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Vashishtha Narayan Singh

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REPRODUCING KERNELS AND OPERATORS WITH A CYCLIC VECTOR I

VASHISHTHA N. SINGH

In this paper a study is begun of the complete unitary invariant $((1-wT)^{-1}e, (1-zT)^{-1}e)$, first considered by Livsic in his paper 'On Spectral Resolution of Linear Nonself Adjoint Operators' Mat. Sb., 34 (76), 1954, 145-199, of a triple (T, H, e) where T is a bounded linear operator on a Hilbert space H and e is a cyclic vector for T in H, as a reproducing kernel. One of the important points is the construction of a subset of the group algebra of the torus closed under pointwise addition and convolution. This obviously will generate a ring called the K-ring. A study of this ring will be done later.

Several other theorems and constructions are also given.

Introduction. Let T be a bounded linear operator on a Hilbert space H with a topologically cyclic vector e in H. In this paper we wish to study certain analytic functions associated with the triple (T, H, e) for the sake of the problem of invariant subspaces of T in H. (See also [8] and [15].)

The paper is divided into six sections. In § 1 we present some facts about reproducing kernels with analyticity properties. In § 2 we consider a triple (T, H, e) of the above type. H can then be represented as a Hilbert space of conjugate analytic functions $\alpha_e[H]$ with a reproducing Kernel K. T^* on H assumes the form of reverse shift on the Taylor coefficients of functions in $\alpha_e[H]$. (See also [11] or [19].) In § 3 we recover (T, H, e) from the reproducing kernel of $\alpha_e[H]$ in two ways. The notion of an analytic function of positive definite type is introduced and it is shown that only these can arise as reproducing kernels of $\alpha_e[H]$. These functions are also related to invariant subspaces (§ 4). In § 5 a category of triples is constructed and it is connected to the harmonic analysis of the two-torus via the analytic functions of positive definite type. Section 6 consists of some examples and counterexamples about the analytic functions of positive definite type.

The paper is based on the author's dissertation written under the guidance of Professor John L. Kelley of the University of California, Berkeley. The author would like to thank Professor Kelley for help and advice.

1. Reproducing kernels and analytic functions of positive type. We start with the definition of a reproducing kernel. Let

H be a Hilbert space of functions on a set X with the inner product $(\,,)$. In Theorem 1.2 we present a few facts of a theory of H due essentially to E. H. Moore and N. Aronszajn. See [1] and [2].

DEFINITION 1.1. A reproducing kernel for H is a complex valued function K on $X \times X$ such that:

(i) For all $y \in X$, $K_y \in H$ where $K_y(x) = K(x, y)$.

(ii) If $g \in H$ then for all $y \in X$, $(g, K_y) = g(y)$, and

(iii) The linear span of the set $\{K_y\}_{y \in X}$ of functions is dense in H.

Note that if K is a reproducing kernel then the map $g \mapsto g(y) = (g, K_y)$ is a bounded linear functional on H; i.e., evaluation at a point is continuous. Also if H is a pre-Hilbert space of funtions on a set X with continuous evaluation then the completion $\bar{H}$ of H can be realized as a space of functions on X by setting $h(x) = (h, K_x)$ for $x \in X$ and $h \in \bar{H}$ where $K_x \in \bar{H}$ is such that $(f, K_x) = f(x)$ for all $f \in H$.

THEOREM 1.2. *Let H be a pre-Hilbert space of functions on a set X so that the evaluation map $f \mapsto f(x)$ is continuous for all $x \in X$, let $\bar{H}$ be the completion of H. Then the following hold*

(i) *There is a unique reproducing kernel K for H.*

(ii) *If, for $x \in X$, $\varepsilon_x \in \bar{H}$, is such that $f(x) = (f, \varepsilon_x)$ for all $f \in H$ then $K(x, y) = (\varepsilon_y, \varepsilon_x)$.*

(iii) *If $\{b_i\}_{i \in I}$ is an orthonormal basis for $\bar{H}$, then $K(x, y) = \sum_{i \in I} b_i(x)\overline{b_i(y)}$, and*

(iv) *If H is the set of finite formal sums $\sum_{i=0}^{n} a_i x_i$, where $x_i \in X$ and a_i's are complex numbers, with an inner product $(\,,)$ given by $(\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{m} b_j y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \bar{b}_j K(y_j, x_i)$, the map $\sum_{i=1}^{n} a_i x_i \mapsto \sum_{i=1}^{n} a_i K_{x_i}$ of formal sums to members of H is inner product preserving and its image is dense in $\bar{H}$.*

See [1] for a proof.

A reproducing kernel is a function of positive type in the following sense.

DEFINITION 1.3. A complex valued function K defined on $X \times X$ is a function of positive type iff $\sum_{i,j=1}^{n} a_i \bar{a}_j K(x_j, x_i) \geqq 0$ for all complex numbers $a_1, a_2, \cdots, a_n$ and all members $x_1, x_2, \cdots, x_n$ of X.

Since $\sum_i a_i K(x, x_i) = (\sum_i a_i K_{x_i}, K_x) \leqq \sqrt{K(x, x)}\ \sqrt{\sum a_i \bar{a}_j K(x_j, x_i)}$ from the Cauchy inequality, a function of positive type K also has a further property, namely that if $\sum_{i,j=1}^{n} a_i \bar{a}_j K(x_j, x_i) = 0$ then $\sum a_i K(x, x_i) = 0$ for all $x \in X$. Thus it is clear from part (iv) of Theorem 1.2 that a function of positive type is a reproducing kernel. (A function of positive type K is necessarily selfadjoint i.e., $K(x, y) = \overline{K(y, x)}$ for all x, y in X.)

