COMMUTANTS OF SOME QUASI-HAUSDORFF MATRICES

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Let $B(c)$ denote the Banach algebra of bounded linear operators over $c$, the space of convergent sequences, and $\Gamma^*$ the subalgebra of conservative infinite matrices. Given an upper triangular matrix $A$ in $\Gamma^*$, a sufficient condition is established for the commutant of $A$ in $\Gamma^*$ to be upper triangular. Also determined is the commutant, in $B(c)$, of certain quasi-Hausdorff matrices.

The spaces of bounded, convergent and null sequences will be denoted by $m$, $c$, $c_0$ respectively, and $l$ will denote the set of sequences $x$ satisfying $\sum_n |x_n| < \infty$. Let $\mathcal{A}$ denote the algebra of conservative upper triangular matrices; i.e., $A \in \mathcal{A}$ implies $A: c \rightarrow c$ and $a_{nk} = 0$ for $n > k$. $\mathcal{Q}^*$ will denote the algebra of conservative quasi-Hausdorff transformations, and $\Gamma$ the algebra of all conservative matrices. $\Gamma^*_1$ is the quasi-Hausdorff transformation generated by $\mu_n = a(n + a)^{-1}$, $a > 1$. For other specialized terminology the reader can consult [3] or [5].

One cannot answer commutant questions for upper or lower triangular matrices in $B(c)$ by taking transposes. For example, let $C$ denote the Cesaro matrix of order 1. $C^* \in \mathcal{Q}^*$ is not conservative. On the other hand, the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1 & \text{for } n = \left( \frac{j + 1}{2} \right), \left( \frac{j}{2} \right) + 1 \leq k \leq n; \quad j = 1, 2, \ldots , \\ 0 & \text{otherwise ,} \end{cases}$$

is conservative, but $A^* \in \mathcal{Q}^*$ is not. It is true that the transpose of any conservative quasi-Hausdorff matrix is a conservative Hausdorff matrix. $C$ shows that the converse is false.

We begin with some results analogous to those of [3] and [5].

**Theorem 1.** Let $A \in \mathcal{A}$. If $A$ has the property that

(1) for each $t \in m$, $n \geq 0$, $(A - a_{nn})t = 0$ implies $t \in \text{linear span } \{e^o, e^t, \ldots , e^n\}$, then every matrix $B$ with finite norm which commutes with $A$ is upper triangular.

$B \rightarrow A$ implies

(2) \[ \sum_{j=0}^{k} b_{nj}a_{jk} = \sum_{j=n}^{\infty} a_{nj}b_{jk} ; \quad n, k = 0, 1, 2, \ldots . \]

Set $k = 0$ to get

$$b_{n0}a_{00} = \sum_{j=n}^{\infty} a_{nj}b_{j0} ; \quad n = 0, 1, 2, \ldots ,$$
which can be written in the form $(A - \alpha_{00}I)t^0 = 0$, where $t^0 = (b_{n0})_{n=0}^\infty$. By hypothesis, $t$ belongs to the linear span of $e^0$, so that $b_{n0} = 0$ for all $n > 0$. By induction one can show that $b_{nk}$ for all $n > k$ and $B$ is upper triangular.

REMARKS. 1. The condition that $A$ be conservative is not needed in the proof. All one needs are restrictions on $A$ and $B$ sufficient to guarantee that the summations in (2) exist for each $n$ and $k$; for example, it would be sufficient to assume that each row of $A$ is in $I$ and each column of $B$ is in $m$.

2. It is an open question whether condition (1) is necessary. (The proof of the necessity of Theorem 1 in [3] is faulty, because it fails to show that $B$ has finite norm.)

An upper triangular matrix is called factorable if $a_{nk} = c_nd_k$, $n \leq k$. Examples of upper triangular factorable matrices in $B(c)$ are the transposes of the weighted mean methods $(N, p_n)$ with $p_n = a^n$, $a > 1$, and the $F^*_a$, $a > 1$.

