

Pacific Journal of Mathematics

COMMUTANTS OF SOME QUASI-HAUSDORFF MATRICES

BILLY E. RHOADES

COMMUTANTS OF SOME QUASI-HAUSDORFF MATRICES

B. E. RHOADES

Let $B(c)$ denote the Banach algebra of bounded linear operators over c , the space of convergent sequences, and Γ^* the subalgebra of conservative infinite matrices. Given an upper triangular matrix A in Γ^* , a sufficient condition is established for the commutant of A in Γ^* to be upper triangular. Also determined is the commutant, in $B(c)$, of certain quasi-Hausdorff matrices.

The spaces of bounded, convergent and null sequences will be denoted by m , c , c_0 respectively, and l will denote the set of sequences x satisfying $\sum_k |x_k| < \infty$. Let \mathcal{L}^* denote the algebra of conservative upper triangular matrices; i.e., $A \in \mathcal{L}^*$ implies $A: c \rightarrow c$ and $a_{nk} = 0$ for $n > k$. \mathcal{H}^* will denote the algebra of conservative quasi-Hausdorff transformations, and Γ the algebra of all conservative matrices. Γ_a^* is the quasi-Hausdorff transformation generated by $\mu_n = a(n+a)^{-1}$, $a > 1$. For other specialized terminology the reader can consult [3] or [5].

One cannot answer commutant questions for upper or lower triangular matrices in $B(c)$ by taking transposes. For example, let C denote the Cesàro matrix of order 1. C^t is not conservative. On the other hand, the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1 \text{ for } n = \binom{j+1}{2}, \binom{j}{2} + 1 \leq k \leq n; & j = 1, 2, \dots, \\ 0 \text{ otherwise,} \end{cases}$$

is conservative, but A^t is not. It is true that the transpose of any conservative quasi-Hausdorff matrix is a conservative Hausdorff matrix. C shows that the converse is false.

We begin with some results analogous to those of [3] and [5].

THEOREM 1. *Let $A \in \mathcal{L}^*$. If A has the property that*

(1) *for each $t \in m$, $n \geq 0$, $(A - a_{nn}I)t = 0$ implies t in linear span $\{e^0, e^1, \dots, e^n\}$, then every matrix B with finite norm which commutes with A is upper triangular.*

$B \leftrightarrow A$ implies

$$(2) \quad \sum_{j=0}^k b_{nj} a_{jk} = \sum_{j=n}^{\infty} a_{nj} b_{jk}; \quad n, k = 0, 1, 2, \dots$$

Set $k = 0$ to get

$$b_{n0} a_{00} = \sum_{j=n}^{\infty} a_{nj} b_{j0}; \quad n = 0, 1, 2, \dots$$

which can be written in the form $(A - a_{00}I)t^0 = 0$, where $t^0 = \{b_{n0}\}_{n=0}^{\infty}$. By hypothesis, t belongs to the linear span of e^0 , so that $b_{n0} = 0$ for all $n > 0$. By induction one can show that $b_{nk} = 0$ for all $n > k$ and B is upper triangular.

REMARKS. 1. The condition that A be conservative is not needed in the proof. All one needs are restrictions on A and B sufficient to guarantee that the summations in (2) exist for each n and k ; for example, it would be sufficient to assume that each row of A is in l and each column of B is in m .

2. It is an open question whether condition (1) is necessary. (The proof of the necessity of Theorem 1 in [3] is faulty, because it fails to show that B has finite norm.)

An upper triangular matrix is called factorable if $a_{nk} = c_n d_k$, $n \leq k$. Examples of upper triangular factorable matrices in $B(c)$ are the transposes of the weighted mean methods (\bar{N}, p_n) with $p_n = a^n$, $a > 1$, and the Γ_a^* , $a > 1$.