DEFINITION 1.4. For each $s > 0$, $D_s(\bar{D}_s)$ is the open (closed) disc of radius s about 0 in the complex plane $\mathbb{C}$. A_s is the space of functions defined and conjugate analytic in D_s and $A_{s,s}$ the space of functions defined on $D_s \times D_s$ which are conjugate analytic in the first variable and analytic in the second variable. $\mathscr{P}$ is the space of polynomials with complex coefficients. If K is a function of positve type and is in $A_{s,s}$ for some s then K is an analytic function of positive type abbreviated a.f.p.t.

Let $K \in A_{s,s}$. The following theorem gives an alternate construction of a Hilbert space which is equivalent to K being a function of positive type.

THEOREM 1.5. *Let* $K \in A_{s,s}$. *Then*

(i) *K is an a.f.p.t. if* $1/4\pi^2 \oint_C \oint_C K(z, w)\overline{p(z^{-1})}p(w^{-1})\overline{(dz/z)} \times (dw/w) \geqq 0$ *for all* $p \in \mathscr{P}$ *where the path of integration C is a simple contour in D_s such that its inverse winds around* $\overline{D_{1/s}}$ *once.*

(ii) *Let* $(H, (\ ,\ ))$ *be a pre-Hilbert space of functions analytic in a neighborhood of* $\overline{D_{1/s}}$ *with the inner product* $(p, q)_K = 1/4\pi^2 \oint_C \oint_C K(z, w)p(w^{-1})\overline{q(z^{-1})(dz/z)(dw/w)}$. *Then the map which takes a member K_α of H, α a member of D_s, into the rational function $1/(1 - \alpha z)$ is inner-product preserving.*

Proof. Consider $1/4\pi^2 \oint \oint K(z, w)\overline{p(z^{-1})}p(w^{-1})\overline{(dz/z)}(dw/w)$ as a limit of Riemann sums $1/4\pi^2 \sum_{i,j=1}^n K(z_j, z_i)\overline{p(z_j^{-1})}p(z_i^{-1})\overline{(z_{j+1} - z_j/z_j)} \times (z_{i+1} - z_i/z_i)$. Since K is a function of positive type it is clear that $1/4\pi^2 \oint \oint K(z, w)\overline{p(z^{-1})}p(w^{-1})\overline{(dz/z)}(dw/w)$ is positive. Let now $a_1, a_?, \cdots, a_n$ be any complex numbers and let $\alpha_1, \cdots, \alpha_n \in D_s$. Consider the rational function f given by $f(z) = \sum_i a_i(1 - \alpha_i z)^{-1}$. Then $\sum_{i,j=1}^n a_i\bar{a}_jK(\alpha_j, \alpha_i) = 1/4\pi^2 \oint_{C_1} \oint_{C_2} K(z, w)\overline{f(z^{-1})}f(w^{-1})\overline{(dz/z)}(dw/w) = \lim_{i\to\infty} 1/4\pi^2 \oint_{C_1} \oint_{C_2} K(z, w) \times \overline{p_i(z^{-1})}p_i(w^{-1})\overline{(dz/z)}(dw/w) \geqq 0$ where p_i is a sequence of polynoimials converging to f uniformly on compact subsets of $D_{1/s}$ and C_1, C_2 are chosen suitably. Thus (i) is proved the proof of (ii) follows from the observation that the norm in the space of rational functions of the function $z \mapsto \sum_{i=1}^n a_i(1 - \alpha_i z)^{-1}$ is $\sum_{i,j=1}^n a_i\bar{a}_jK(\alpha_j, \alpha_i)$ which is the same as the norm of $\sum_{i=1}^n a_iK_{\alpha_i}$ in H.

2. The kernel function of cyclic triple. Consider now a Hilbert space H and a bounded linear operator T on H. We construct certain functions of positive type associated with T and H. Some facts from the functional calculus of T are used. (See [12].) If b is a complex valued function analytic in the neighborhood of the spectrum $\sigma(T)$ of T define $b(T)$ as follows.

DEFINITION 2.1. Let C be a contour lying in the domain of analyticity of b which winds around each point of $\sigma(T)$ once. Then $b(T) = 1/2\pi i \oint_C b(z)(zI - T)^{-1}dz$. It is known that $b \mapsto b(T)$ is a linear and multiplicative homomorphism of the space of functions analytic in a neighborhood of $\sigma(T)$ into the space of bounded linear operators on H. (See [12], page 199.)

The following result is probably well-known and will be used later.

THEOREM 2.2. *Let T be a bounded linear operator on a Hilbert space H and suppose that there is an analytic function b defined and nonzero on a connected neighborhood of the spectrum $\sigma(T)$ of T such that $b(T) = 0$. Then there is a nonzero polynomial p such that $p(T) = 0$.*

We leave the proof of this fact to the reader. See for example [20].

Now we make the following definition.

DEFINITION 2.3. For $r > 0$. B_r is the space of functions analytic in a neighborhood of the closed disc $\bar{D}_r$ with topology given as the inductive limit topology of the spaces $A(U)$ of the functions analytic in s neighborhood U of $\bar{D}_r$. (See [14], page 219, problem D.)

It is known that B_r is a Montel space. (See [14], page 196, problem F(e).)

For $r > 0$, define $\bar{A}_r$ as follows.

DEFINITION 2.4. $\bar{A}_r$ is the space consisting of functions which are complex conjugates of the functions in A_r.

Let E be a linear space with a locally convex topology τ and let E' be its dual. Then the strong topology for E' is the topology of uniform convergence on τ-bounded sets.

The following result will be important in the sequel.

LEMMA 2.5. *Let $r > 0$. Then $\bar{A}_{1/r}$ and B_r are strong duals for the pairing given by $[f, g] = \oint_C f(z)g(z^{-1})dz/z$ for $f \in B_r$ and $g \in \bar{A}_{1/r}$ where C is the contour $t \mapsto (r + \varepsilon)e^{it}$, $0 \leqq t \leqq 2\pi$, for some small ε depending on f and g.*

Proof. We present an outline of the proof and refer to [14] for more information. Since B_r is a Montel space it is reflexive ([14], 20 F(a)) and hence it suffices to show that the dual of B_r, with strong topology is $\bar{A}_{1/r}$, for the pairing above.

