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**ON AN INVERSION THEOREM FOR THE GENERAL
MEHLER-FOCK TRANSFORM PAIR**

PETER MICHAEL ROSENTHAL

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Let $P_m^k(y)$ be the Legendre function of the first kind and let $\Gamma(z)$ be the Gamma function. Then the general Mehler-Fock transform of complex order k of a function $g(y)$ is defined by the equation

$$f(x) = L_2(g) = \pi^{-1}x \sin h(\pi x) \Gamma\left(\frac{1}{2} - k - ix\right) \\
 \times \Gamma\left(\frac{1}{2} - k + ix\right) \int_1^\infty g(y) P_{ix-1/2}^k(y) dy,$$

the inversion theorem states

$$g(y) = L_1(f) = \int_0^\infty f(x) P_{ix-1/2}^k(y) dx.$$

It is stated on page 416 of I. N. Sneddon's book 'The Use of Integral Transforms, (1972) that apparently a class of functions $g(y)$ for which this result is valid is not yet clearly defined. The purpose of this paper is to define a class of functions $g(y)$ as well as a class $f(x)$ and give proofs that the above inversion formula hold for these classes.

Introduction. The theorem and proofs presented in the paper are basically a generalization of those in a paper of V. Fock [4] who treated the case $k = 0$, the Mehler-Fock transform. Some applications of the Mehler-Fock transform and general Mehler-Fock transform are given in [7], [8]. Tables of these transforms are given in [6].

All integrals are taken in the improper (complex) Riemann sense. $x \sim +\infty$ means x positive and sufficiently large, $x \sim +1 \equiv$ sufficiently close to 1, $x > 1$.

THEOREM 1. *Let G be the class of complex valued functions such that $g \in G$ if and only if*

1. $g(y) = (y - 1)^{-k/2} g_1(y)$, $y > 1$, $g_1(y)$ is twice differentiable and continuous for $y \geq 1$, the real and imaginary parts of $g_1''(y)$ are of bounded variation on any closed and bounded interval contained in $\infty > y \geq 1$.

2. $d^n g_1 / dy^n = O(y^{-(1/2)-n+(k/2)-\epsilon})$, $y \geq 1$, $1/4 > \epsilon > 0$, $0 \equiv$ large order relation, $n = 0, 1, 2$ (the case $n = 0$ means g_1).

Then $L_1(L_2(g)) = g$, $y > 1$, $|\operatorname{Re} k| < 1/4$.

Proof of Theorem 1.

LEMMA 1. *Let*

$$g \in G, h(t) = \int_0^t p(t, q) dq, p = (\sinh q)^{1-k} (\cosh t - \cosh q)^{-1/2+k} g(\cosh q),$$

$$f(x) = \int_0^\infty \cos(xt) h'(t) dt, |\operatorname{Re} k| < \frac{1}{4}.$$

Then

1. $f(x) = O(x^{-2}), x \sim +\infty, \int_0^\infty |f(x)| dx < \infty.$
- 2a. $h'(t)$ is continuous for $t \geq 0.$
- 2b. $h'(t)$ satisfies the conditions of a Fourier inversion theorem [9, p. 13], h', h'' are both absolutely integrable over the infinite interval $\infty \geq t \geq 0, \lim_{t \rightarrow +0, +\infty} h = 0, \lim_{t \rightarrow +\infty} h' = 0.$
3. $\int_0^\infty \left(\int_0^t |p| dq \right) dt < \infty.$

Proof of Lemma 1. Let $s = \cosh t, r = \cosh q, r = (s-1)w + 1.$
Then

$$p = (s-1)^{(1+k)/2} ((s-1)w + 2)^{-k/2} g((s-1)w + 1) c(w),$$

$$c(w) = (1-w)^{-(1/2+k)} w^{-k/2}.$$

Hence there exists $c_n(w)$ independent of t such that

$$\left| \frac{\partial^n p}{\partial t^n} \right| \leq e^{-\varepsilon t} |c_n(w)|, t \sim +\infty, \int_0^1 |c_n| dw < \infty, \frac{1}{4} > \varepsilon > 0, n$$

