ON AN INVERSION THEOREM FOR THE GENERAL MEHLER-FOCK TRANSFORM PAIR

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Let $P^k_m(y)$ be the Legendre function of the first kind and let $\Gamma(z)$ be the Gamma function. Then the general Mehler-Fock transform of complex order $k$ of a function $g(y)$ is defined by the equation

$$f(x) = L_2(g) = \pi^{-1} x \sin h(\pi x) \Gamma \left( \frac{1}{2} - k - i x \right)$$

$$\times \Gamma \left( \frac{1}{2} - k + i x \right) \int_1^\infty g(y) P^k_{ix-1/2}(y) dy ,$$

the inversion theorem states

$$g(y) = L_1(f) = \int_0^\infty f(x) P^k_{ix-1/2}(y) dx .$$

It is stated on page 416 of I. N. Sneddon's book 'The Use of Integral Transforms, (1972) that apparently a class of functions $g(y)$ for which this result is valid is not yet clearly defined. The purpose of this paper is to define a class of functions $g(y)$ as well as a class $f(x)$ and give proofs that the above inversion formula hold for these classes.

Introduction. The theorem and proofs presented in the paper are basically a generalization of those in a paper of V. Fock [4] who treated the case $k = 0$, the Mehler-Fock transform. Some applications of the Mehler-Fock transform and general Mehler-Fock transform are given in [7], [8]. Tables of these transforms are given in [6].

All integrals are taken in the improper (complex) Riemann sense. $x \sim + \infty$ means $x$ positive and sufficiently large, $x \sim + 1$ sufficiently close to 1, $x > 1$.

**Theorem 1.** Let $G$ be the class of complex valued functions such that $g \in G$ if and only if

1. $g(y) = (y - 1)^{-k/2} g_\epsilon(y), y > 1, g_\epsilon(y)$ is twice differentiable and continuous for $y \geq 1$, the real and imaginary parts of $g''_\epsilon(y)$ are of bounded variation on any closed and bounded interval contained in $\infty > y \geq 1$.

2. $d^n g_\epsilon / dy^n = O(y^{-(1/2) - n + (k/2) - \epsilon}), y \geq 1, 1/4 > \epsilon > 0, 0 \equiv$ large order relation, $n = 0, 1, 2$ (the case $n = 0$ means $g_\epsilon$).

Then $L_1(L_2(g)) = g, y > 1, |\text{Re } k| < 1/4$. 

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Proof of Theorem 1.

Lemma 1. Let

\[ g \in G, \ h(t) = \int_0^t p(t, q) dq, \ p = (\sinh q)^{1-k}(\cosh t - \cosh q)^{-1/2+k}g(\cosh q), \]

\[ f(x) = \int_0^\infty \cos (xt)h'(t)dt, \ |\text{Re} \ k| < \frac{1}{4}. \]

Then

1. \( f(x) = 0(x^{-\varepsilon}), \ x \sim +\infty, \int_0^\infty |f(x)| dx < \infty. \)
2a. \( h'(t) \) is continuous for \( t \geq 0. \)
2b. \( h'(t) \) satisfies the conditions of a Fourier inversion theorem \[9, \text{p. 13}], \ h', h'' \ are both absolutely integrable over the infinite interval \( \infty \geq t \geq 0, \lim_{t \to 0, +\infty} h = 0, \lim_{t \to +\infty} h' = 0. \)
3. \( \int_0^\infty (\int_0^t |p| dq) dt < \infty. \)

Proof of Lemma 1. Let \( s = \cosh t, \ r = \cosh q, \ r = (s - 1)w + 1. \)

Then

\[ p = (s - 1)^{(1+k)/2}((s - 1)w + 2)^{-k/2}g((s - 1)w + 1)c(w), \]
\[ c(w) = (1 - w)^{-(1/2+k)}w^{-k/2}. \]

Hence there exists \( c_n(w) \) independent of \( t \) such that

\[ \left| \frac{\partial^n p}{\partial t^n} \right| \leq e^{-\varepsilon t} |c_n(w)|, \ t \sim +\infty, \int_0^1 |c_n| dw < \infty, \frac{1}{4} > \varepsilon > 0, \]
\[ = 0, 1, 2, \ |\text{Re} \ k| < \frac{1}{4}. \]