THEOREM 2. If $A$ is a factorable upper triangular matrix with $a_{nn} \neq 0$ for all $n$, then $B \rightarrow A$ implies $B$ is upper triangular.

Proof. Set $n = k = 0$ in (2) to get $\sum_{j=1}^{\infty} a_{0j}b_{j0} = 0$. From (2) with $k = 0$, $n = 1$, we have

$$b_{10}a_{00} = \sum_{j=1}^{\infty} a_{1j}b_{j0} = \frac{c_1}{c_0} \sum_{j=1}^{\infty} a_{0j}b_{j0} = 0.$$ 

Since $a_{00} \neq 0$, $b_{10} = 0$. By induction, $b_{n0} = 0$ for all $n > 0$. Then by induction on $k$, we can show $b_{nk} = 0$ for all $n > k$, and $B$ is upper triangular.

COROLLARY 1. If $A \in D^*$, $A$ is factorable and has exactly one zero on the main diagonal, then $B \rightarrow A$ implies $B$ is upper triangular.

Proof. Let $N$ be such that $a_{NN} = 0$. If $N > 0$, then the proof of Theorem 2 forces $b_{nk} = 0$ for $n > k$, $k < N$. For $n > N$, $k = N$ in (1) we have

$$\sum_{j=n}^{\infty} a_{nj}b_{jN} = \sum_{j=0}^{N} b_{nj}a_{jN} = b_{nN}a_{NN} = 0,$$

or, $-a_{nn}b_{nN} = \sum_{j=n+1}^{\infty} a_{nj}b_{jN}$; i.e., $-d_nb_{nN} = \sum_{j=n+1}^{\infty} b_{jN}d_j$, which leads to $d_nb_{nN} = 0$. Since $d_n \neq 0$, $b_{nN} = 0$. By induction, $b_{nk} = 0$ for $n > k > N$.

COROLLARY 2. If $A \in D^*$, is factorable, and has at least two
nonadjacent zeros on the main diagonal, then there exists a matrix $B \leftrightarrow A$, $B$ not upper triangular.

Let $M$ and $N$ satisfy $a_{MM} = a_{NN} = 0$, $N > M + 1$. There are four possibilities: (i) $c_M = c_N = 0$, (ii) $c_M = d_N = 0$, (iii) $d_M = c_N = 0$, and (iv) $d_M = d_N = 0$.

If $d_N \neq 0$ the system (1) with $n = M$ has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_M = 0$, $t_k = -\sum_{j=k+1}^N a_{kj} t_j / a_{kk}$, $k \neq M$, $k < N$. If $d_N = 0$, then (1), with $n = M$, has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_{N-1} = 0$, $t_M = 0$,

$$t_k = -\sum_{j=k+1}^{N-1} a_{kj} t_j / a_{kk}, \quad k \neq M, \quad k < N - 1.$$

Define $B$ by $b_{nm} = t_n$, $b_{n,m+1} = -c_M t_n / c_{M+1}$, $n \leq N$, $b_{nk} = 0$ otherwise. Then $B \leftrightarrow A$, $B \in \Gamma$, but $B \notin \Delta^*$.

Suppose $A \in \Delta^*$, is factorable, and satisfies $a_{NN} = a_{N+1,N+1} = 0$, $a_{nn} \neq 0$ for $n \neq N, N + 1$. If $d_{N+1} = 0$ or $c_N = 0$, then an examination of the proof of Corollary 2 shows that we can find a matrix $B$ which commutes with $A$ and which is not upper triangular. If, however, $c_{N+1} = d_N = 0$, but $c_N d_{N+1} \neq 0$, then $B$ must be upper triangular.

**Corollary 3.** Let $A$ be a factorable upper triangular matrix such that, for some integer $N$, $d_N = c_{N+1} = 0$, and $c_N d_{N+1} \neq 0$, and $a_{nn} \neq 0$ for $n \neq N, N + 1$. Then $B \leftrightarrow A$ implies $B$ is upper triangular.