$$= 0, 1, 2, |\operatorname{Re} k| < \frac{1}{4}.$$

Again by dominated convergence we conclude $d^n h/dt^n = \int_0^1 (\partial^n p / \partial t^n) dw,$
 $\infty > t \geq 0, n = 1, 2, |\operatorname{Re} k| < 1/4.$ Hence parts 2, 3 of Lemma 1 hold. We are now permitted to integrate by parts with respect to t the right-hand side of the defining formula for $f(x)$ in the hypothesis of Lemma 1 to conclude $f(x) = x^{-1} F(x), F(x) = \int_0^\infty \sin(xt) h''(t) dt.$ Since $h''(t) = O(e^{-\varepsilon t}), t \sim +\infty, 1/4 > \varepsilon > 0,$ we conclude the real and imaginary parts of $h''(t)$ are of bounded variation in the infinite interval $\infty \geq t \geq 0$ (see I.P. Natanson "Theory of Functions of a Real Variable", p. 238, for definitions and theorem). This implies $F(x) = O(x^{-1}), x \sim +\infty.$ This completes the proof of Lemma 1.

LEMMA 2. *Let $g \in G.$ Then*

$$\lim_{A \rightarrow +\infty} \int_0^{A>0} \left(\int_0^t \hat{f} dq \right) dt = \lim_{A \rightarrow +\infty} \int_0^A \left(\int_q^A \hat{f} dt \right) dq = \int_0^\infty \left(\int_q^\infty \hat{f} dt \right) dq,$$

$$\hat{f} = p \sin(xt), \quad x \geq 0, \quad |\operatorname{Re} k| < \frac{1}{4}.$$

(See Lemma 1 for the definition of p .)

Proof of Lemma 2. Since $g \in G$, the iterated integrals in Lemma 2 are equal for finite A . Part 3 of Lemma 1 implies absolute integrability of the first iterated integral in Lemma 2. Hence we satisfy Fubini's theorem which implies Lemma 2.

LEMMA 3. *Let*

$$F(v) = \int_1^v (v-s)^{-1/2+k} r ds, \quad r = (s^2-1)^{-k/2} g(s), \quad g \in G.$$

Then

$$\frac{d}{dt} \int_1^t (t-v)^{-1/2-k} F(v) dv = \int_1^t (t-v)^{-1/2-k} \frac{dF}{dv} dv, \quad |\operatorname{Re} k| < \frac{1}{4}.$$

Proof of Lemma 3. Part 2 of Lemma 1 implies $F(v)$, $F'(v)$ are both continuous for $v > 1$, $\lim_{v \rightarrow +1} F(v) = 0$. Hence we satisfy a theorem (relating to the Abel integral equation) [1, p. 5] (this theorem can be modified to include singularities of the type $(x-1)^a$, $x \sim +1$, $\operatorname{Re} a > -1$, our case, see [1, p. 6]), which implies the conclusion of Lemma 3.

The rest of the proof of Theorem 1 consists mainly in applying the above lemmas to show that all the operations we use to show that (2) is a solution to (1) are valid.

Using the integral representation for $P_{ix-1/2}^k$ from [5, p. 165], we obtain from (2), the iterated integral,

$$(3) \quad f(x) = a(k)x \int_0^\infty \left(\int_t^\infty p \sin(xs) ds \right) dt$$

(see Lemma 1 for the definition of p)

$$a(k) = 2^{1/2} \pi^{-3/2} \Gamma\left(\frac{1}{2} - k\right) \sin\left(\left(\frac{1}{2} + k\right)\pi\right), \quad x \geq 0, \quad |\operatorname{Re} k| < \frac{1}{4}.$$

(We note (3) is valid by Lemma 2.)

We now apply to the right-hand side of (3) the following operations in this order,

1. integration over a triangular domain (see Lemma 2),
2. integration by parts with respect to s ,
3. the Fourier cosine transform.

Since operations 1, 2, 3 are now permissible by Lemmas 1, 2 ($g \in G$),

we obtain from (3) the valid identity

$$\int_0^\infty \cos(tx)f(x)dx = a_1(k)\frac{dh}{dt} \quad (\text{see Lemma 1 for definition of } h)$$

$$(4) \quad a_1(k) = (2\pi)^{-1/2}\Gamma\left(\frac{1}{2} - k\right) \sin\left(\left(\frac{1}{2} + k\right)\pi\right),$$

$$t > 0, |\operatorname{Re} k| < \frac{1}{4}.$$

Lemma 3 implies all the operations (those indicated in Lemma 3) to show the right-hand side of (4) is a solution to an Abel integral equation are now permissible [1, p. 9]. (Again we note only real k are treated on p. 9, but the theory can be extended to complex k , our case.) Hence applying these operations (those indicated in Lemma 3 to the right-hand side of (4), we obtain the valid identity

$$(5) \quad g(\cosh t) = \int_0^t \left(\int_0^\infty u dx \right) ds, \quad u = a_2(k)(\sinh t)^k (\cosh h t - \cosh s)^{-1/2-k}$$

$$\cos(sx)f(x), \quad a_2(k) = (2^{-1}\pi)^{-1/2} \left(\Gamma\left(\frac{1}{2} - k\right) \right)^{-1}, \quad t > 0, |\operatorname{Re} k| < \frac{1}{4}.$$

