

# Pacific Journal of Mathematics

**ON AN INVERSION THEOREM FOR THE GENERAL  
MEHLER-FOCK TRANSFORM PAIR**

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## ON AN INVERSION THEOREM FOR THE GENERAL MEHLER-FOCK TRANSFORM PAIR

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Let  $P_m^k(y)$  be the Legendre function of the first kind and let  $\Gamma(z)$  be the Gamma function. Then the general Mehler-Fock transform of complex order  $k$  of a function  $g(y)$  is defined by the equation

$$f(x) = L_2(g) = \pi^{-1}x \sin h(\pi x) \Gamma\left(\frac{1}{2} - k - ix\right) \\
 \times \Gamma\left(\frac{1}{2} - k + ix\right) \int_1^\infty g(y) P_{ix-1/2}^k(y) dy,$$

the inversion theorem states

$$g(y) = L_1(f) = \int_0^\infty f(x) P_{ix-1/2}^k(y) dx.$$

It is stated on page 416 of I. N. Sneddon's book 'The Use of Integral Transforms, (1972) that apparently a class of functions  $g(y)$  for which this result is valid is not yet clearly defined. The purpose of this paper is to define a class of functions  $g(y)$  as well as a class  $f(x)$  and give proofs that the above inversion formula hold for these classes.

**Introduction.** The theorem and proofs presented in the paper are basically a generalization of those in a paper of V. Fock [4] who treated the case  $k = 0$ , the Mehler-Fock transform. Some applications of the Mehler-Fock transform and general Mehler-Fock transform are given in [7], [8]. Tables of these transforms are given in [6].

All integrals are taken in the improper (complex) Riemann sense.  $x \sim +\infty$  means  $x$  positive and sufficiently large,  $x \sim +1$   $\equiv$  sufficiently close to 1,  $x > 1$ .

**THEOREM 1.** *Let  $G$  be the class of complex valued functions such that  $g \in G$  if and only if*

1.  $g(y) = (y-1)^{-k/2} g_1(y)$ ,  $y > 1$ ,  $g_1(y)$  is twice differentiable and continuous for  $y \geq 1$ , the real and imaginary parts of  $g_1''(y)$  are of bounded variation on any closed and bounded interval contained in  $\infty > y \geq 1$ .

2.  $d^n g_1/dy^n = O(y^{-(1/2)-n+(k/2)-\epsilon})$ ,  $y \geq 1$ ,  $1/4 > \epsilon > 0$ ,  $0 \equiv$  large order relation,  $n = 0, 1, 2$  (the case  $n = 0$  means  $g_1$ ).

Then  $L_1(L_2(g)) = g$ ,  $y > 1$ ,  $|\operatorname{Re} k| < 1/4$ .

*Proof of Theorem 1.*

LEMMA 1. *Let*

$$g \in G, h(t) = \int_0^t p(t, q) dq, p = (\sinh q)^{1-k} (\cosh t - \cosh q)^{-1/2+k} g(\cosh q),$$

$$f(x) = \int_0^\infty \cos(xt) h'(t) dt, |\operatorname{Re} k| < \frac{1}{4}.$$

*Then*

1.  $f(x) = O(x^{-2}), x \sim +\infty, \int_0^\infty |f(x)| dx < \infty.$
- 2a.  $h'(t)$  is continuous for  $t \geq 0.$
- 2b.  $h'(t)$  satisfies the conditions of a Fourier inversion theorem [9, p. 13],  $h', h''$  are both absolutely integrable over the infinite interval  $\infty \geq t \geq 0, \lim_{t \rightarrow +0, +\infty} h = 0, \lim_{t \rightarrow +\infty} h' = 0.$
3.  $\int_0^\infty \left( \int_0^t |p| dq \right) dt < \infty.$

*Proof of Lemma 1.* Let  $s = \cosh t, r = \cosh q, r = (s-1)w + 1.$   
Then

$$p = (s-1)^{(1+k)/2} ((s-1)w + 2)^{-k/2} g((s-1)w + 1) c(w),$$

$$c(w) = (1-w)^{-(1/2+k)} w^{-k/2}.$$

Hence there exists  $c_n(w)$  independent of  $t$  such that

$$\left| \frac{\partial^n p}{\partial t^n} \right| \leq e^{-\varepsilon t} |c_n(w)|, t \sim +\infty, \int_0^1 |c_n| dw < \infty, \frac{1}{4} > \varepsilon > 0, n$$