To show this, let for each $z \in D_{1/r}$, h_z be the member of B_r given by $h_z(\zeta) = 1/(1 - \zeta z)$ for $\zeta \in \bar{D}_r$. For any continuous linear functional g on B_r let $i_0(g)$ be defined by $i_0(g)(z) = g(h_z)$. Then $i_0(g)$ is a well-defined function and is a member of $\bar{A}_{1/r}$. Moreover, $g(f) = 1/2\pi i \oint_C f(z) i_0(g)(z^{-1}) dz/z = 1/2\pi i[f, i_0(g)]$, where C is a contour as in the statement of the proposition ε being so small that it is contained in the domain of holomorphy of f. Conversely any member of $\bar{A}_{1/r}$ defines a continuous linear functional on B_r by the above formula and thus $\bar{A}_{1/r}$ and B'_r are algebraically isomorphic.

To complete the proof it remains to show that the topology of u.c.c. on $\bar{A}_{1/r}$ coincides with the strong topology of B'_r. To prove this first observe that each bounded set of B_r is contained in a bounded set of $\bar{A}_{r'}$ for some $r' > r$ ([14], 17G (6) (iii)). Hence it follows that B'_r with strong topology is metrizable ([14], 18.4). Since each bounded subset of $A_{r'}$ is uniformly bounded on every compact subset of $D_{r'}$, a Cauchy sequence in the u.c.c. topology of $\overline{A_{1/r}}$ is also a Cauchy sequence in its strong topology. That a Cauchy sequence in the strong topology of B'_r is also a Cauchy sequence in its u.c.c. topology follows from the observation that for any compact subset K of $D_{1/r}$ the family of functions $\{h_z\}_{z \in K}$ is bounded in B_r and also the fact that for any $h \in \overline{A_{1/r}}$, $[h_z, h] = 2\pi i h(z)$ where $z \in K$.

Consider now an operator T on a Hilbert space H and let $e \in H$.

DEFINITION 2.6. A Hilbert triple is a triple $(T.H.e)$ where H is a Hilbert space. T a bounded linear operator on H and e a member of H. A Hilbert triple is a cyclic triple if the orbit of e (the linear span of the set $\{T^i e\}_{i=0}^{\infty}$ is dense in H. If $r = \|T\|$ the map $\beta_e \colon B_r \to H$ is given by $\beta_e(b) = b(T)e$.

The map β_e depends on T, H, and e. Note that $\beta_e(B_r)$ is dense in H iff (T, H, e) is cyclic.

The following is a consequence of Theorem 2.2.

THEOREM 2.7. *If (T, H, e) is cyclic then H is infinite dimensional if and only if β_e is injective.*

Proof. If H is finite dimensional then it follows from the Caley-Hamilton theorem of linear algebra that $p(T) = 0$ for some $p \in \mathscr{P}$ ([4], page 320) and thus β_e is not injective. Since (T, H, e) is cyclic the converse is the statement of Theorem 2.2.

Now we establish commutativity of certain maps and construct a reproducing kernel which is an a.f.p.t. The reason for this round about construction will be clear from Theorem 4.3.

DEFINITION 2.8. Multiplication by z, $M_z\colon B_r \to B_r$ is the operator given by $M_z f(\zeta) = \zeta f(\zeta)$ for $\zeta \in \operatorname{Domain}(f)$.

The following lemma is an easy consequence of the functional calculus.

LEMMA 2.9. *β_e and M_z are continuous linear maps such that $\beta_e(1) = e$ and $T\cdot\beta_e = \beta_e\cdot M_z$.*

DEFINITION 2.10. $\phi\colon H \to H'$ is defined by $\phi(x)(y) = (y, x)$ for all x, yH.

H' is conjugate linearly isomorphic to H via the map ϕ and $\phi\cdot T^* = T'\cdot\phi$ where T^* is the Hilbert space adjoint of T on H and T' is the Banach space dual of T. Also from $T\cdot\beta_e = \beta_e\cdot M_z$ we obtain $\beta'_e\cdot T' = M'_z\cdot\beta'_e$ where β'_e, M'_z, etc. are Banach space duals of β_e, M_z, etc. We know from Lemma 2.5 that B'_r can be identified with $A_{1/r}$ via the map i_0.

DEFINITION 2.11. The map $\alpha_e\colon H \to A_{1/r}$ is the composition of the maps ϕ, β'_e, i_0 and — as given in the diagram

$$H \xrightarrow{\varphi} H' \xrightarrow{\beta'_e} B'_r \xrightarrow{i_0} A_{1/r} \xrightarrow{-} A_{1/r}$$

where — takes a function into its complex conjugate.

From Lemma 2.9 and the discussion following it we see that there is a map $S^*\colon A_{1/r} \to A_{1/r}$ so that $\alpha_e\cdot T^* = S^*\cdot\alpha_e$ where α_e is as above.

THEOREM 2.12. *Let (T, H, e) be a Hilbert triple with $\|T\| = r$. Then $S^*\alpha_e = \alpha_e\cdot T^*$ where α_e is such that $\alpha_e(x)(z) = (x, (1 - zT)^{-1}e)$ for all $z \in D_{1/r}$ and all $x \in H$, and $S^*f(z) = f(z) - f(0)/z$. Moreover, α_e is injective iff (T, H, e) is cyclic and in that case, if $\dim H = \infty$, the range of α_e is dense in $A_{1/r}$.*

Proof. We show that α_e is such that $\alpha_e(x)(z) = (x, (1 - zT)^{-1}e)$ for all $z \in D_{1/r}$ and $S^*f(z) = f(z) - f(0)/z$, because if this were so then $\alpha_e(T^*x)(z) = (T^*x, (1 - zT)^{-1}e) = (x, (1 - zT)^{-1}Te) = S^*(\alpha_e(x))(z)$ and hence $S^*\cdot\alpha_e = \alpha_e\cdot T^*$. Now $-0i_0 0\beta' 0\phi(x)(z) = \overline{\beta' 0\phi(x)(h_z)} = (x)(\beta_e(h_z)) = \overline{((1 - zT)^{-1}e, x)} = (x, (1 - zT)^{-1}e)$.