Interchanging the order of integration of the iterated integral on the right-hand side of (5) (which is now permissible by part 1 of Lemma 1), then using the integral representation for $P_{ix-1/2}^k$ from [2, p. 156], we obtain the valid identity $L_1(L_2(g)) = g, t > 0, |\operatorname{Re} k| < 1/4$. This completes the proof of Theorem 1.

COROLLARY 1. *Let $g_1, g_2 \in G$ such that $L_2(g_1) = L_2(g_2)$, then $g_1(t) = g_2(t), t > 0, |\operatorname{Re} k| < 1/4$.*

Proof. Let $u = g_1 - g_2$. Then $u \in G$. Hence $L_2(u) = 0$ by linearity of L_2 . Hence $f(x)$ (of (3)) = 0, $x \geq 0$. We then obtain from (5) the conclusion of Corollary 1.

THEOREM 2. *Let F be the class of real valued functions such that $f \in F$ if and only if*

1. $f(x) = x^2 f_1(x), f_1'(x)$ is continuous for $x \geq 0$, and of bounded variation on any closed and bounded interval contained in $\infty > x \geq 0$.
2. $f, f' = O(x^{-\epsilon}), x \sim +\infty, \epsilon > 0$.

Then $L_2(L_1(f)) = f, x \geq 0, |\operatorname{Re} k| < 1/2$.

Proof of Theorem 2.

LEMMA 4. *Let $f \in F, g = L_1(f)$, then*

1. $\int_1^A |g(x)| dy$ exists for any $A > 1$.

2. $g = 0((\cosh^{-1} y)^{-2}(y^2 - 1)^{-1/4})$, $y \sim + \infty$,
 providing $|\operatorname{Re} k| < 1/2$.

Proof of Lemma 4. From formula 26 [2, p. 129],

$$(a) \quad P_{ix-1/2}^k(\cosh t) = (2\pi \sinh t)^{-1/2}(e^{-itx}f_1 + e^{itx}f_2),$$

$$f_1 = \frac{\Gamma(-ix)}{\Gamma\left(\frac{1}{2} - k - ix\right)}f_3, \quad f_3 = F\left(\frac{1}{2} + k, \frac{1}{2} - k, 1 + ix; -\frac{1}{2}e^{-t} \cosh t\right),$$

$$f_2(x) = f_1(-x), \quad F(a, b, c; z) = M \int_0^1 wds, \quad w = s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a},$$

$\operatorname{Re} b, \operatorname{Re}(c - b) > 0, |z| < 1, M$ independent of z [2, p. 59].

(b) $z^{b-a}(\Gamma(z + a)/\Gamma(z + b)) \sim a_1 + a_2z^{-1} + \dots$ (an asymptotic series), $|z| \sim + \infty$ uniformly for $|\arg z| \leq \pi - \varepsilon, \pi/2 > \varepsilon > 0$ [2, p. 47], so differentiation of the right-hand side of (b) is permissible [3, p. 21]. From (a) we conclude $(1 + x)^{-1/2+k}f_3'(x), (1 + x)^{-1/2+k}f_3''(x)$ are uniformly bounded for $x \geq 0$ and $t \geq 1$, providing $|\operatorname{Re} k| < 1/2$. In (1) we now use the integral representation from (a), then integrate by parts with respect to x , which is permissible ($f \in F$) to conclude $g^{(j)}(y) = (\cosh^{-1} y)^{-1}(y^2 - 1)^{-1/4} \int_0^\infty e^{\pm itx}c^{(j)}(y, x, k)dx, y \geq 2, |\operatorname{Re} k| < 1/2$, further the real and imaginary parts $c^{(j)}$ are of bounded variation in x on the infinite interval $\infty \geq x \geq 0, y \geq 2, |\operatorname{Re} k| < 1/2$. Hence the real and imaginary parts of $c^{(j)}$ can each be written as the difference of two monotonically decreasing functions $c_n^{(j)}(x), x \geq 0, \lim_{x \rightarrow +\infty} c_n^{(j)}(x) = 0$ uniformly in $y \geq 2, c_n^{(j)}$ are uniformly bounded, $x \geq 0, y \geq 2, |\operatorname{Re} k| < 1/2, n = 1, 2, j = 1, 2$, since $f(x) = O(x^{-1-\varepsilon}), x \sim + \infty$. Also $g(y) = O((y - 1)^{-1/4}), 2 > y > 1, |\operatorname{Re} k| < 1/2$, by (5) (in the proof of Theorem 1), $f \in F$. Hence Lemma 4 holds.