From the proof of Theorem 2, $b_{nk} = 0$ for each $k < N$, $n > k$. For $k = N$, $n \geq N$, we have, from (2),

$$\sum_{j=n}^N a_{nj} b_{jn} = \sum_{j=0}^N b_{nj} a_{jN} = b_{nN} a_{NN} = 0. \quad (3)$$

For $n > N + 1$, (3) becomes $c_n \sum_{j=n}^N d_j b_{jN} = 0$, which leads to $b_{nN} = 0$ since $c_n, d_n \neq 0$. With $n = N$, (3) now becomes $a_{NN} b_{NN} + a_{N,N+1} b_{N+1,N} = 0$. By induction it can be shown that $b_{nk} = 0$ for $n > k > N + 1$, so that $B$ is upper triangular.

To determine the commutants of various quasi-Hausdorff matrices in the algebras $\Delta^*$, $\Gamma$ and $B(c)$, we shall use $\Gamma_\omega^*$, which is a member of $\Delta^*$.

**Corollary 4.** $\text{Com}(\Gamma_\omega^*)$ in $\Delta^*$ = $\text{Com}(\Gamma_\omega^*)$ in $\Gamma = H^\omega*$.

The first equality follows from Theorem 2, since $\Gamma_\omega^*$ is factorable. The second equality comes from the following Lemma and Theorem 4.1 of [2].
LEMMA. Let \( H \) be a quasi-Hausdorff method with distinct diagonal entries, \( B \) any upper triangular matrix, \( B \rightarrow H \). Then \( B \) is quasi-Hausdorff.

**Proof.** From (2) we get
\[
\sum_{j=n}^{k} h_{ij} b_{jk} = \sum_{j=n}^{k} b_{ij} h_{jk}, \quad k \geq n.
\]

Denote the diagonal entries of \( B \) by \( \lambda_n \). Then, it can be shown by induction that \( b_{n,n+p} = \binom{n+p}{p} \Delta^p \lambda_n, \ p = 0, 1, \ldots \), and \( B \) is quasi-Hausdorff.

Leviatan [2] has shown that every matrix which commutes formally with the inverse of \( C^\tau \) is a quasi-Hausdorff matrix.

For any \( T \in B(c) \) one can define continuous linear functionals \( \chi \) and \( \chi_i \) by \( \chi(T) = \lim T e - \Sigma_k \lim (T e^k) \) and \( \chi_i(T) = (T e_i - \Sigma_k (T e^k)), \ i = 1, 2, \ldots \). Any \( T \in B(c) \) has the representation \( T x = v \lim x + B x \) for each \( x \in c \), where \( B \) is the matrix representation of the restriction of \( T \) to \( c_0 \), and \( v \) is the bounded sequence \( v = \{\chi_i(T)\} \). (See, e.g. [1].)

**Theorem 3.** For each \( a > 1 \), \( \text{Com}(I_n^\prime) \) in \( B(c) = \{T \in B(c): v = v, e \text{ and } B \in \mathcal{L}^*\} \).

**Proof.** From Corollary 1 of [5] we must have \( Av = \chi(A)v \).

Therefore, for each \( n \), \( \sum_{k=n}^{\infty} h_{nk}^* v_k = a v_n/(a - 1) \). But
\[
h_{nk}^* = \frac{ak! \Gamma(n + a)}{n! \Gamma(k + a + 1)}.
\]

Thus
\[
v_n = \frac{(a - 1) \Gamma(n + a)}{n!} \sum_{k=n}^{\infty} \frac{k! v_k}{\Gamma(k + a + 1)},
\]
which leads to \( v_n = v_i \) for all \( n > 1 \).

That \( B \in \mathcal{L}^* \) comes from the lemma.

Theorems 3 and 4 of [5] are not extendable to upper triangular matrices because the system of equations \( Av = \chi(A)v \) is now much more complicated.

It is an open question whether having distinct diagonal entries is a sufficient condition for a conservative quasi-Hausdorff matrix \( H^* \) to have the same commutant in \( \mathcal{L}^* \) and \( I' \).

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