Now we prove the rest of the theorem. Since $\alpha_e(x) = 0$ iff $(x, (1 - zT)^{-1}e) = 0$ for all $z \in D_{1/r}$ it follows that $\alpha_e(x) = 0$ iff $(x, p(T)e) = 0$ for all $p \in \mathscr{P}$. If e is cyclic then this holds iff $x = 0$. The statement that if e is cyclic and if $\dim H = \infty$ then the range of α_e is dense in $A_{1/r}$ is a mere dualisation of the statement in Theorem 2.2.

Note that in general α_e is injective on the orbit of e and zero on its orthogonal complement. If e is cyclic then we can assign an inner product $(\ )_{\alpha_e}$ on $\alpha_e[H]$ by $(\alpha_e(x), \alpha_e(y))_{\alpha_e} = (x, y)$. This inner product makes T^* on H unitarily equivalent to S^* on $\alpha_e[H]$. Moreover, and this is the point of the construction, $\alpha_e[H]$ has a reproducing kernel.

THEOREM 2.13. *Let (T, H, e) be cyclic. Then for each $z \in D_{1/r}$ the evaluation map $\alpha_e(x) \mapsto \alpha_e(x)(z)$ defined on $\alpha_e[H]$ is continuous and $\alpha_e(x)(z) = (\alpha_e(x), \varepsilon_z)$ where $\varepsilon_z = \alpha_e((1-zT)^{-1}e)$. Consequently $\alpha_e[H]$ has a reproducing kernel K given explicitly as $K(z, w) = ((1 - wT)^{-1}e, (1 - zT)^{-1}e) = \sum_{m,n=0}^{\infty} (T^n e, T^m e)\bar{z}^m w^n$.*

Proof. Since $\alpha_e(x)(z) = (x, (1 - zT)(^{-1}e) = (\alpha_e(x), \alpha_e((1 - zT)^{-1}e))_{\alpha_e}$ for all $\alpha_e(x) \in \alpha_e[H]$, we see that the evaluation at z is explicitly given by inner product with a member of $\alpha_e[H]$ and is hence continuous. Furthermore, the particular member of $\alpha_e[H]$ corresponding to evaluation at z is $\alpha_e((1 - zT)^{-1})$ and hence $\varepsilon_z = \alpha_e((1 - zT)^{-1}e)$. Now in view of Theorem 1.2 it follows that $\alpha_e[H)$ has a reproducing kernel given explicitly as $K(z, w) = (\alpha_e((1 - wT)^{-1}e), \alpha_e((1 - zT)^{-1}e))_{\alpha_e} = ((1 - wT)^{-1}e, (1 - zT)^{-1}e)$.

DEFINITION 2.14. The kernel function of a cyclic triple (T, H, e) is the reproducing kernel of $\alpha_e[H]$. If (T, H, e) is just a Hilbert triple, not necessarily cyclic, then the kernel function for (T, H, e) is defined to be the kernel function for $(T_0, 0(e), e)$ where $0(e)$ is the closed orbit of e and T_0 is the restriction of T to $0(e)$.

Two cyclic triples (T_1, H_1, e_1) and (T_2, H_2, e_2) are defined to be unitarily equivalent, $(T_1, H_1, e_1) \sim (T_2, H_2, e_2)$, iff there exists a unitary map $U: H_1 \to H_2$ with $U \cdot T_1 = T_2 \cdot U$ and $U(e_1) = e_2$.

Let (T, H, e) be a cyclic triple with kernel function K. Consider the Hilbert space $\alpha_e[H]$ with the inner product $(\ )_{\alpha_e}$ obtained from (T, H, e). We can construct $\alpha_e[H]$ from K in the manner of Theorem 1.2, since K is the reproducing kernel for $\alpha_e[H]$. Since $\alpha_e(e) = K_0$ we have the following as a corollary to the preceding.

COROLLARY 2.15. *Let (T, H, e) be a cyclic triple. Then the kernel function K for (T, H, e) is a complete unitary invariant for (T, H, e), i.e., (T^*, H, e) is unitarily equivalent via α_e to $(S^*, \alpha_e[H], K_0)$.*

The above corollary can also be deduced directly. See for example [15].

3. The triple of an A.F.P.T. In this section we study the kernel function of a triple more closely. First we mention some properties of a kernel function. Note that if K is the kernel function of (T, H, e) where $\|T\| = r$ then K can be written as $\sum_{m,n=0}^{\infty}(T^n e, T^m e)\bar{z}^m w^n$ for $z, w \in D_{1/r}$. The boundedness of T is reflected in a special property of K. The infinite dimensionality of H is also reflected in another property of K. We describe these properties. First the following definitions.

DEFINITION 3.1. The map $\mathscr{S}\colon A_{s,s} \to A_{s,s}$ is given by $\mathscr{S}(a)(z, w) = [a(z, w) - a(z, 0) - a(0, w) + a(0, 0)]/(\bar{z}w)$ for $z, w \in D_s$. A member K of $A_{s,s}$ which is an a.f.p.t. is an analytic function of positive definite type (abbreviated a.f.p.d.t.) iff there is a positive real r so that $r^2K - \mathscr{S}(K)$ is also an a.f.p.t. We write $\rho(K)$ for the least such r. $K \in A_{s,s}$ is a degenerate a.f.p.t. iff K is an f.p.t. and there is a polynomial p so that $\oint_C \oint_C K(z, w)\overline{p(z^{-1})}p(w^{-1})\overline{(dz/z)}(dw/w) = (p, p) = 0$, where C is such that its inverse contains $\overline{D_{1/s}}$ and is preferably a circle with center 0.