LEMMA 5. *The g of Lemma 4 implies $\int_0^\infty \left(\int_q^\infty |\hat{f}| dt\right) dq < \infty, x \geq 0, |\operatorname{Re} k| < 1/2$ (see Lemma 2 of Theorem 1 for the definition of \hat{f}).*

Proof. Using the change of variable $(\cosh t - \cosh q) = (\cosh q + 1)w$, we conclude $\int_q^\infty |\hat{f}| dt \leq M(\sinh q/2)^{-1}|(\sinh q)^{1-k}(\cosh q)^k g(\cosh q)|, q > 0, x \geq 0, M$ a constant, $|\operatorname{Re} k| < 1/2$. Hence the conclusion of Lemma 5 follows.

The rest of the proof of Theorem 2 consists mainly in justifying in reverse order all the formulas arising from the solution of the integral equation $L_1(f) = g$ in the proof of Theorem 1. Hence we will point only where the rest of the proof of Theorem 2 must be modified from that of Theorem 1.

REMARK 1. The inversion theorem for the solution to the Abel integral equation [1, p. 9] appealed to in the proof of Theorem 1 has been modified to include functions which have singularities of the type $(x - 1)^a$, $x \sim +1$, $\operatorname{Re} a > -1$. Hence this modified form of the theorem applies again to our case (see (5) in the proof of Theorem 1) since we have a singularity of this type when we use the change of variable $s = \cosh q$.

REMARK 2. Lemma 5, $f \in F$ imply the sum $\hat{h}(+\infty) - \hat{h}(+0)$, $x \geq 0$, $|\operatorname{Re} k| < 1/2$, of the upper and lower limits (both are finite) (arising when one does an integration by parts, i.e., the reverse operation corresponding to the one of part 2 of (3) in the proof of Theorem 1) is zero.

REMARK 3. Lemma 5 implies the g of Lemma 4 satisfies the conclusion of Lemma 2 of Theorem 1. Hence the reverse operation of integrating over a triangular domain (see Lemma 2 of Theorem 1) is now permissible. Hence we conclude all the reverse formulas are valid. This completes the proof of Theorem 2.

COROLLARY 2. Let $f_1, f_2 \in F$ such that $L_1(f_1) = L_1(f_2)$. Then $f_1(x) = f_2(x)$, $x \geq 0$, $|\operatorname{Re} k| < 1/2$.

Proof. Let $r = f_1 - f_2$. Then $r \in F$. Hence by linearity $L_1(r) = 0$. Then by (3) of Theorem 1 (see also Lemma 5 of Theorem 2) we obtain the conclusion of Corollary 2.

We note in closing, using the change of variable $(\cosh t - \cosh q) = (\cosh q + \cos a)s$, the integral representations for $P_{iz-1/2}^k$ in Theorem 1 and [5], we obtain a pair of reciprocal transforms

1. $g(\cosh q) = \sin a(\cosh q + \cos a)^{-3/2+k}(\sinh q)^{-k}$, $|a| < \pi/2$,
2. $f(x) = 2^{1/2}\pi^{-1/2}(\Gamma(1/2 - k))^{-1}\beta(1/2 - k, 1)x\Gamma(1/2 - k + ix)\Gamma(1/2 - k - ix)\sinh ax$, $|\operatorname{Re} k| < 1/2$. (The case $k = 0$ specializes to the example in [4].) $\beta \equiv$ Beta function. Further, $g \in G$ of Theorem 1 and $f \in F$ of Theorem 2.

If in Theorem 1, part 1, we now assume g_1 is analytic for $y \geq 1$, $\operatorname{Re} k < 1/2$, in 2 we assume $n \geq 0$ and arbitrary, then by the methods in the proofs of Theorems 1 and 2 (we use the integral representation for $P_{iz-1/2}^k$ from (5) in L_2), we conclude $c(k) = L_1(L_2(g))$ is an analytic function in k for $\operatorname{Re} k < 1/2$, $y > 1$. Hence by analytic continuation, Theorem 1 and Corollary 1 are now valid for $\operatorname{Re} k < 1/2$.

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