Again by dominated convergence we conclude \( d^nh/dt^n = \int_0^t (\partial^n p/\partial t^n) dw, \)
\( \infty > t \geq 0, \ n = 1, 2, \ |\text{Re} \ k| < 1/4. \) Hence parts 2, 3 of Lemma 1 hold. We are now permitted to integrate by parts with respect to \( t \) the right-hand side of the defining formula for \( f(x) \) in the hypothesis of Lemma 1 to conclude \( f(x) = x^{-1}F(x), \)
\( F(x) = \int_0^\infty \sin (xt)h'(t)dt. \) Since \( h''(t) = O(e^{-\varepsilon t}), \ t \sim +\infty, 1/4 > \varepsilon > 0, \) we conclude the real and imaginary parts of \( h''(t) \) are of bounded variation in the infinite interval \( \infty \geq t \geq 0 \) (see I.P. Natanson “Theory of Functions of a Real Variable”, p. 238, for definitions and theorem). This implies \( F(x) = O(x^{-1}), \ x \sim +\infty. \) This completes the proof of Lemma 1.

Lemma 2. Let \( g \in G. \) Then
\[
\lim_{A \to +\infty} \left( \int_{A}^{\infty} \left( \int_{0}^{A} \hat{f} dt \right) dq \right) = \lim_{A \to +\infty} \left( \int_{0}^{A} \hat{f} dt \right) dq = \int_{0}^{\infty} \left( \int_{0}^{\infty} \hat{f} dt \right) dq,
\]
\[
\hat{f} = p \sin (xt), \quad x \geq 0, \quad |\text{Re} \ k| < \frac{1}{4}.
\]

(See Lemma 1 for the definition of \( p \).)

**Proof of Lemma 2.** Since \( g \in G \), the iterated integrals in Lemma 2 are equal for finite \( A \). Part 3 of Lemma 1 implies absolute integrability of the first iterated integral in Lemma 2. Hence we satisfy Fubini's theorem which implies Lemma 2.

**Lemma 3.** Let
\[
F(v) = \int_{1}^{v} (v - s)^{-1/2 + k} ds, \quad r = (s^2 - 1)^{-1/2} g(s), \quad g \in G.
\]
Then
\[
\frac{d}{dt} \int_{1}^{t} (t - v)^{-1/2 - k} F(v) dv = \int_{1}^{t} (t - v)^{-1/2 - k} \frac{dF}{dv} dv, \quad |\text{Re} \ k| < \frac{1}{4}.
\]

**Proof of Lemma 3.** Part 2 of Lemma 1 implies \( F(v), F''(v) \) are both continuous for \( v > 1 \), \( \lim_{v \to +1} F(v) = 0 \). Hence we satisfy a theorem (relating to the Abel integral equation) [1, p. 5] (this theorem can be modified to include singularities of the type \( (x - 1)^a, \quad x \sim +1, \quad \text{Re} \ a > -1 \), our case, see [1, p. 6]), which implies the conclusion of Lemma 3.

The rest of the proof of Theorem 1 consists mainly in applying the above lemmas to show that all the operations we use to show that (2) is a solution to (1) are valid.

Using the integral representation for \( P_{t/2 - 1/2}^{k} \) from [5, p. 165], we obtain from (2), the iterated integral,
\[
(3) \quad f(x) = a(k)x \int_{0}^{\infty} \left( \int_{1}^{\infty} p \sin (xs) ds \right) dt
\]
(see Lemma 1 for the definition of \( p \))
\[
a(k) = 2^{1/2} \pi^{-3/2} \Gamma \left( \frac{1}{2} - k \right) \sin \left( \pi \left( \frac{1}{2} + k \right) \right), \quad x \geq 0, \quad |\text{Re} \ k| < \frac{1}{4}.
\]
(We note (3) is valid by Lemma 2.)

We now apply to the right-hand side of (3) the following operations in this order,
1. integration over a triangular domain (see Lemma 2),
2. integration by parts with respect to \( s \),
3. the Fourier cosine transform.
Since operations 1, 2, 3 are now permissible by Lemmas 1, 2 (\( g \in G \)),
we obtain from (3) the valid identity
\[
\int_0^\infty \cos (tx)f(x)dx = a_\ell(k) \frac{d h}{dt} \quad \text{(see Lemma 1 for definition of } h) \tag{4}
\]

\[
a_\ell(k) = (2\pi)^{-1/2} \Gamma\left(\frac{1}{2} - k\right) \sin \left(\left(\frac{1}{2} + k\right)\pi\right),
\]

\[
t > 0, \ |\Re k| < \frac{1}{4}.
\]

