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For Banach spaces X, Z, let $B(X, Z)$ denote the space of bounded linear operators from X into Z and $K(X, Z)$ (resp. $W(X, Z)$) the subspace of compact (resp. weakly compact) operators. It is shown that (a) if X contains an isomorph of c_0, then $K(X, l^\infty)$ is not complemented in $B(X, l^\infty)$, (b) if S is a compact Hausdorff space which is not scattered, then $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ for $Z = c_0$ or l^∞. In particular, $K(l^\infty, c_0)$ is not complemented in $B(l^\infty, c_0)$, which gives a negative answer to a question proposed by Arterburn and Whitley.

A subspace Y of a Banach space X is complemented if there is a projection $P: X \to X$ with range Y, i.e., a bounded linear operator of X such that $P^2 = P$ and $P(X) = Y$. There is a general conjecture afoot that if $K(X, Z)$ is a proper subspace of $B(X, Z)$ (resp. $W(X, Z)$) then it is not complemented in $B(X, Z)$ (resp. $W(X, Z)$). This conjecture was first studied by Thorp in [8], where he proved that $K(X, Z)$ is not complemented in $B(X, Z)$ when X, Z are certain Banach spaces of sequences. Later, various types of pairs X, Z for which the conjecture is known to be true were exhibited in [1] and [9]. We only recall that if weak and norm sequential convergence are not the same in the dual of a separable Banach space X, then $K(X, Z)$ is not complemented in $W(X, Z)$ for $Z = c_0$ or l^∞.

Let S be a compact Hausdorff space. S is called *scattered* if it contains no nonempty perfect subset. From the known results, we shall first establish some basic tools to determine certain situation where $K(X, Z)$ or $W(X, Z)$ is uncomplemented, then restrict ourselves to the projections in $B(X, Z)$ when X contains an $\mathscr{L}^\infty$-space in the sense of [4] and especially when $X = C(S)$. To avoid lengthy statements, we only discuss below the projections of $B(X, Z)$ onto $K(X, Z)$ and remark here that the statements in Proposition 1 through Theorem 6 remain true if we replace $B(\cdot, \cdot)$ by $W(\cdot, \cdot)$ everywhere; and also if, instead, we replace $K(\cdot, \cdot)$ by $W(\cdot, \cdot)$ everywhere. Our results are consistent with the conjecture. Furthermore, no counterexamples to the conjecture are known at present. In the sequel, let X^* denote the dual space of a Banach space X and let X be embedded into X^{**} under the canonical isometry.

PROPOSITION 1. *Let Z be a Banach space such that Z is comple-*

mented in Z^{**}. *Suppose* $K(X, Z)$ *is not complemented in* $B(X, Z)$, *then* $K(Z^*, X^*)$ *is not complemented in* $B(Z^*, X^*)$.

Proof. The map $T \to T^*$ is an isometrical isomorphism of $B(X, Z)$ into $B(Z^*, X^*)$ such that T^* is compact if and only if T is. Also T^{**} is a linear extension of T. Suppose now Q is a projection of Z^{**} onto Z and R is a projection of $B(Z^*, X^*)$ onto $K(Z^*, X^*)$; define $P: B(X, Z) \to B(X, Z)$ by

$$(PT)(x) = Q((RT^*)^*(x)) .$$

P is then a projection of $B(X, Z)$ onto $K(X, Z)$, a contradiction.

As an application, since $K(l^1, l^1)$ is not complemented in $B(l^1, l^1)$ [8], it follows that $K(l^\infty, l^\infty)$ is not complemented in $B(l^\infty, l^\infty)$, a simple result which is not contained in previous work.

PROPOSITION 2. *There exists an isometrical isomorphism of* $B(X, Z^*)$ *onto* $B(Z, X^*)$ *such that* $K(X, Z^*)$ *corresponds to* $K(Z, X^*)$. *Thus if* $K(X, Z^*)$ *is not complemented in* $B(X, Z^*)$, *neither is* $K(Z, X^*)$ *in* $B(Z, X^*)$.

Proof. Consider $T \in B(X, Z^*)$. Since Z is weak* dense in Z^{**}, the map $\tau: T \to T^* |_Z$, the restriction of T^* to Z, is an isometrical isomorphism. τ is also surjective, for given any $U \in B(Z, X^*)$, we have $\tau(U^* |_X) = U$. The correspondence of the subspaces of compact operators is trivial.

REMARK. In particular, $K(c_0, l^\infty)$ is thus uncomplemented in $B(c_0, l^\infty)$ because $K(l^1, l^1)$ is uncomplemented in $B(l^1, l^1)$. This proof avoids direct expressions for the norms of operators in terms of matrix coefficients as in the original proof of [8].