It follows from Theorem 1.5 that if 'a' is a function of positive type so is $\mathscr{S}(a)$, since $\oint_C \oint_C \mathscr{S}(a)(z, w)\overline{p(z^{-1})}p(w^{-1})\overline{(dz/z)}(dw/w) = \oint_C \oint_C a(z, w)p_1(z^{-1})p_1(w^{-1})\overline{(dz/z)}(dw/w)$, where p_1 is the polynomial such that $p_1(z) = zp(z)$. As shown by the following result the kernel function of a Hilbert triple is always an a.f.p.d.t.

PROPOSITION 3.2. *If K is the kernel function for the triple (T, H, e) then K is an a.f.p.d.t. and $\mathscr{S}(K)$ is the kernel function for the triple (T, H, Te). Moreover, if (T, H, e) is cyclic, then $= \rho(K) = \|T\|$ and K is nondegenerate iff* $\dim H = \infty$.

We omit the straightforward proof of this proposition.

Recall that if (T, H, e) is a cyclic triple then we have seen in § 2 (Theorem 2.15) that (T^*, H, e) is unitarily equivalent via α_e to $(S^*, \alpha_e[H], K_0)$ and that $\alpha_e[H]$ with $(\ )_{\alpha_e}$ is a space of functions with continuous evaluations at points, thus the space $\alpha_e[H]$ and in fact the triple (T^*, H, e) and hence (T, H, e), is determined by the kernel function K. We want to describe explicitly in two different ways corresponding to the two Theorems 1.2 and 1.5 of § 1, the construction of (T, H, e) from K.

First it follows from $\alpha_e \cdot T^* = S^* \cdot \alpha_e$ and the fact that $\alpha_e^* = \alpha_e^{-1}$ that $\alpha_e \cdot T = S \cdot \alpha_e$ where S is the adjoint of S^* relative to the inner product $(\ )_{\alpha_e}$ of $\alpha_e[H]$. We construct $\alpha_e[H]$ in terms of the kernel K of (T, H, e). We make the construction for an arbitrary a.f.p.t.

$K \in A_{s,s}$. A formal for S is given in Proposition 3.4. First a definition.

DEFINITION 3.3. For K an a.f.p.t., $K \in A_{r,r}$, let H^K be the completion of the space of functions which are finite linear combinations $\sum_i a_i K_{\alpha_i}$ for $a_i \in \mathbb{C}$ and $\alpha_i \in D_r$ with respect to the inner product $(\sum_i a_i K_{\alpha_i}, \sum_j b_j K_{\beta_j})^k = \sum_{i,j} a_i \bar{b}_j K(\beta_j, \alpha_i)$.

We have the following result.

PROPOSITION 3.4. *Let $K \in A_{r,r}$ be an a.f.p.t. Then $H^K \subset A_r$. If K is the kernel function for a cyclic triple (T, H, e) then $\alpha_e[H] = H^K$ and the adjoint S of S^*: $H^K \to H^K$ is given by $S(K_w) = K_w - K_0/w$ for $w \neq 0$ and $S(K_0) = dK/dw\,|_{w=0}$.*

Proof. The fact that H^K consists of functions is known from §1. Let $f \in H^K$ and let $f_n = \sum_i a_i^n K_{\alpha_i^n}$ be a sequence of finite sums converging to f. Then since $|\sum a_i K_{\alpha_i}(z)|^2 \leqq K(z, z)\,\|a_i K_{\alpha_i}\|^2$ it follows that the sequence f_n of conjugate analytic functions is a Cauchy sequences in the topology of uniform convergence on compact sets of D_r. Hence the limit function f is also conjugate analytic. Thus $H^K \subset A_r$.

Since, if K is the kernel function for (T, H, e) then $K(z, w) = ((1 - wT)^{-1}e, (1 - zT)^{-1}e)$, $\alpha_e[H] = H^K$. Since e is cyclic the adjoint S of S^* is given by

$$S(K_w)(z) = ((1 - wT)^{-1}Te, (1 - zT)^{-1}e)$$
$$= \left(\frac{(1 = wT)^{-1}e - e}{w}, (1 - zT)^{-1}e\right) = \frac{K_w - K_0}{w}(z)$$

for $w \neq 0$ and $S(K_0)(z) = (Te, (1 - zT)^{-1}e) = dK/dw\,|_{w=0(z)}$.

COROLLARY 3.5. *If K is the kernal function of the cyclic triple (T, H, e) then $(T, H, e)$is unitarily equivalent via α_e to (S, H^K, K_0).*

Also this,

COROLLARY 3.6. *(S, H^K, K_0) is a cyclic triple with kernel function K.*

Now we make the second construction corresponding to Theorem 1.5 of §1. First a definition.

DEFINITION 3.7. If $K \in A_{1/r,1/r}$ and K is an a.f.p.t., H_K is the completion of the space B_r of functions analytic in a neighborhood of $\overline{D_r}$ with respect to $(\ )_K$ given by $(b_1, b_2) = 1/4\pi^2 \oint_{C_1} \oint_{C_2} K(z, w) \times b_1(w^{-1})\overline{b_2(z^{-1})(dz/z)}(dw/w)$, where b_1, b_2 are members of B_r and where

C_i is a contour such that its inverse is rectifiable in the complex plane, contains $\bar{D}_r$ and has winding number 1 about $\bar{D}_r$ and is contained in the domain of holomorphy of b_i. If multiplication by z, M_z is bounded on B_r relative to $(\ \ )_K$ then $M_z^\wedge$ is its bounded extension to H_K.

