic functions imitating the behavior of the sequence of integral powers. In particular, it is shown that the sequence  $\{a^n\}$  converges iff the spectrum of  $a$  (with the possible exception of the point  $\lambda = 1$ ) lies in the open unit disc and  $\lambda = 1$  is a pole of  $(\lambda - a)^{-1}$  of order  $\leq 1$ .**

The spectral characterization of power convergent operators on Hilbert (or Banach) spaces given in [3] can be extended to elements of Banach algebras, however, the methods of [3], based on the direct decomposition of the underlying space are no longer applicable. The main purpose of this note is to prove certain convergence theorems in a complex unital Banach algebra  $\mathcal{A}$ , which will yield, as a special case, the following result (cf. [3] for operator formulation):

**THEOREM 0.** *Let  $a \in \mathcal{A}$ . The sequence  $\{a^n\}$  converges iff*

- (i)  $\text{Sp}(a) - \{1\}$  lies in the open unit disc, and
- (ii) 1 is a pole of  $(\lambda - a)^{-1}$  of order  $\leq 1$ .

( $\text{Sp}(a)$  denotes the spectrum of the element  $a \in \mathcal{A}$ .) Rephrasing the theorem slightly, we may say that the sequence  $\{f_n(a)\}$  converges in  $\mathcal{A}$  iff  $\{f_n(\lambda)\}$  converges uniformly to zero on  $\text{Sp}(a) - \{1\}$  and 1 is a pole of  $(\lambda - a)^{-1}$  of order  $\leq 1$ , where  $f_n(\lambda) = \lambda^n$ . In the sequel, we shall consider functions more general than  $f_n(\lambda) = \lambda^n$ , employing the operational calculus in a Banach algebra (cf. [2, Chapter V] or [1, Chapter VII]).

A complex function  $f$  of complex variable will be called (in this paper) *power-like* if the following two conditions are fulfilled:

- (1)  $f$  is analytic in a disc  $\Delta(f) = \{\lambda: |\lambda| < \delta\}$ ,  $\delta > 1$ ,
- (2)  $(1 - f(\lambda))(1 - \lambda)^{-1}$  has a removable singularity at  $\lambda = 1$ .

A sequence  $\{f_n\}$  of power-like functions will be called *admissible for  $\mathcal{A}$*  if

- (3)  $(1 - x)f_n(x) \rightarrow 0$  for each  $x \in \mathcal{A}$  with  $\text{Sp}(x) \subset \bigcap_n \Delta(f_n)$  and with  $\{f_n(x)\}$  convergent,

and

$$(4) \quad f_n(0) \longrightarrow 0.$$

We offer some examples of sequences of power-like functions admissible for any algebra  $\mathcal{A}$ :

(i) The very prototype of such sequences, the sequence  $\{\lambda^n\}$  of integral powers of  $\lambda$ .

(ii) The sequence of Cesàro means of the integral powers,

$$\frac{1}{n}(1 + \lambda + \dots + \lambda^{n-1}).$$

(iii) Let  $\{\gamma_n\}$  be any sequence of complex numbers convergent to 0. We may define  $f_n$  inductively by one of the following formulae [5, Proposition 2.1]:

$$\begin{aligned} f_{n+1}(\lambda) &= (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n, & f_1(\lambda) &\equiv 1, \\ f_{n+1}(\lambda) &= (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n\lambda, & f_1(\lambda) &\equiv 1, \\ f_{n+1}(\lambda) &= ((1 - \gamma_n)\lambda + \gamma_n)f_n(\lambda), & f_1(\lambda) &\equiv 1. \end{aligned}$$

In each of the three formulae,  $f_n$  is a polynomial of the form

$$(5) \quad f_n(\lambda) = 1 + (\lambda - 1)g_n(\lambda),$$

where  $g_n$  is a polynomial of degree  $\leq n - 2$ .

We observe that, by virtue of (2), each power-like function  $f_n$  can be written in the form (5) with  $g_n$  analytic in  $\Delta(f_n)$ .