$$= 0, 1, 2, |\operatorname{Re} k| < \frac{1}{4}.$$

Again by dominated convergence we conclude  $d^n h/dt^n = \int_0^1 (\partial^n p / \partial t^n) dw,$   
 $\infty > t \geq 0, n = 1, 2, |\operatorname{Re} k| < 1/4.$  Hence parts 2, 3 of Lemma 1 hold.  
We are now permitted to integrate by parts with respect to  $t$  the right-hand side of the defining formula for  $f(x)$  in the hypothesis of Lemma 1 to conclude  $f(x) = x^{-1} F(x), F(x) = \int_0^\infty \sin(xt) h''(t) dt.$  Since  $h''(t) = O(e^{-\varepsilon t}), t \sim +\infty, 1/4 > \varepsilon > 0,$  we conclude the real and imaginary parts of  $h''(t)$  are of bounded variation in the infinite interval  $\infty \geq t \geq 0$  (see I.P. Natanson "Theory of Functions of a Real Variable", p. 238, for definitions and theorem). This implies  $F(x) = O(x^{-1}), x \sim +\infty.$  This completes the proof of Lemma 1.

LEMMA 2. *Let  $g \in G.$  Then*

$$\lim_{A \rightarrow +\infty} \int_0^{A>0} \left( \int_0^t \hat{f} dq \right) dt = \lim_{A \rightarrow +\infty} \int_0^A \left( \int_q^A \hat{f} dt \right) dq = \int_0^\infty \left( \int_q^\infty \hat{f} dt \right) dq ,$$

$$\hat{f} = p \sin(xt), x \geq 0, |\operatorname{Re} k| < \frac{1}{4} .$$

(See Lemma 1 for the definition of  $p$ .)

*Proof of Lemma 2.* Since  $g \in G$ , the iterated integrals in Lemma 2 are equal for finite  $A$ . Part 3 of Lemma 1 implies absolute integrability of the first iterated integral in Lemma 2. Hence we satisfy Fubini's theorem which implies Lemma 2.

LEMMA 3. *Let*

$$F(v) = \int_1^v (v - s)^{-1/2+k} r ds, r = (s^2 - 1)^{-k/2} g(s), g \in G .$$

*Then*

$$\frac{d}{dt} \int_1^t (t - v)^{-1/2-k} F(v) dv = \int_1^t (t - v)^{-1/2-k} \frac{dF}{dv} dv, |\operatorname{Re} k| < \frac{1}{4} .$$

*Proof of Lemma 3.* Part 2 of Lemma 1 implies  $F(v), F'(v)$  are both continuous for  $v > 1, \lim_{v \rightarrow +1} F(v) = 0$ . Hence we satisfy a theorem (relating to the Abel integral equation) [1, p. 5] (this theorem can be modified to include singularities of the type  $(x - 1)^a, x \sim +1, \operatorname{Re} a > -1$ , our case, see [1, p. 6]), which implies the conclusion of Lemma 3.

The rest of the proof of Theorem 1 consists mainly in applying the above lemmas to show that all the operations we use to show that (2) is a solution to (1) are valid.

Using the integral representation for  $P_{ix-1/2}^k$  from [5, p. 165], we obtain from (2), the iterated integral,

$$(3) \quad f(x) = a(k)x \int_0^\infty \left( \int_t^\infty p \sin(xs) ds \right) dt$$

(see Lemma 1 for the definition of  $p$ )

$$a(k) = 2^{1/2} \pi^{-3/2} \Gamma\left(\frac{1}{2} - k\right) \sin\left(\left(\frac{1}{2} + k\right)\pi\right), x \geq 0, |\operatorname{Re} k| < \frac{1}{4} .$$

(We note (3) is valid by Lemma 2.)

We now apply to the right-hand side of (3) the following operations in this order,

1. integration over a triangular domain (see Lemma 2),
2. integration by parts with respect to  $s$ ,
3. the Fourier cosine transform.

Since operations 1, 2, 3 are now permissible by Lemmas 1, 2 ( $g \in G$ ),

we obtain from (3) the valid identity

$$\int_0^\infty \cos(tx) f(x) dx = a_1(k) \frac{dh}{dt} \quad (\text{see Lemma 1 for definition of } h)$$

$$(4) \quad a_1(k) = (2\pi)^{-1/2} \Gamma\left(\frac{1}{2} - k\right) \sin\left(\left(\frac{1}{2} + k\right)\pi\right),$$

$$t > 0, |\operatorname{Re} k| < \frac{1}{4}.$$

Lemma 3 implies all the operations (those indicated in Lemma 3) to show the right-hand side of (4) is a solution to an Abel integral equation are now permissible [1, p. 9]. (Again we note only real  $k$  are treated on p. 9, but the theory can be extended to complex  $k$ , our case.) Hence applying these operations (those indicated in Lemma 3 to the right-hand side of (4), we obtain the valid identity

$$(5) \quad g(\cosh t) = \int_0^t \left( \int_0^\infty u dx \right) ds, \quad u = a_2(k) (\sinh t)^k (\cosh h t - \cosh s)^{-1/2-k}$$

$$\cos(sx) f(x), \quad a_2(k) = (2^{-1}\pi)^{-1/2} \left( \Gamma\left(\frac{1}{2} - k\right) \right)^{-1}, \quad t > 0, |\operatorname{Re} k| < \frac{1}{4}.$$

Interchanging the order of integration of the iterated integral on the right-hand side of (5) (which is now permissible by part 1 of Lemma 1), then using the integral representation for  $P_{ix-1/2}^k$  from [2, p. 156], we obtain the valid identity  $L_1(L_2(g)) = g$ ,  $t > 0$ ,  $|\operatorname{Re} k| < 1/4$ . This completes the proof of Theorem 1.

