STOPPING TIMES FOR BERNOUlli AUTOMORPHISMS

ALAN SALESKI
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The purpose of this note is to study a certain class of stopping times for Bernoulli automorphisms by means of the Friedman-Ornstein results concerning weakly Bernoulli partitions.

1. Introduction. Let $T$ be an automorphism of the non-atomic Lebesgue space $(X, \mathcal{A}, \mu)$ and let $\theta: X \to \mathbb{Z}^+$ be a measurable function. If the transformation $S = T^\theta$ defined by $S(x) = T^{\theta(x)}(x)$, for $x \in X$, is an automorphism of $X$ then $\theta$ is called a stopping time for $T$. Such a stopping time will be said to be of $n$th order (where $n$ is a positive integer or $\infty$) if there exists a decreasing sequence $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$ of measurable subsets of $X$ satisfying:

(a) $\mu(D_n) > 0$ and $\mu(D_{n+1}) = 0$ if $n$ is finite

or

(b) $\mu(D_i) > 0$ for all $i$ and $\mu(\bigcap_i D_i) = 0$ if $n$ is infinite such that $T^\theta$ coincides (modulo 0) with the automorphism $M$ defined by

$$M(x) = T_{D_0} \circ T_{D_1} \circ T_{D_2} \circ \cdots \circ T_{D_{n-1}} \circ T_{D_n}(x), \quad \text{for } x \in D_s - D_{s+1},$$

for $s = 0, 1, 2, \cdots, n - 1$, where $D_0 = X$ and $T_{D_i}$ denotes the automorphism induced by $T$ on $D_i$.

Neveu has shown [3] that every stopping time $\theta$ for which $T^\theta$ is ergodic is an $n$th order stopping time for a unique $n$. Moreover, the sets $D_1, D_2, \cdots$ are also unique (modulo 0). It follows from the work of Belinskaya [1] that if $\theta$ is an $n$th order stopping time for $T$ then $h(T^\theta) = n h(T)$.

The purpose of this note is to study certain ergodic properties of $T^\theta$ under the hypothesis that $T$ is a Bernoulli automorphism. For definitions and notation of entropy theory the reader is referred to [4] and [6]. For convenience of notation we shall let $P_\infty = \bigvee_\infty T^\theta P$, $P^+ = \bigvee_{\theta=0}^\infty T^\theta P$, and $P^- = \bigvee_{\theta=-\infty}^0 T^\theta P$ where $m > n$ and $P$ is a partition of $X$.

2. We now establish a result, using a technique developed in [7], concerning a special class of stopping times for a Bernoulli automorphism.

Let $T$ be a Bernoulli automorphism of $X$, and let $B$ be a Bernoulli partition for $T$, i.e., $B$ is a generator and the orbit of $B$ under $T$, $\{T^i B: i \in \mathbb{Z}\}$, is a jointly independent sequence of partitions. We let $\mathcal{F}_B$ denote the collection of all measurable partitions $P$ of $X$ for which $H(P^+ | B^+) + H(P^- | B^-) < \infty$. 
THEOREM. Let $T$ be a Bernoulli automorphism of $X$ and $B$ be a Bernoulli partition. Let $\theta$ be an $n$th order stopping time for $T$ and let $D_0, D_1, \cdots$ be the sets corresponding to $\theta$. Let $P$ denote the partition $\{X - D_i, D_i - D_{i+1}; i = 1, 2, \cdots\}$. Suppose $P \in \mathcal{F}_n$. Then $S = T^\theta$ is weakly mixing if and only if $S$ is a Bernoulli automorphism having entropy $n h(T)$.

To prove this theorem we will require the following lemma.

**Lemma 1.** Let $A, F$ and $C$ be measurable partitions of $X$ such that $F$ is independent of $A$ and $H(C \mid A) < \epsilon$. Let $D \subseteq C$ be finite. Then

$$H(D \mid F) \geq H(D) - \epsilon.$$ 

**Proof.** Choose $F_n \subseteq F$ such that $F_n \uparrow F$ and $H(F_n) < \infty$. Then

$$H(F_n \mid D) \geq H(F_n \mid A) - H(D \mid A) \geq H(F_n) - \epsilon.$$ 

Hence

$$H(D \mid F_n) \geq H(D) - \epsilon.$$ 

Letting $n \to \infty$ we obtain the desired result.