**THEOREM 1.** *Let  $\{f_n\}$  be an admissible sequence of power-like functions, and let  $\text{Sp}(a) \subset \bigcap_n \Delta(f_n)$ . Then  $\{f_n(a)\}$  converges iff*

$$(6) \quad a = p + c,$$

where

$$(7) \quad p^2 = p, \quad pc = cp = 0, \quad f_n(c) \longrightarrow 0.$$

*Proof.* Suppose first that  $f_n(a) \rightarrow p$ . Then  $(1 - a)p = p(1 - a) = 0$  in view of (3), and  $ap = pa = p$ . More generally,

$$(8) \quad a^k p = p a^k = p, \quad k \geq 0.$$

For each complex  $\lambda \notin \text{Sp}(a) \cup \{1\}$ ,

$$(9) \quad (\lambda - a)^{-1} p = (\lambda - 1)^{-1} p.$$

This shows that  $p = 0$  whenever  $\lambda = 1$  is a regular point for  $(\lambda - a)^{-1}$ . Let  $C_n$  be a contour in  $\Delta(f_n)$  enclosing  $\text{Sp}(a) \cup \{1\}$ . ( $C_n$  is a boundary of an open set  $U_n \supset \text{Sp}(a) \cup \{1\}$  consisting of a finite number

of closed rectifiable Jordan curves positively oriented with respect to  $U_n$ .) Then

$$\begin{aligned} pf_n(a) &= \frac{1}{2\pi i} \int_{c_n} f_n(\lambda)(\lambda - a)^{-1} p d\lambda \\ &= \frac{p}{2\pi i} \int_{c_n} f_n(\lambda)(\lambda - 1)^{-1} d\lambda = pf_n(1) = p ; \end{aligned}$$

we have used (9), and then (5) to get  $f_n(1) = 1$ . Consequently,

$$p^2 = p \lim_{n \rightarrow \infty} f_n(a) = \lim_{n \rightarrow \infty} pf_n(a) = p .$$

More generally,  $p^k = p$  for each  $k \geq 1$ , and induction (utilizing (8)) yields

$$(10) \quad (a - p)^k = a^k - p, \quad k \geq 1 .$$

Let us write  $\alpha_{nk}$  for  $f_n^{(k)}(0)/k!$ , and set  $c = a - p$ . Then

$$\begin{aligned} f_n(a) - (1 - f_n(0))p &= \sum_{k=0}^{\infty} \alpha_{nk} a^k - \left[ \sum_{k=1}^{\infty} \alpha_{nk} \right] p \\ &= \sum_{k=0}^{\infty} \alpha_{nk} (a - p)^k = f_n(a - p) = f_n(c) , \end{aligned}$$

using the analyticity of  $f_n$  on  $\Delta(f_n) (\supset \text{Sp}(a))$ , and the identity (10). Therefore,  $f_n(c)$  is defined, and

$$f_n(c) = (f_n(a) - p) + f_n(0)p \longrightarrow 0$$

by virtue of (4). Finally,

$$cp = pc = p(a - p) = pa - p^2 = 0 .$$

Assume, conversely, that (6) and (7) hold. Then

$$a^k = (p + c)^k = p + c^k ,$$

and

$$\begin{aligned} f_n(a) = f_n(p + c) &= \sum_{k=0}^{\infty} \alpha_{nk} (p + c)^k = f_n(0) + \sum_{k=1}^{\infty} \alpha_{nk} c^k + \left[ \sum_{k=1}^{\infty} \alpha_{nk} \right] p \\ &= f_n(c) + (1 - f_n(0))p \longrightarrow p \quad \text{as } n \longrightarrow \infty . \end{aligned}$$

If  $f_n(\lambda) = \lambda^n$  in the preceding theorem, we obtain the following result.