H_K is well-defined despite the arbitrariness of the contours and r. This is so because of Cauchy's theorem. Also note that H_K does not consist of functions in general (see counterexample 1, § 6).

PROPOSITION 3.8. *Let K be an a.f.p.t. $\in A_{1/r,1/r}$. Then*
(i) *M_z on B_r has bounded extension $M_z^\wedge$ to H_K iff K is an a.f.p.d.t.*
(ii) *If K is an a.f.p.d.t. then K is the kernel function of $(M_z^\wedge, H_K, 1)$.*
(iii) *If K is the kernel function of a cyclic triple (T, H, e) then $(M_z^\wedge, H_K, 1)$ is unitarily equivalent to (T, H, e) under the map $p \mapsto p(T)e$ for all polynomials p.*

Proof.
(i) K is an a.f.p.d.t. iff there is $t \geqq 0$ so that $t^2K - \mathscr{S}(K)$ is of positive type. However, $\oint\oint \mathscr{S}(K)(z, w)\overline{a(z^{-1})}a(w^{-1})\overline{(dz/z)}(dw/w) = \oint\oint K(z, w)\overline{(M_z a)(z^{-1})}(M_w a)(w^{-1})\overline{(dz/z)}(dw/w), a \in B_r$. Hence $t^2K - \mathscr{S}(K)$ is a function of positive definite type iff $t \geqq \| M_z^\wedge \|$, B_r being dense in H_K. To prove
(ii) note that if $\lambda \in D_{1/r}$ then the function $R_\lambda\colon z \to (1 - \lambda z)^{-1} \in B_r$ and $(R_\lambda, R_\mu) = 1/4\pi^2 \oint_{C_1}\oint_{C_2} K(z, w)1/((1 - \lambda w^{-1})\overline{(1 - \mu z^{-1})})\overline{(dz/z)}(dw/w)$ where C_1, C_2 are contours specified as in Definition 3.7. It follows from the Cauchy formula (see [13], page 26) that $(R_\lambda, R_\mu) = K(\lambda, \mu)$ and thus (ii) is proved.
The proof of (iii) is straightforward and is left to the reader.

COROLLARY 3.9. *A function K is the kernel function for some cyclic triple iff K is an a.f.p.d.t.*

4. Further properties of an A.F.P.T.

DEFINITION 4.1. The linear map $C'\colon H^K \to H_K$ is given on K_α's by $C'(K_\alpha)(z) = (1 - \alpha z)^{-1}$ and the map $J\colon H_K \to H^K$ is given on polynomials by $J(p) = 1/2\pi i \oint_C K_w p(w^{-1})dw/w$ for a simple contour C which lies in the domain of analyticity of the function $w \mapsto K_w$ and whose inverse contains $\overline{D_r}$ and has winding number 1 about it.

To be sure we should refer to vector valued integration but here we mean that $J(p)$ is the function so that $J(p)(z) = 1/2\pi i \oint_C K(z, w) \times p(w^{-1})dw/w$. Clearly $J(p)$ is independent of the contour C so long it has the properties given in the definition.

We have the following proposition.

PROPOSITION 4.2. *If (T, H, e) is a cyclic triple then α_e and C' are unitary and the diagram*

$$\begin{array}{ccccc} H & \xrightarrow{\alpha_e} & H^K & \xrightarrow{C'} & H_K \\ {\scriptstyle T}\downarrow & & {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle M_z^{\wedge}} \\ H & \xrightarrow{\alpha_e} & H^K & \xrightarrow{C'} & H_K \end{array}$$

is commutative. Moreover, $C'^{-1} = J$ for any a a.f.p.t. K and C' is unitary for any a.f.p.t.

Proof. We know that α_e is unitary. That C' is unitary for any a.f.p.t. K is a consequence of Theorem 1.5. Since $\alpha_e[H] = H^K$ we know that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\alpha_e} & H^K \\ {\scriptstyle T}\downarrow & & {\scriptstyle S}\downarrow \\ H & \xrightarrow{\alpha_e} & H^K \end{array}$$

is commutative. To prove that the rest of the diagram is also commutative it suffices to prove that $C'\cdot S = M_z$. C' on elements of the type K_α of H^K. However, this is true since $C'\cdot S(K_\alpha)(z) = C'$.

$$\frac{K_\alpha - K_0}{\alpha}(z) = \frac{(1-\alpha z)^{-1} - 1}{\alpha} = z(1-\alpha z)^{-1} = M_z\cdot C'(K)(z) .$$

Now we prove that $C'^{-1} = J$ for any a.f.p.t. K. To do so it suffices to consider elements of the form $C'(K_\alpha)$ of H_K. For such an element

$$J\cdot C'(K_\alpha) = \frac{1}{2\pi i}\oint_C K_w(1-\alpha w^{-1})(dw/w) = \frac{1}{2\pi i}\oint_C \frac{K_w}{w-\alpha}dw = K_\alpha .$$

Now let H be the closure of B_r under an inner product. Then we have the

THEOREM 4.3. *$H = H_K$ for some a.f.p.t. $K \in A_{1/r,\,1/r}$ iff $H^* = H^K$.*

Proof. If $H = H_K$ then the inclusion of B_r in H is continuous where B_r is taken with its u.c.c. topology. Hence the dual of H can be identified with H^K as in Proposition 4.2. Conversely if $H^* = H^K$ for some a.f.p.t. K in $A_{1/r,\,1/r}$, then the dual can be identified with H is the same as $H_K \subset B_r$ via the map C'.

The following two theorems give characteristics of an a.f.p.d.t. supplementing Definition 3.1 and furnishing a connection between an a.f.p.d.t. and an invariant subspace.