Lemma 3 implies all the operations (those indicated in Lemma 3) to show the right-hand side of (4) is a solution to an Abel integral equation are now permissible [1, p. 9]. (Again we note only real \( k \) are treated on p. 9, but the theory can be extended to complex \( k \), our case.) Hence applying these operations (those indicated in Lemma 3 to the right-hand side of (4), we obtain the valid identity
\[
g(\cosh t) = \int_0^t \left(\int_0^\infty udx\right)ds, u = a_\ell(k)(\sinh t)^k(\cosh t - \cosh s)^{-1/2 - k}
\]

\[
\cos (sx)f(x), \ a_\ell(k) = (2^{-1/2})^{-1/2} \left(\frac{1}{2} - k\right)^{-1}, \ t > 0, \ |\Re k| < \frac{1}{4}.
\]

Interchanging the order of integration of the iterated integral on the right-hand side of (5) (which is now permissible by part 1 of Lemma 1), then using the integral representation for \( P_{1/2-1/2} \) from [2, p. 156], we obtain the valid identity \( L_\ell(L_\ell(g)) = g, \ t > 0, \ |\Re k| < 1/4 \). This completes the proof of Theorem 1.

**Corollary 1.** Let \( g_1, g_2 \in G \) such that \( L_\ell(g_1) = L_\ell(g_2) \), then \( g_1(t) = g_2(t), \ t > 0, \ |\Re k| < 1/4 \).

**Proof.** Let \( u = g_1 - g_2 \). Then \( u \in G \). Hence \( L_\ell(u) = 0 \) by linearity of \( L_\ell \). Hence \( f(x) \) (of (3)) = 0, \( x \geq 0 \). We then obtain from (5) the conclusion of Corollary 1.

**Theorem 2.** Let \( F \) be the class of real valued functions such that \( f \in F \) if and only if

1. \( f(x) = x^2 f'(x) \), \( f'(x) \) is continuous for \( x \geq 0 \), and of bounded variation on any closed and bounded interval contained in \( \infty > x \geq 0 \).
2. \( f, f' = O(x^{-1-\varepsilon}), \ x \sim + \infty, \ \varepsilon > 0 \).

Then \( L_\ell(L_\ell(f)) = f, \ x \geq 0, \ |\Re k| < 1/2 \).

**Proof of Theorem 2.**

**Lemma 4.** Let \( f \in F, g = L_\ell(f) \), then

1. \( \int_1^A |g(x)|dy \) exists for any \( A > 1 \).
2. \( g = 0((\cosh^{-1} y)^{-2}(y^2 - 1)^{-1/4}), y \sim + \infty, \)

\[
\text{providing } |\text{Re } k| < 1/2.
\]

**Proof of Lemma 4.** From formula 26 [2, p. 129],

(a) \( P^k_{x-1/2}(\cos t) = (2\pi \sinh t)^{-1/2}(e^{-i\pi} f_1 + e^{i\pi} f_2), \)

\[
f_1(x) = \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - k - ix)} f_3, f_3 = F\left(\frac{1}{2} + k, \frac{1}{2} - k, 1 + ix; - \frac{1}{2}e^{-t} \cosh t\right),
\]

\[
f_2(x) = f_1(-x), F(a, b, c; z) = M \int_0^1 w^s d w, w = s^{1/2} - (1 - s)^{1/2} - (1 - z s)^{-s},
\]

\[
\text{Re } b, \text{Re } (c - b) > 0, |z| < 1, M \text{ independent of } z[2, p. 59].
\]

(b) \( z^{b-a}(\Gamma(z + a)/\Gamma(z + b)) \sim a_n + a_z z^{-1} + \cdots \) (an asymptotic series), \( |z| \sim + \infty \) uniformly for \( |\arg z| \leq \pi - \epsilon, \pi/2 > \epsilon > 0 [2, p. 47], \)

so differentiation of the right-hand side of (b) is permissible [3, p. 21]. From (a) we conclude \( (1 + x)^{-1/2+k} f'_3(x), (1 + x)^{-1/2+k} f''_3(x) \) are uniformly bounded for \( x \geq 0 \) and \( t \geq 1, \) providing \( |\text{Re } k| < 1/2. \) In (1) we now use the integral representation from (a), then integrate by parts with respect to \( x, \) which is permissible \( (f \in F) \) to conclude

\[
g^{(i)}(y) = (\cosh^{-1} y)^{-1}(y^2 - 1)^{-1/4} \int_0^\infty e^{\pm it} e^{(øj) (y, x, k)} dx, y \geq 2, |\text{Re } k| < 1/2, \)

further the real and imaginary parts \( c^{(j)} \) are of bounded variation in \( x \) on the infinite interval \( \infty \geq x \geq 0, y \geq 2, |\text{Re } k| < 1/2. \) Hence the real and imaginary parts of \( c^{(j)} \) can each be written as the difference of two monotonically decreasing functions \( c^{(j)}(x), x \geq 0, \)

\[
\lim_{x \to +\infty} c^{(j)}(x) = 0 \text{ uniformly in } y \geq 2, c^{(j)}(x) \text{ uniformly bounded}, x \geq 0, y \geq 2, |\text{Re } k| < 1/2, \)

Also \( g(y) = O((y - 1)^{-1/4}), 2 > y > 1, |\text{Re } k| < 1/2, \) by (5) (in the proof of Theorem 1), \( f \in F. \) Hence Lemma 4 holds.