COROLLARY.  $\{a^n\}$  converges iff  $a = p + c$ , where

$$p^2 = p, \quad pc = cp = 0, \quad \lim_{n \rightarrow \infty} \|c^n\|^{1/n} < 1 .$$

The following theorem gives a sufficient condition for the con-

vergence of  $\{f_n(a)\}$  if  $\{f_n\}$  is an admissible sequence of power-like functions. A brief glance at Theorem 3 will tell the reader how far this condition is from being also necessary. The proof of the theorem could be based on our Theorem 1, on Theorem 5.5.1 [1, p. 174], and on Theorem VII.3.22 [2, p. 576]. We give a direct proof which appears to be fairly simple and straightforward.

**THEOREM 2.** *Let  $\{f_n\}$  be an admissible sequence of power-like functions. If*

(i) *all  $f_n$  are analytic and uniformly convergent to zero on a fixed open neighborhood  $\Omega$  of  $\text{Sp}(a) - \{1\}$ ,*

*and*

(ii) *1 is a pole of  $(\lambda - a)^{-1}$  of order  $\leq 1$ ,*  
*then  $\{f_n(a)\}$  converges.*

*Proof.* For a certain  $\delta > 0$ ,

$$(11) \quad (\lambda - a)^{-1} = (\lambda - 1)^{-1}p + h(\lambda), \quad 0 < |\lambda - 1| < \delta,$$

where  $h$  is analytic in an open neighborhood of  $\text{Sp}(a)$ . We can select a contour  $C$  in  $\Omega$  enclosing  $\text{Sp}(a) - \{1\}$ , and for each  $n$  we can find a positively oriented circle  $C_n = \{\lambda: |\lambda - 1| = \varepsilon < \delta\}$  that misses  $C$  and such that  $f_n$  is analytic in an open neighborhood of  $C_n$ . Using (11), we get

$$\begin{aligned} f_n(a) - p &= \frac{1}{2\pi i} \int_{C+C_n} f_n(\lambda)(\lambda - a)^{-1}d\lambda - \frac{1}{2\pi i} \int_{C_n} (\lambda - a)^{-1}d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda + \frac{1}{2\pi i} \int_{C_n} (f_n(\lambda) - 1)(\lambda - a)^{-1}d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda + \frac{p}{2\pi i} \int_{C_n} g_n(\lambda)d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{C_n} (f_n(\lambda) - 1)h(\lambda)d\lambda \\ &= \frac{1}{2\pi i} \int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda, \end{aligned}$$

where  $g_n$  is specified in (5). Hence

$$\|f_n(a) - p\| \leq \frac{1}{2\pi} \sup_{\lambda \in C} \|f_n(\lambda)(\lambda - a)^{-1}\| \cdot l(C) \leq K \sup_{\lambda \in \Omega} |f_n(\lambda)|,$$

with

$$K = \frac{l(C)}{2\pi} \sup_{\lambda \in C} \|(\lambda - a)^{-1}\| < +\infty, \quad l(C) \text{ the length of } C.$$

This gives  $f_n(a) \rightarrow p$ , and completes the proof.

Theorem 2 has a partial converse which will be proved after the following two auxiliary results.

**LEMMA 1.** *If  $x_n x \rightarrow 1$  and  $xx_n \rightarrow 1$ , then  $x$  is invertible, and  $x_n \rightarrow x^{-1}$ .*

*Proof.* Let  $N$  be a fixed positive integer such that

$$\|1 - x_N x\| < \frac{1}{2}.$$

For each  $\varepsilon > 0$  we can find a positive integer  $n_0$  such that

$$\|xx_n - xx_m\| < \varepsilon/(2\|x_N\|) \quad \text{whenever } n, m > n_0.$$

Since

$$x_n - x_m = (1 - x_N x)(x_n - x_m) + x_N(xx_n - xx_m),$$

we get

$$\|x_n - x_m\| < \frac{1}{2}\|x_n - x_m\| + \frac{1}{2}\varepsilon,$$

and

$$\|x_n - x_m\| < \varepsilon \quad \text{whenever } n, m > n_0.$$

Hence  $x_n \rightarrow y$  for some  $y \in \mathcal{A}$ , and  $yx = xy = 1$ .