**COROLLARY 1.** *Let  $g_1, g_2 \in G$  such that  $L_2(g_1) = L_2(g_2)$ , then  $g_1(t) = g_2(t)$ ,  $t > 0$ ,  $|\operatorname{Re} k| < 1/4$ .*

*Proof.* Let  $u = g_1 - g_2$ . Then  $u \in G$ . Hence  $L_2(u) = 0$  by linearity of  $L_2$ . Hence  $f(x)$  (of (3)) = 0,  $x \geq 0$ . We then obtain from (5) the conclusion of Corollary 1.

**THEOREM 2.** *Let  $F$  be the class of real valued functions such that  $f \in F$  if and only if*

1.  $f(x) = x^2 f_1(x)$ ,  $f_1'(x)$  is continuous for  $x \geq 0$ , and of bounded variation on any closed and bounded interval contained in  $\infty > x \geq 0$ .

2.  $f, f' = O(x^{-1-\epsilon})$ ,  $x \sim +\infty$ ,  $\epsilon > 0$ .

Then  $L_2(L_1(f)) = f$ ,  $x \geq 0$ ,  $|\operatorname{Re} k| < 1/2$ .

*Proof of Theorem 2.*

**LEMMA 4.** *Let  $f \in F$ ,  $g = L_1(f)$ , then*

1.  $\int_1^A |g(x)| dy$  exists for any  $A > 1$ .

2.  $g = 0((\cosh^{-1} y)^{-2}(y^2 - 1)^{-1/4})$ ,  $y \sim + \infty$ ,  
 providing  $|\operatorname{Re} k| < 1/2$ .

*Proof of Lemma 4.* From formula 26 [2, p. 129],

$$(a) \quad P_{ix-1/2}^k(\cosh t) = (2\pi \sinh t)^{-1/2}(e^{-itx} f_1 + e^{itx} f_2),$$

$$f_1 = \frac{\Gamma(-ix)}{\Gamma\left(\frac{1}{2} - k - ix\right)} f_3, \quad f_3 = F\left(\frac{1}{2} + k, \frac{1}{2} - k, 1 + ix; -\frac{1}{2} e^{-t} \cosh t\right),$$

$$f_2(x) = f_1(-x), \quad F(a, b, c; z) = M \int_0^1 w ds, \quad w = s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a},$$

$\operatorname{Re} b, \operatorname{Re}(c-b) > 0, |z| < 1, M$  independent of  $z$  [2, p. 59].

(b)  $z^{b-a}(\Gamma(z+a)/\Gamma(z+b)) \sim a_1 + a_2 z^{-1} + \dots$  (an asymptotic series),  $|z| \sim + \infty$  uniformly for  $|\arg z| \leq \pi - \varepsilon, \pi/2 > \varepsilon > 0$  [2, p. 47], so differentiation of the right-hand side of (b) is permissible [3, p. 21]. From (a) we conclude  $(1+x)^{-1/2+k} f_3'(x), (1+x)^{-1/2+k} f_3''(x)$  are uniformly bounded for  $x \geq 0$  and  $t \geq 1$ , providing  $|\operatorname{Re} k| < 1/2$ . In (1) we now use the integral representation from (a), then integrate by parts with respect to  $x$ , which is permissible ( $f \in F$ ) to conclude  $g^{(j)}(y) = (\cosh^{-1} y)^{-1}(y^2 - 1)^{-1/4} \int_0^\infty e^{\pm itx} c^{(j)}(y, x, k) dx, y \geq 2, |\operatorname{Re} k| < 1/2$ , further the real and imaginary parts  $c^{(j)}$  are of bounded variation in  $x$  on the infinite interval  $\infty \geq x \geq 0, y \geq 2, |\operatorname{Re} k| < 1/2$ . Hence the real and imaginary parts of  $c^{(j)}$  can each be written as the difference of two monotonically decreasing functions  $c_n^{(j)}(x), x \geq 0, \lim_{x \rightarrow +\infty} c_n^{(j)}(x) = 0$  uniformly in  $y \geq 2, c_n^{(j)}$  are uniformly bounded,  $x \geq 0, y \geq 2, |\operatorname{Re} k| < 1/2, n = 1, 2, j = 1, 2$ , since  $f(x) = O(x^{-1-\varepsilon}), x \sim + \infty$ . Also  $g(y) = O((y-1)^{-1/4}), 2 > y > 1, |\operatorname{Re} k| < 1/2$ , by (5) (in the proof of Theorem 1),  $f \in F$ . Hence Lemma 4 holds.