THEOREM 4.4. *Let $K \in A_{r,r}$ be an a.f.p.t. Then K is an a.f.p.d.t. if and only if H^K is S^*-invariant.*

The above theorem is an immediate consequence, of the fact that the inclusion of H^K in A_r is continuous, and the closed graph theorem. We leave the proof to the reader.

Let (T, H, e) be a cyclic triple. We prove another proposition which relates invariant subspaces of T in H to functions of positive definite type arising from the kernel function of (T, H, e) and the projection corresponding to the invariant subspace.

THEOREM 4.5. *Let (T, H, e) be a cyclic triple. Then P is an orthogonal projection so that $P(H)$ is invariant under T if and only if the function K' defined by $K'(z, w) = ((I - P)(1 - wT)^{-1}e, (I - P) \times (1 - zT)^{-1}e)$ is an a.f.p.d.t.*

We have to show that $P(H)$ is invariant under T if and only if the operator $(I - P)x \mapsto (I - P)Tx$ is bounded. We prove this fact as a consequence of a more general lemma.

LEMMA 4.6. *Let (T, H, e) be a Hilbert triple and $\tilde{H}$ a first countable Hausdorff linear topological space, $\tilde{T}$ a continuous linear operator defined on $\tilde{H}$ and $\tilde{e}$ a member of $\tilde{H}$. Let $H_{\tilde{e}} = \{x \in H;$ there is a sequence $\{p_i(T)e\}_i \to x$ and $\{p_i(\tilde{T})\tilde{e}\}_i \to 0$ where each $p_i \in P$, $i = 1, 2, \cdots\}$. Then $H_{\tilde{e}}$ is a closed invariant subspace for T.*

Proof. $H_{\tilde{e}}$ is clearly a linear subspace of H invariant under T. The whole difficulty lies in showing that it is closed. This is also quite straightforward and we do it as follows. Let $\{p_i^m(T)e\}_i$ be a sequence converging to x_m and let also the $\{x_m\}_m$ converge to x. Also let $\{p_i^m(\tilde{T})\tilde{e}\}_i$ converge to 0 for each m. Let $B(x, 2^{-n})(\bar{B}(x, 2^{-n}))$ be the open (closed) ball of radius 2^{-n} with center at x and let $\{n_i\}_i$ be the sequence for which $\{x_m\}_m \cap B(x, 2^{-n_i}) - \bar{B}(x, 2^{-n_i-1}) \neq \emptyset$. Such a sequence $\{n_i\}_i \to \infty$ exists since $\{x_m\}_m \to x$. For each n_i choose an $x_{n_i} \in B(x, 2^{-n_i}) - \bar{B}(x, 2^{-n_i-1})$. Suppose also that $\{U_{n_i}\}_i$ is a countable neighborhood basis at 0 for $\tilde{H}$. For each n_i choose a $p_{j(n_i)}^{ni}$ so that $p_{j(n_i)}^{ni}(\tilde{T})\tilde{e} \in U_{n_i}$ and $p_{j(n_i)}^{ni}(T)e$ is in $B(x, z^{-n_i}) - \bar{B}(x, 2^{-n_i-1})$. It is possible to do so since $\{p_j^{n_i}(T)e\}_j \to x_{n_i}$ and $\{p_j^{n_i}(\tilde{T})\tilde{e}\}_j \to 0$ for each n_i by supposition.

Now the sequence $\{p_{j(n_i)}^{n_i}(\tilde{T})\tilde{e}\}_{n_i}$ has 0 as a cluster point in $\tilde{H}$ so there is a proper infinite subsequence of this sequence which tends to 0. Since $\{p_{j(n_i)}^{n_i}(T)e\}_{n_i} \to x$ in that subsequence if we replece T and e by T and e it tends to x. Thus $H_{\tilde{e}}$ is closed.

The proof of this lemma can perhaps be simplified by construction of a bounded intertwining operator from H to $\tilde{H}$ whose kernel coincides with the subspace given by the limits of $\{p_i(T)e\}_i$ as above.

Now we complete the proof of Theorem 4.5 as follows.

Proof of Theorem 4.5. Suppose that $P(H)$ is invariant for T. We know that $\|p\|_{H_{K'}} = \|(I-P)p(T)e\|_H$. Thus $\|M_zp\|_{H_{K'}} = \|(I-P)Tp(T)e\|_H = \|(I-P)TPp(T)e + (I-P)T(I-P)p(T)e\|_H = \|(I-P)T(I-P)p(T)e\|_H \leqq \|T(I-P)p(T)e\|_H \leqq M\|(I-P)p(T)e\|_H = M\|p\|_{H_{K'}}$, where M is the norm of T, and the suffix indicates the space in which norm is to be taken. It follows that M_z is bounded on $H_{K'}$ and hence K' is an a.f.p.d.t.

In order to prove the converse we use Lemma 4.6 and put for $\tilde{H}$, $\tilde{T}$, $\tilde{e}$ $H_{K'}$, $M_z^{\wedge}$ and 1 respectively.

We conclude by characterizing interlacing maps of cyclic triples to another triple in terms of kernel functions as a preparation for § 5.

THEOREM 4.7. *Let (T_1, H_1, e_1) be a cyclic triple and let (T_2, H_2, e_2) be another triple and ϕ_{12} the unique map given by $\phi_{12}(p(T_1)e_1) = p(T_2)e_2$ for all $p \in \mathscr{P}$. Then ϕ_{12} extends to a unique bounded linear operator $\phi_{12}\colon H_1 \to H_2$ iff there exists a real number r so that $r^2K_1 - K_2$ is an a.f.p.t. where K_1 and K_2 are the kernel functions of the triples (T_1, H_1, e_1) and (T_2, H_2, e_2) respectively.*

5. A category of triples and harnomic analysis. Let $\mathscr{C}$ be the class of all triples (T, H, e) so that T is a proper contraction. Then $\mathscr{C}$ becomes a category if we define morphism between triples (T_1, H_1, e_1) and (T_2, H_2, e_2) to be a bounded linear map $\phi_{12}\colon H_1 \to H_2$ so that $\phi_{12}(e_1) = e_2$ and $T_2 \cdot \phi_{12} = \phi_{12} \cdot T_1$. We discuss some elementary facts about this category. See [16] for terminology.