**Proof of theorem.** Let $K$ be any positive integer, $\epsilon > 0$, and $Q = B^{\infty}_K$ (this notation will be employed only with respect to the automorphism $T$). Choose $N > 0$ such that $H(P^0_N \mid B^\infty_N) < \epsilon/4$ and $H(P^0_0 \mid B^-N) < \epsilon/4$. Let $R = \max\{N, K\}$. If $S$ is weakly mixing then there is an integer $M > R$ for which

$$H(S^{-M}Q \mid B^-R) \geq H(S^{-M}Q) - \frac{\epsilon}{4}.$$ 

Since

$$S^{-M}Q \subseteq B^K_\infty \vee P^0_\infty,$$ 

and

$$H(B^R_\infty \vee P^0_\infty \mid B^-R) < \frac{\epsilon}{4},$$

Lemma 1 implies that

$$H(S^{-M}Q \vee B^R_{-R} \mid B^R_{R+1}) \geq H(S^{-M}Q \vee B^R_{-R}) - \frac{\epsilon}{4}.$$ 

Using the fact that
we obtain:

\[
H\left( S^{-M} Q \bigg| \bigvee_0^\infty S^i Q \right) \geq H(S^{-M} Q \mid B_{-R}^\infty \vee P_0^\infty)
\]
\[
\geq H(S^{-M} Q \mid B_{-R}^\infty) - H(P_0^\infty \mid B_{-R}^\infty)
\]
\[
\geq H(S^{-M} Q \mid B_{-R}^\infty) - \frac{\epsilon}{4}
\]
\[
= H(S^{-M} Q \vee B_{-R}^R \mid B_{R+1}^R) - H(B_{-R}^R \mid B_{R+1}^R) - \frac{\epsilon}{4}
\]
\[
\geq H(S^{-M} Q \vee B_{-R}^R) - H(B_{-R}^R) - \frac{\epsilon}{2}
\]
\[
= H(S^{-M} Q \mid B_{-R}^R) - \frac{\epsilon}{2}
\]
\[
\geq H(S^{-M} Q) - \epsilon.
\]

Since \( K \) and \( \epsilon \) were arbitrary, there exists an integer \( p > N \) for which

\[
H\left( S^{-p} (B_{-R}^R) \bigg| \bigvee_0^\infty S^i(B_{-R}^R) \right) \geq H(B_{-R}^R) - \frac{\epsilon}{2}.
\]

Now, for all \( t > p \),

\[
H\left( \bigvee_p^t S^i Q \bigg| \bigvee_0^\infty S^i Q \right) \geq H\left( \bigvee_p^t S^i Q \bigg| B_{-R}^R \vee P_0^\infty \right)
\]
\[
\geq H\left( \bigvee_p^t S^i Q \bigg| B_{-R}^R \right) - H(P_0^\infty \mid B_{-R}^R)
\]
\[
\geq H\left( \bigvee_p^t S^i Q \bigg| B_{-R}^R \right) - \frac{\epsilon}{4}
\]
\[
= H\left( \bigvee_p^t S^i Q \vee B_{-R}^{R-1} \mid B_{-R}^{R-1} \right) - \frac{\epsilon}{4}
\]
\[
= H\left( \bigvee_p^t S^i Q \vee B_{-R}^R \mid B_{R+1}^R \right) - H(B_{-R}^R) - \frac{\epsilon}{4}
\]
\[
\geq H\left( \bigvee_p^t S^i Q \vee B_{-R}^R \right) - \frac{\epsilon}{2} - H(B_{-R}^R)
\]
\[
\geq H\left( \bigvee_p^t S^i Q \right) + H(B_{-R}^R) - \epsilon - H(B_{-R}^R)
\]
\[
= H\left( \bigvee_p^t S^i Q \right) - \epsilon.
\]

This verifies that \( Q \) is a weakly Bernoulli partition for \( S \) and thus, applying the Friedman-Ornstein theorem [2], \( Q \) generates a Bernoulli
factor. As a result of Ornstein’s theorem 2 of [5], letting $K \to \infty$, we find that $S$ is actually a Bernoulli automorphism.