THEOREM 2. *If A is a factorable upper triangular matrix with $a_{nn} \neq 0$ for all n , then $B \leftrightarrow A$ implies B is upper triangular.*

Proof. Set $n = k = 0$ in (2) to get $\sum_{j=1}^{\infty} a_{0j} b_{j0} = 0$. From (2) with $k = 0$, $n = 1$, we have

$$b_{10} a_{00} = \sum_{j=1}^{\infty} a_{1j} b_{j0} = \frac{c_1}{c_0} \sum_{j=1}^{\infty} a_{0j} b_{j0} = 0.$$

Since $a_{00} \neq 0$, $b_{10} = 0$. By induction, $b_{n0} = 0$ for all $n > 0$. Then by induction on k , we can show $b_{nk} = 0$ for all $n > k$, and B is upper triangular.

COROLLARY 1. *If $A \in \mathcal{A}^*$, A is factorable and has exactly one zero on the main diagonal, then $B \leftrightarrow A$ implies B is upper triangular.*

Proof. Let N be such that $a_{NN} = 0$. If $N > 0$, then the proof of Theorem 2 forces $b_{nk} = 0$ for $n > k$, $k < N$. For $n > N$, $k = N$ in (1) we have

$$\sum_{j=n}^{\infty} a_{nj} b_{jN} = \sum_{j=0}^N b_{nj} a_{jN} = b_{nN} a_{NN} = 0,$$

or, $-a_{nn} b_{nN} = \sum_{j=n+1}^{\infty} a_{nj} b_{jN}$; i.e., $-d_n b_{nN} = \sum_{j=n+1}^{\infty} b_{jN} d_j$, which leads to $d_n b_{nN} = 0$. Since $d_n \neq 0$, $b_{nN} = 0$. By induction, $b_{nk} = 0$ for $n > k > N$.

COROLLARY 2. *If $A \in \mathcal{A}^*$, is factorable, and has at least two*

nonadjacent zeros on the main diagonal, then there exists a matrix $B \leftrightarrow A$, B not upper triangular.

Let M and N satisfy $a_{MM} = a_{NN} = 0$, $N > M + 1$. There are four possibilities: (i) $c_M = c_N = 0$, (ii) $c_M = d_N = 0$, (iii) $d_M = c_N = 0$, and (iv) $d_M = d_N = 0$.

If $d_N \neq 0$ the system (1) with $n = M$ has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_M = 0$, $t_k = -\sum_{j=k+1}^N a_{kj}t_j/a_{kk}$, $k \neq M$, $k < N$. If $d_N = 0$, then (1), with $n = M$, has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_{N-1} = 0$, $t_M = 0$,

$$t_k = -\sum_{j=k+1}^{N-1} a_{kj}t_j/a_{kk}, \quad k \neq M, \quad k < N - 1.$$

Define B by $b_{nM} = t_n$, $b_{n,m+1} = -c_M t_n/c_{M+1}$, $n \leq N$, $b_{nk} = 0$ otherwise. Then $B \leftrightarrow A$, $B \in \Gamma$, but $B \notin \Delta^*$.

Suppose $A \in \Delta^*$, is factorable, and satisfies $a_{NN} = a_{N+1,N+1} = 0$, $a_{nn} \neq 0$ for $n \neq N, N + 1$. If $d_{N+1} = 0$ or $c_N = 0$, then an examination of the proof of Corollary 2 shows that we can find a matrix B which commutes with A and which is not upper triangular. If, however, $c_{N+1} = d_N = 0$, but $c_N d_{N+1} \neq 0$, then B must be upper triangular.

COROLLARY 3. *Let A be a factorable upper triangular matrix such that, for some integer N , $d_N = c_{N+1} = 0$, and $c_N d_{N+1} \neq 0$, and $a_{nn} \neq 0$ for $n \neq N, N + 1$. Then $B \leftrightarrow A$ implies B is upper triangular.*

From the proof of Theorem 2, $b_{nk} = 0$ for each $k < N$, $n > k$. For $k = N$, $n \geq N$, we have, from (2),

$$(3) \quad \sum_{j=n}^{\infty} a_{nj}b_{jN} = \sum_{j=0}^N b_{nj}a_{jN} = b_{nN}a_{NN} = 0.$$

For $n > N + 1$, (3) becomes $c_n \sum_{j=n}^{\infty} d_j b_{jN} = 0$, which leads to $b_{nN} = 0$ since $c_n, d_n \neq 0$. With $n = N$, (3) now becomes $a_{NN}b_{NN} + a_{N,N+1}b_{N+1,N} = 0$. By induction it can be shown that $b_{nk} = 0$ for $n > k > N + 1$, so that B is upper triangular.