Let Y be a subspace of X. A bounded linear operator $E: B(Y, Z) \to B(X, Z)$ is called a *simultaneous extension* if $R_Y E(T) = T$ for every $T \in B(Y, Z)$, where R_Y denotes the restriction to Y. Suppose in addition that $E(K(Y, Z)) \subset K(X, Z)$ and that P is a projection of $B(X, Z)$ onto $K(X, Z)$; then $R_Y PE$ is a projection of $B(Y, Z)$ onto $K(Y, Z)$. Hence we have:

LEMMA 3. *Suppose* $K(Y, Z)$ *is not complemented in* $B(Y, Z)$ *and that there exists a simultaneous extension* $E: B(Y, Z) \to B(X, Z)$ *such that* $E(K(Y, Z)) \subset K(X, Z)$; *then* $K(X, Z)$ *is not complemented in* $B(X, Z)$.

LEMMA 4. *If* Y *is complemented in* X, *then there exists a simultaneous extension* E *such that* $E(K(Y, Z)) \subset K(X, Z)$.

LEMMA 5. *If Z is complemented in Z^{**} and $Y \subset Y_1 \subset Y^{**}$, then there exists a simultaneous extension E from $B(Y, Z)$ to $B(Y_1, Z)$ with $E(K(Y, Z)) \subset K(Y_1, Z)$.*

Proof. The map $T \to T^{**}$ is an isometrical isomorphism from $B(Y, Z)$ into $B(Y^{**}, Z^{**})$ such that T^{**} is an extension of T and T^{**} is compact if and only if T is. Let P be a projection of Z^{**} onto Z. Define $E: B(Y, Z) \to B(Y_1, Z)$ by $(ET)(y) = P(T^{**}(y))$, $y \in Y_1$; then E is the desired simultaneous extension.

THEOREM 6. *If Z is complemented in Z^{**} and Y is an $\mathscr{L}^{\infty}$-space such that $K(Y, Z)$ is not complemented in $B(Y, Z)$ then $K(X, Z)$ is not complemented in $B(X, Z)$ for any X containing a subspace isomorphic to Y.*

Proof. We can assume without loss of generality that $Y \subset X$, because if Y is isomorphic to $\tilde{Y}$ then $K(Y, Z)$ is complemented in $B(Y, Z)$ if and only if $K(\tilde{Y}, Z)$ is complemented in $B(\tilde{Y}, Z)$. Then Y^{**} can be regarded as a subspace of X^{**}. Since Y^{**} is an injective space [4, p. 291], there exists a projection Q from X^{**} onto Y^{**}. Let P be the projection from Z^{**} onto Z. On account of Lemma 4 and Lemma 5, we define $E: B(Y, Z) \to B(X, Z)$ by $(ET)(x) = P(T^{**}(Q(x)), x \in X$. Then E is a simultaneous extension such that $E(K(Y, Z)) \subset K(X, Z)$, which in turn proves that $K(X, Z)$ is not complemented in $B(X, Z)$.

REMARKS. (a) Z is complemented in Z^{**} if and only if Z is isomorphic to a complemented subspace of a dual space. (b) A bounded linear operator $T \in B(Y, Z)$ is weakly compact if and only if T^{**} maps Y^{**} into Z, i.e., $T \in W(Y, Z) \Leftrightarrow T^{**} \in W(Y^{**}, Z)$. Hence if $B(Y, Z) = W(Y, Z)$, or if we are merely looking for a projection of $W(X, Z)$ onto $K(X, Z)$, the assumption that Z is complemented in Z^{**} is redundant.

Observe that c_0 is an $\mathscr{L}^{\infty}$-space [4, p. 283]. Therefore, since there exists no projection of $B(c_0, l^{\infty})$ onto $K(c_0, l^{\infty})$ and since every infinite-dimensional Banach space whose dual is an L^1 space contains a subspace isomorphic to c_0 [10], we have

COROLLARY 7. *If X contains a subspace isomorphic to c_0, which is in particular the case when X is isomorphic to a $C(S)$ space or X is an infinite-dimensional Banach space whose dual is an L^1 space, then $K(X, l^{\infty})$ is not complemented in $B(X, l^{\infty})$.*

REMARK. An infinite-dimensional Banach space whose dual is an

L^1 space need not be isomorphic to a $C(S)$ space. As an example, given by Benyamini and Lindenstrauss, there exists a predual of l^1 which is not isomorphic to any $C(S)$ space [2].