3. We now illustrate some consequences of the theorem in the case of second order stopping times.

We omit the proofs of the following two elementary lemmas.

**Lemma 2.** If $R$ is an automorphism of $X$ and $A$ is a measurable subset of $X$ for which $\bigcup^n_0 R^iA = X$ then $R$ is ergodic if and only if $R_A$ is ergodic.

**Lemma 3.** Let $R$ be an automorphism of $X$. If $R^2$ is weakly mixing then so is $R$.

**Proposition 1.** Let $T$ be an automorphism of $X$ and let $\theta$ be a second order stopping time for $T$. Then $S = T^\theta$ is ergodic if and only if both $T$ and $(T^D_1)^2$ are ergodic.

*Proof.* If $S = T^\theta$ is ergodic it is well-known that $S_{D_1} = (T_{D_1})^2$ is also ergodic. Hence $T_{D_1}$ is ergodic. From $\bigcup^{n\infty}_0 S^iD_1 = X$ it follows that $\bigcup^{n\infty}_0 T^iD_1 = X$. Applying Lemma 2 we obtain the ergodicity of $T$.

Conversely suppose $T$ and $(T^D_1)^2$ are ergodic. In view of Lemma 2, it suffices to show $\bigcup^{n\infty}_0 S^iD_1 = X$. One easily verifies that

$$S\left(\bigcup^n_0 T^iD_1\right) \cup D_1 = \bigcup^{n+1}_0 T^iD_1 \quad (\text{for } n \geq 0)$$

from which is obtained $\bigcup^{n\infty}_0 S^iD_1 = \bigcup^{n\infty}_0 T^iD_1 = X$.

**Corollary 1.** Let $T$ be a Bernoulli automorphism of $X$, $B$ be a Bernoulli partition for $T$, and $\theta$ be a second order stopping time defined by choosing $D_1$ to be an atom of $\bigvee^K_X T^iB$ for any integer $K$. Then $S = T^\theta$ is ergodic.

*Proof.* It follows from a corollary of Theorem 1 of [7] that $T_{D_1}$ is Bernoulli and hence, of course, $(T_{D_1})^2$ is ergodic. Thus Proposition 1 yields that $S$ is ergodic.

**Proposition 2.** Let $T$ be a Bernoulli automorphism of $X$ and $B$ be a Bernoulli partition for $T$. Let $\theta$ be a second order stopping time for which $\{D_n, X - D_1\} \in \mathcal{F}_B$. Then the following are equivalent:

(a) $T_{D_1}$ is weakly mixing.
(b) $T_{D_1}$ is Bernoulli.
(c) $S_{D_1}$ is Bernoulli.
(d) $S_{D_1}$ is weakly mixing.
Proof. Using Lemma 3 together with the observation that $S_{D_i} = (T_{D_i})^i$ and Theorem 1 of [7] the proof is immediate.

**Proposition 3.** Let $T$ be a Bernoulli automorphism of $X$ and $B = \{B^1, B^2, \cdots, B^K\}$ be a Bernoulli partition for $T$. Let $\theta$ be the second order stopping time for $T$ defined by choosing $D_i = B^i$. Then $S = T^\theta$ is mixing.

Proof. Let $K$ be a fixed positive integer and set $Q = B^K$. As a consequence of the definition of $S$ one can verify that

$$\not\exists_k S'^k Q \leq B_k^\infty$$ and $$\not\exists_\infty S'^k Q \leq B_\infty^\infty.$$

So if $A$ and $B$ are members of the algebra generated by the atoms of $Q$ then $\mu(S'^k A \cap B) \rightarrow \mu(A)\mu(B)$. Now a standard approximation argument will show that $S$ is mixing.

**Corollary 2.** Under the hypotheses of Proposition 3 the automorphism $S = T^\theta$ is Bernoulli.

Proof. This follows immediately from our theorem.

**References**


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