To determine the commutants of various quasi-Hausdorff matrices in the algebras Δ^* , Γ and $B(c)$, we shall use Γ_a^{n*} , which is a member of Δ^* .

COROLLARY 4. $\text{Com}(\Gamma_a^{n*}) \text{ in } \Delta^* = \text{Com}(\Gamma_a^{n*}) \text{ in } \Gamma = \mathcal{H}^*$.

The first equality follows from Theorem 2, since Γ_a^{n*} is factorable. The second equality comes from the following Lemma and Theorem 4.1 of [2].

LEMMA. Let H be a quasi-Hausdorff method with distinct diagonal entries, B any upper triangular matrix, $B \leftrightarrow H$. Then B is quasi-Hausdorff.

Proof. From (2) we get

$$\sum_{j=n}^k h_{nj} b_{jk} = \sum_{j=n}^k b_{nj} h_{jk}, \quad k \geq n.$$

Denote the diagonal entries of B by λ_n . Then, it can be shown by induction that $b_{n,n+p} = \binom{n+p}{p} \Delta^p \lambda_n$, $p = 0, 1, \dots$, and B is quasi-Hausdorff.

Leviatan [2] has shown that every matrix which commutes formally with the inverse of C^x is a quasi-Hausdorff matrix.

For any $T \in B(c)$ one can define continuous linear functionals χ and χ_i by $\chi(T) = \lim Te - \sum_k \lim (Te^k)$ and $\chi_i(T) = (Te)_i - \sum_k (Te^k)_i$, $i = 1, 2, \dots$. Any $T \in B(c)$ has the representation $Tx = v \lim x + Bx$ for each $x \in c$, where B is the matrix representation of the restriction of T to c_0 , and v is the bounded sequence $v = \{\chi_i(T)\}$. (See, e.g. [1].)

THEOREM 3. For each $a > 1$, $\text{Com}(\Gamma_a^*)$ in $B(c) = \{T \in B(c) : v = v_1 e \text{ and } B \in \mathcal{H}^*\}$.

Proof. From Corollary 1 of [5] we must have $Av = \chi(A)v$. Therefore, for each n , $\sum_{k=n}^\infty h_{nk}^* v_k = av_n / (a - 1)$. But

$$h_{nk}^* = \frac{ak! \Gamma(n+a)}{n! \Gamma(k+a+1)}.$$

Thus

$$v_n = \frac{(a-1)\Gamma(n+a)}{n!} \sum_{k=n}^\infty \frac{k! v_k}{\Gamma(k+a+1)},$$

which leads to $v_n = v_1$ for all $n > 1$.

That $B \in \mathcal{H}^*$ comes from the lemma.

Theorems 3 and 4 of [5] are not extendable to upper triangular matrices because the system of equations $Av = \chi(A)v$ is now much more complicated.

It is an open question whether having distinct diagonal entries is a sufficient condition for a conservative quasi-Hausdorff matrix H^* to have the same commutant in Δ^* and Γ .

ACKNOWLEDGEMENTS. 1. To Professor W. Meyer-König who, after hearing a presentation of [3] and [5], requested that I consider the question of commutants for quasi-Hausdorff matrices.

2. To the referee, for his careful reading of the original version of the paper, and for his comments, which are incorporated in Corollaries 2 and 3, and Remark 2.