**LEMMA 2.** *Let  $\{f_n\}$  be an arbitrary sequence of power-like functions with  $\bigcap_n \Delta(f_n) \supset \text{Sp}(c)$ . If  $f_n(c) \rightarrow 0$ , then 1 is a regular point for  $(\lambda - c)^{-1}$ , and*

$$g_n(c) \longrightarrow (1 - c)^{-1},$$

with  $g_n$  defined in (5).

*Proof.* If  $f_n(c) \rightarrow 0$ , then

$$g_n(c)(1 - c) = (1 - c)g_n(c) \longrightarrow 1.$$

The result follows on taking  $x_n = g_n(c)$  and  $x = 1 - c$  in Lemma 1.

A special case of Lemma 1 for the algebra of bounded linear operators on a Banach space and with  $f_n$  polynomials of a certain form has been proved in [4, Proposition 5]. A particularly simple form of Lemma 2 is the following well known result: If  $c^n \rightarrow 0$ , then the series  $\sum_n c^n$  converges to  $(1 - c)^{-1}$ . Also:

$$n^{-1}c^n + 1 \longrightarrow 0 \implies n^{-1}(1 + c + \dots + c^{n-1}) \longrightarrow (1 - c)^{-1},$$

$$n^{-1}(1 + c + \cdots + c^{n-1}) \longrightarrow 0 \implies \sum_{k=0}^{n-2} \frac{n-k-1}{n} c^k \longrightarrow (1-c)^{-1},$$

etc.

**THEOREM 3.** *Let  $\{f_n\}$  be an admissible sequence of power-like functions with  $\bigcap_n \Delta(f_n) \supset \text{Sp}(a)$ . If  $\{f_n(a)\}$  converges, then*

(i)  $f_n(\lambda) \rightarrow 0$  uniformly on  $\text{Sp}(a) - \{1\}$ ,

and

(ii) 1 is a pole of  $(\lambda - a)^{-1}$  of order  $\leq 1$ .

*Proof.* Suppose  $f_n(a) \rightarrow p$ . The elements  $p$  and  $c = a - p$  satisfy the conditions (6) and (7), in particular,  $f_n(c) \rightarrow 0$ . By Lemma 2, 1 is a regular point for  $(\lambda - c)^{-1}$ , and hence the function

$$h(\lambda) = (\lambda - c)^{-1}(1 - p)$$

is analytic in a certain open neighborhood of 1. The function

$$u(\lambda) = h(\lambda) + (\lambda - 1)^{-1}p$$

has a pole of order  $\leq 1$  at  $\lambda = 1$ . The elements  $\lambda - a$  and  $u(\lambda)$  commute (whenever the latter is defined). Moreover,

$$\begin{aligned} (\lambda - a)u(\lambda) &= (\lambda - a)(\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}(\lambda - a)p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c) + p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c + (\lambda - c)p) \\ &= (\lambda - c)^{-1}(\lambda - c) \\ &= 1. \end{aligned}$$

Hence,  $u(\lambda) = (\lambda - a)^{-1}$ , and

$$(12) \quad (\lambda - a)^{-1} = (\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}p.$$

The identity (12) shows that

$$\text{Sp}(a) - \{1\} = \text{Sp}(c).$$

Finally,  $f_n(c) \rightarrow 0$  implies  $f_n(\lambda) \rightarrow 0$  uniformly on  $\text{Sp}(c)$  [1, p. 584], and the proof is complete.

#### REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear Operators I*, Interscience, New York, 1957.
2. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, AMS Colloquium Publications XXXI, AMS, Providence, Rhode Island, 1957.
3. J. J. Koliha, *Convergent and stable operators and their generalization*, J. Math. Anal. Appl., **43** (1973), 778-794.

4. J. J. Koliha, *Ergodic theory and averaging iterations*, *Canad. J. Math.*, **25** (1973), 14-23.
5. ———, *The solution of linear equations in normed spaces by averaging iteration*, *SIAM J. Math. Anal.*, **5** (1974).