LEMMA 5. *The  $g$  of Lemma 4 implies  $\int_0^\infty \left(\int_q^\infty |\hat{f}| dt\right) dq < \infty, x \geq 0, |\operatorname{Re} k| < 1/2$  (see Lemma 2 of Theorem 1 for the definition of  $\hat{f}$ ).*

*Proof.* Using the change of variable  $(\cosh t - \cosh q) = (\cosh q + 1)w$ , we conclude  $\int_q^\infty |\hat{f}| dt \leq M (\sinh q/2)^{-1} |(\sinh q)^{1-k} (\cosh q)^k g(\cosh q)|, q > 0, x \geq 0, M$  a constant,  $|\operatorname{Re} k| < 1/2$ . Hence the conclusion of Lemma 5 follows.

The rest of the proof of Theorem 2 consists mainly in justifying in reverse order all the formulas arising from the solution of the integral equation  $L_1(f) = g$  in the proof of Theorem 1. Hence we will point only where the rest of the proof of Theorem 2 must be modified from that of Theorem 1.

REMARK 1. The inversion theorem for the solution to the Abel integral equation [1, p. 9] appealed to in the proof of Theorem 1 has been modified to include functions which have singularities of the type  $(x-1)^a$ ,  $x \sim +1$ ,  $\operatorname{Re} a > -1$ . Hence this modified form of the theorem applies again to our case (see (5) in the proof of Theorem 1) since we have a singularity of this type when we use the change of variable  $s = \cosh q$ .

REMARK 2. Lemma 5,  $f \in F$  imply the sum  $\hat{h}(+\infty) - \hat{h}(+0)$ ,  $x \geq 0$ ,  $|\operatorname{Re} k| < 1/2$ , of the upper and lower limits (both are finite) (arising when one does an integration by parts, i.e., the reverse operation corresponding to the one of part 2 of (3) in the proof of Theorem 1) is zero.

REMARK 3. Lemma 5 implies the  $g$  of Lemma 4 satisfies the conclusion of Lemma 2 of Theorem 1. Hence the reverse operation of integrating over a triangular domain (see Lemma 2 of Theorem 1) is now permissible. Hence we conclude all the reverse formulas are valid. This completes the proof of Theorem 2.

COROLLARY 2. Let  $f_1, f_2 \in F$  such that  $L_1(f_1) = L_1(f_2)$ . Then  $f_1(x) = f_2(x)$ ,  $x \geq 0$ ,  $|\operatorname{Re} k| < 1/2$ .

*Proof.* Let  $r = f_1 - f_2$ . Then  $r \in F$ . Hence by linearity  $L_1(r) = 0$ . Then by (3) of Theorem 1 (see also Lemma 5 of Theorem 2) we obtain the conclusion of Corollary 2.

We note in closing, using the change of variable  $(\cosh t - \cosh q) = (\cosh q + \cos a)s$ , the integral representations for  $P_{ix-1/2}^k$  in Theorem 1 and [5], we obtain a pair of reciprocal transforms

1.  $g(\cosh q) = \sin a (\cosh q + \cos a)^{-3/2+k} (\sinh q)^{-k}$ ,  $|a| < \pi/2$ ,
2.  $f(x) = 2^{1/2} \pi^{-1/2} (\Gamma(1/2 - k))^{-1} \beta(1/2 - k, 1) x \Gamma(1/2 - k + ix) \Gamma(1/2 - k - ix) \sinh ax$ ,  $|\operatorname{Re} k| < 1/2$ . (The case  $k = 0$  specializes to the example in [4].)  $\beta \equiv$  Beta function. Further,  $g \in G$  of Theorem 1 and  $f \in F$  of Theorem 2.

If in Theorem 1, part 1, we now assume  $g_1$  is analytic for  $y \geq 1$ ,  $\operatorname{Re} k < 1/2$ , in 2 we assume  $n \geq 0$  and arbitrary, then by the methods in the proofs of Theorems 1 and 2 (we use the integral representation for  $P_{ix-1/2}^k$  from (5) in  $L_2$ ), we conclude  $c(k) = L_1(L_2(g))$  is an analytic function in  $k$  for  $\operatorname{Re} k < 1/2$ ,  $y > 1$ . Hence by analytic continuation, Theorem 1 and Corollary 1 are now valid for  $\operatorname{Re} k < 1/2$ .

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