REFERENCES

1. H. I. Brown, D. R. Kerr, and H. H. Stratton, *The structure of $B[c]$ and extensions of the concept of conull matrix*, Proc. Amer. Math. Soc., **22** (1969), 7-14.
2. D. Leviatan, *Moment problems and quasi-Hausdorff transformations*, Canad. Math. Bull., **11** (1968), 225-236.
3. B. E. Rhoades, *Commutants of some Hausdorff matrices*, Pacific J. Math., **42** (1972), 715-719.
4. ———, *Commutants of some Hausdorff matrices*, corrections, to appear in the Pacific J. Math.
5. B. E. Rhoades and A. Wilansky, *Some commutants in $B(c)$ which are almost matrices*, Pacific J. Math., **42** (1973), 211-217.

Received January 8, 1973.

INDIANA UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1973 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Pacific Journal of Mathematics

Vol. 52, No. 2

February, 1974

Harm Bart, <i>Spectral properties of locally holomorphic vector-valued functions</i>	321
J. Adrian (John) Bondy and Robert Louis Hemminger, <i>Reconstructing infinite graphs</i>	331
Bryan Edmund Cain and Richard J. Tondra, <i>Biholomorphic approximation of planar domains</i>	341
Richard Carey and Joel David Pincus, <i>Eigenvalues of seminormal operators, examples</i>	347
Tyrone Duncan, <i>Absolute continuity for abstract Wiener spaces</i>	359
Joe Wayne Fisher and Louis Halle Rowen, <i>An embedding of semiprime P.I.-rings</i>	369
Andrew S. Geue, <i>Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators</i>	377
Charles Lemuel Hagopian, <i>Locally homeomorphic λ connected plane continua</i>	403
Darald Joe Hartfiel, <i>A study of convex sets of stochastic matrices induced by probability vectors</i>	405
Yasunori Ishibashi, <i>Some remarks on high order derivations</i>	419
Donald Gordon James, <i>Orthogonal groups of dyadic unimodular quadratic forms. II</i>	425
Geoffrey Thomas Jones, <i>Projective pseudo-complemented semilattices</i>	443
Darrell Conley Kent, Kelly Denis McKennon, G. Richardson and M. Schroder, <i>Continuous convergence in $C(X)$</i>	457
J. J. Koliha, <i>Some convergence theorems in Banach algebras</i>	467
Tsang Hai Kuo, <i>Projections in the spaces of bounded linear operations</i>	475
George Berry Leeman, Jr., <i>A local estimate for typically real functions</i>	481
Andrew Guy Markoe, <i>A characterization of normal analytic spaces by the homological codimension of the structure sheaf</i>	485
Kunio Murasugi, <i>On the divisibility of knot groups</i>	491
John Phillips, <i>Perturbations of type I von Neumann algebras</i>	505
Billy E. Rhoades, <i>Commutants of some quasi-Hausdorff matrices</i>	513
David W. Roeder, <i>Category theory applied to Pontryagin duality</i>	519
Maxwell Alexander Rosenlicht, <i>The nonminimality of the differential closure</i>	529
Peter Michael Rosenthal, <i>On an inversion theorem for the general Mehler-Fock transform pair</i>	539
Alan Saleski, <i>Stopping times for Bernoulli automorphisms</i>	547
John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces. II</i>	553
Vashishtha Narayan Singh, <i>Reproducing kernels and operators with a cyclic vector. I</i>	567
Peggy Strait, <i>On the maximum and minimum of partial sums of random variables</i>	585
J. L. Brenner, <i>Maximal ideals in the near ring of polynomials modulo 2</i>	595
Ernst Gabor Straus, <i>Remark on the preceding paper: "Ideals in near rings of polynomials over a field"</i>	601
Masamichi Takesaki, <i>Faithful states on a C^*-algebra</i>	605
R. Michael Tanner, <i>Some content maximizing properties of the regular simplex</i>	611
Andrew Bao-hwa Wang, <i>An analogue of the Paley-Wiener theorem for certain function spaces on $SL(2, \mathbb{C})$</i>	617
James Juei-Chin Yeh, <i>Inversion of conditional expectations</i>	631