Received August 7, 1973 and in revised form November 28, 1973.

UNIVERSITY OF MELBOURNE





# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunkin Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1973 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

Harm Bart, <i>Spectral properties of locally holomorphic vector-valued functions</i> . . . . .	321
J. Adrian (John) Bondy and Robert Louis Hemminger, <i>Reconstructing infinite graphs</i> . . . . .	331
Bryan Edmund Cain and Richard J. Tondra, <i>Biholomorphic approximation of planar domains</i> . . . . .	341
Richard Carey and Joel David Pincus, <i>Eigenvalues of seminormal operators, examples</i> . . . . .	347
Tyrone Duncan, <i>Absolute continuity for abstract Wiener spaces</i> . . . . .	359
Joe Wayne Fisher and Louis Halle Rowen, <i>An embedding of semiprime P.I.-rings</i> . . . . .	369
Andrew S. Geue, <i>Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators</i> . . . . .	377
Charles Lemuel Hagopian, <i>Locally homeomorphic <math>\lambda</math> connected plane continua</i> . . . . .	403
Darald Joe Hartfiel, <i>A study of convex sets of stochastic matrices induced by probability vectors</i> . . . . .	405
Yasunori Ishibashi, <i>Some remarks on high order derivations</i> . . . . .	419
Donald Gordon James, <i>Orthogonal groups of dyadic unimodular quadratic forms. II</i> . . . . .	425
Geoffrey Thomas Jones, <i>Projective pseudo-complemented semilattices</i> . . . . .	443
Darrell Conley Kent, Kelly Denis McKennon, G. Richardson and M. Schroder, <i>Continuous convergence in <math>C(X)</math></i> . . . . .	457
J. J. Koliha, <i>Some convergence theorems in Banach algebras</i> . . . . .	467
Tsang Hai Kuo, <i>Projections in the spaces of bounded linear operations</i> . . . . .	475
George Berry Leeman, Jr., <i>A local estimate for typically real functions</i> . . . . .	481
Andrew Guy Markoe, <i>A characterization of normal analytic spaces by the homological codimension of the structure sheaf</i> . . . . .	485
Kunio Murasugi, <i>On the divisibility of knot groups</i> . . . . .	491
John Phillips, <i>Perturbations of type I von Neumann algebras</i> . . . . .	505
Billy E. Rhoades, <i>Commutants of some quasi-Hausdorff matrices</i> . . . . .	513
David W. Roeder, <i>Category theory applied to Pontryagin duality</i> . . . . .	519
Maxwell Alexander Rosenlicht, <i>The nonminimality of the differential closure</i> . . . . .	529
Peter Michael Rosenthal, <i>On an inversion theorem for the general Mehler-Fock transform pair</i> . . . . .	539
Alan Saleski, <i>Stopping times for Bernoulli automorphisms</i> . . . . .	547
John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces. II</i> . . . . .	553
Vashishtha Narayan Singh, <i>Reproducing kernels and operators with a cyclic vector. I</i> . . . . .	567
Peggy Strait, <i>On the maximum and minimum of partial sums of random variables</i> . . . . .	585
J. L. Brenner, <i>Maximal ideals in the near ring of polynomials modulo 2</i> . . . . .	595
Ernst Gabor Straus, <i>Remark on the preceding paper: "Ideals in near rings of polynomials over a field"</i> . . . . .	601
Masamichi Takesaki, <i>Faithful states on a <math>C^*</math>-algebra</i> . . . . .	605
R. Michael Tanner, <i>Some content maximizing properties of the regular simplex</i> . . . . .	611
Andrew Bao-hwa Wang, <i>An analogue of the Paley-Wiener theorem for certain function spaces on <math>SL(2, \mathbb{C})</math></i> . . . . .	617
James Juei-Chin Yeh, <i>Inversion of conditional expectations</i> . . . . .	631