In connection with the linear extension of operators, we have the following corollary, which will serve as a lemma for the next theorem.

COROLLARY 8. *If Y is an $\mathscr{L}^\infty$-space and X contains Y, then there exists a simultaneous extension E from $W(Y, Z)$ to $W(X, Z)$ such that $E(K(Y, Z)) \subset K(X, Z)$. If in addition Z is complemented in Z^{**}, then there exists a simultaneous extension from $B(Y, Z)$ to $B(X, Z)$ with $K(Y, Z)$ and $W(Y, Z)$ corresponding to subspaces of $K(X, Z)$ and $W(X, Z)$ respectively.*

THEOREM 9. *Let S be a compact Hausdorff space which is not scattered, then $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ for $Z = c_0$ or $Z = l^\infty$.*

Proof. Consider the space $C([0, 1])$. Since weak and norm sequential convergence are not the same in $C([0, 1])^*$, it is known by the aforementioned result in [1] that $K(C([0, 1]), Z)$ is not complemented in $W(C([0, 1]), Z)$ when Z is c_0 or l^∞. Now if S is not scattered, the interval $[0, 1]$ is a continuous image of S [7], hence $C(S)$ contains an isometric copy of $C([0, 1])$. Therefore by Corollary 8, there exists a simultaneous extension from $W(C([0, 1]), Z)$ to $W(C(S), Z)$ such that $K(C([0, 1]), Z)$ corresponds to a subspace of $K(C(S), Z)$. It follows then from Lemma 3 that $K(C(S), Z)$ is not complemented in $W(C(S), Z)$.

In answer to a question raised by Arterburn and Whitley in [1], where they asked whether $K(l^\infty, c_0)$ is complemented in $B(l^\infty, c_0)$, we have the following corollary, though an independent proof has been given in [9].

COROLLARY 10. *$K(l^\infty, c_0)$ is not complemented in $B(l^\infty, c_0)$.*

Proof. Since l^∞ can be identified with $C(\beta N)$ and βN is not scattered, the desired result follows immediately from Theorem 9.

Finally, to complete the examples studied by Tong and Wilken in [9], we consider the space of bounded linear operators $B(C(S), Z)$, $Z = c_0$ or $Z = l^p$, $1 \leqq p \leqq \infty$. Suppose S is scattered; then, since $C(S)^*$ is isometric to $l^1(S)$, $K(C(S), Z) = W(C(S), Z)$. (Recall that weak convergent sequences in $l^1(S)$ are norm convergent [3, p. 33] and a bounded linear operator T is compact if and only if T^* has the same property.) But it is well known that $W(C(S), Z) = B(C(S), Z)$ for an arbitrary Banach space Z containing no subspace isomorphic to c_0 [6], hence $K(C(S), Z) = B(C(S), Z)$ for $Z = l^p, 1 \leqq p < \infty$. When $Z = c_0$ or $Z = l^\infty$, then since $C(S)$ contains a complemented copy of c_0, $K(C(S), Z)$ is not complemented in $B(C(S), Z)$. If S is not scattered and $Z = c_0$ or $Z = l^\infty$, it is clear that $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ (and hence not complemented in $B(C(S), Z)$) by Theorem 9. For $Z = l^p, 2 \leqq p \leqq \infty$, $K(C(S), Z)$ is not complemented in $B(C(S), Z)$ by the main theorem in [9] and the fact that there exists a noncompact operator from $C(S)$ into Z as indicated there. When $Z = l^p, 1 \leqq p < 2$, the question of the existence of a noncompact operator was left open in the same reference; the answer is no, as follows from a factorization theorem:

THEOREM 11. *Every bounded linear operator from an $\mathscr{L}^\infty$-space into $l^p, 1 \leqq p < 2$ is compact.*

Proof. By Theorem 5.2 in [4], every bounded linear operator from an $\mathscr{L}^\infty$-space into $l^p, 1 \leqq p < 2$ can be factorized through a Hilbert space. Indeed, since l^p is separable, the Hilbert space H can further be chosen to be l^2. For if $T: H \rightarrow l^p$, then T can be factorized as $H \xrightarrow{\Phi} H/N \xrightarrow{\hat{T}} l^p$, where N is the null space of T, Φ is the quotient map and $\hat{T}$ is the induced injective map. Now since $\hat{T}^*: l^q \rightarrow H/N$ has a weak* dense image (hence weakly dense, since H/N is reflexive), H/N must be separable, which implies that H/N is isomorphic to l^2. The desired result then follows from the fact that every bounded linear operator from l^2 into $l^p, 1 \leqq p < 2$ is compact.

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