**Lemma 5.** The \( g \) of Lemma 4 implies \( \int_0^\infty \left( \int_q^\infty |\hat{f}| dt \right) dq < \infty, x \geq 0, |\text{Re } k| < 1/2 \) (see Lemma 2 of Theorem 1 for the definition of \( \hat{f} \)).

**Proof.** Using the change of variable \( (\cosh t - \cosh q) = (\cosh q + 1)w, \) we conclude

\[
\int_q^\infty |\hat{f}| dt \leq M (\sinh q/2)^{-1} |(\sinh q)^{1-k}(\cosh q)^{1-k} g(\cosh q)|, q > 0, x \geq 0, M \text{ a constant, } |\text{Re } k| < 1/2. \) Hence the conclusion of Lemma 5 follows.

The rest of the proof of Theorem 2 consists mainly in justifying in reverse order all the formulas arising from the solution of the integral equation \( L_x(f) = g \) in the proof of Theorem 1. Hence we will point only where the rest of the proof of Theorem 2 must be modified from that of Theorem 1.
REMARK 1. The inversion theorem for the solution to the Abel integral equation [1, p. 9] appealed to in the proof of Theorem 1 has been modified to include functions which have singularities of the type \((x - 1)^\alpha, x \sim +1, \Re \alpha > -1\). Hence this modified form of the theorem applies again to our case (see (5) in the proof of Theorem 1) since we have a singularity of this type when we use the change of variable \(s = \cosh q\).

REMARK 2. Lemma 5, \(f \in F\) imply the sum \(\hat{h}(+\infty) - \hat{h}(0), x \geq 0, |\Re k| < 1/2, \) of the upper and lower limits (both are finite) (arising when one does an integration by parts, i.e., the reverse operation corresponding to the one of part 2 of (3) in the proof of Theorem 1) is zero.

REMARK 3. Lemma 5 implies the \(g\) of Lemma 4 satisfies the conclusion of Lemma 2 of Theorem 1. Hence the reverse operation of integrating over a triangular domain (see Lemma 2 of Theorem 1) is now permissible. Hence we conclude all the reverse formulas are valid. This completes the proof of Theorem 2.

COROLLARY 2. Let \(f_1, f_2 \in F\) such that \(L_i(f_i) = L_i(f_2)\). Then \(f_1(x) = f_2(x), x \geq 0, |\Re k| < 1/2\).

Proof. Let \(r = f_1 - f_2\). Then \(r \in F\). Hence by linearity \(L_i(r) = 0\). Then by (3) of Theorem 1 (see also Lemma 5 of Theorem 2) we obtain the conclusion of Corollary 2.

We note in closing, using the change of variable \((\cosh t - \cosh q) = (\cosh q + \cos a)s,\) the integral representations for \(P^k_{\alpha - 1/2}\) in Theorem 1 and [5], we obtain a pair of reciprocal transforms

1. \(g(\cosh q) = \sin a(\cosh q + \cos a)^{-2/2+k}(\sinh q)^{-k}, |a| < \pi/2,\)
2. \(f(x) = 2^{1/2} \pi^{-1/2}(\Gamma(1/2 - k))^{-1} \beta(1/2 - k, 1)x \Gamma(1/2 - k + ix) \Gamma(1/2 - k - ix) \sinh ax, |\Re k| < 1/2.\) (The case \(k = 0\) specializes to the example in [4].) \(\beta \equiv\) Beta function. Further, \(g \in G\) of Theorem 1 and \(f \in F\) of Theorem 2.

If in Theorem 1, part 1, we now assume \(g_i\) is analytic for \(y \geq 1, \Re k < 1/2,\) in 2 we assume \(n \geq 0\) and arbitrary, then by the methods in the proofs of Theorems 1 and 2 (we use the integral representation for \(P^k_{\alpha - 1/2}\) from (5) in \(L_2\)), we conclude \(c(k) = L_i(L_i(g))\) is an analytic function in \(k\) for \(\Re k < 1/2, y > 1\). Hence by analytic continuation, Theorem 1 and Corollary 1 are now valid for \(\Re k < 1/2\).
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