Let H_1 and H_2 be two Hilbert spaces with linear operators T_1 and T_2. Then $H_1 \oplus H_2$ and $T_1 \oplus H_2$ on $H_1 \oplus H_2$ are defined as usual. (See [17].) It is well-known that if $T_1\colon H_1 \to H_1$ and $T_2\colon H_2 \to H_2$ are bounded linear operators then $T_1 \oplus T_2$ is also bounded and infact

$$\|T_1 \oplus T_2\| \leqq \max(\|T_1\|, \|T_2\|) .$$

We summarize some properties of this category $\mathscr{C}$ in the following lemma.

LEMMA 5.1. *The following hold in the category $\mathscr{C}$:*

(i) *A morphism $\phi_{12}: (T_1, H_1, e_1) \to (T_2, H_2, e_2)$ is an epimorphism iff $\phi_{12}(H_1)$ is dense in H_2.*

(ii) *If (T, H, e) is cyclic then there is at most one morphism from (T, H, e) to any other object (T_1, H_1, e_1) of $\mathscr{C}$.*

(iii) *(T, H, e) is cyclic iff every morphism to it is an epimorphism.*

(iv) *$\mathscr{C}$ has a terminal object, namely the triple* $(0, 0, 0)$.

(v) *The operation $\oplus$ is a sum as well as a product in $\mathscr{C}$.*

(vi) *A morphism $\phi_{12}: (T_1, H_1, e_1) \to (T_2, H_2, e_2)$ is a monomorphism iff* $\operatorname{Ker} \phi_{12} = \{0\}$.

Proof. We prove only (vi) and leave the proof of the rest of the proposition to the reader. If $\phi_{12}: H_1 \to H_2$ is an injection then clearly $\phi_{12}: (T_1, H_1, e_1) \to (T_2, H_2, e_2)$ is a monomorphism. To prove the converse let ϕ_{12} be a monomorphism. Then we have to show that $\operatorname{Ker}(\phi_{12}) = \{0\}$. It is easy to see that $\operatorname{Ker}(\phi_{12})$ is a closed subspace of H_1 invariant under T_1. Consider the triple $(T_1 \oplus T_1, H_1 \oplus \operatorname{Ker}(\phi_{12}), e_1 \oplus 0)$. Define the morphisms f and $g: (T_1 \oplus T_1, H_1 \oplus \operatorname{Ker}(\phi_{12}), e_1 \oplus 0) \to (T_1, H_1, e_1)$ by setting $f = p_1$ and $g = \phi_{11} + \phi_{11}$ where p_1 is the projection on the first coordinate and ϕ_{11} is the identity mapping of (T_1, H_1, e_1) into (T_1, H_1, e_1) and $\tilde{\phi}_{11}: (T_1, \operatorname{Ker}(\phi_{12}), 0) \to (T_1, H_1, 0)$ is the injection of $\operatorname{Ker}(\phi_{12})$ into H_1. Thus $g(x \oplus y) = x + y$. We see now that the morphisms $\phi_{12}g$ and $\phi_{12}f$ are equal. However $f \neq g$. Thus ϕ_{12} is not a monomorphism if $\operatorname{Ker}(\phi_{12}) \neq \{0\}$.

The category $\mathscr{C}$ also admits the usual tensor product operation $\otimes$. $H_1 \otimes H_2$ and its l^2-completion $H_1 \hat{\otimes} H_2$, $T_1 \otimes T_2$ and its extension $T_1 \hat{\otimes} T_2$ are defined as usual. It is well-known that if T_1 and T_2 are bounded linear operaters then so is $T_1 \otimes T_2$ and in fact $\| T_1 \otimes T_2 \| \leqq \| T_1 \| \| T_2 \|$. (See [6].)

LEMMA 5.2. *The operation $\otimes$ is a product in $\mathscr{C}$.*

We leave the proof of this statement to the reader. It is easy to check that $\otimes$ is not a sum in $\mathscr{C}$ (see counterexample 2, § 6).

Unfortunately we do not know an abstract characterization of the category $\mathscr{C}$.

DEFINITION 5.3. An atom in a catogory is an object such that every morphism from it is either zero or a monomorphism.

Theorem 5.4. *Let $(T, H, e) \in \mathscr{C}$ be a cyclic triple then T has no proper invariant subspaces in H iff (T, H, e) is an atom in $\mathscr{C}$.*

Proof. If T has no proper invariant subspaces then it is clear from (vi) Lemma 5.1 that (T, H, e) is an atom. If T has a proper invariant subspace let it be given by the range of an orthogonal projection P. Let (T, H, e) be written as $(T, \overline{\{p(T)e \mid p \in \mathscr{P}\}}, e)$ consider also the triple $(M_z^\wedge \{\mathscr{P}\}_P, 1)$ where the inner product on $\{\mathscr{P}\}_P$ is given by $(p, q) = ((I - P)p(T)e, (I - P)q(T)e)$. Then $(M_z^\wedge, \{\mathscr{P}\}_P, 1)$ is a cyclic triple (Theorem 4.5). Now the desired morphism $\phi_{12}: (T, \overline{\{p(T)e \mid p \in \mathscr{P}\}}, e) \to (M_z^\wedge, \{\mathscr{P}\}_P, 1)$ is given by $\phi_{12}(p(T)e) = p$.

It is not clear whether the category $\mathscr{C}$ also has an initial object. However, if we adjoin to $\mathscr{C}$ the triple $(M_z, H^2, 1)$ where H^2 is the Hardy space of analytic functions defined on D_1 whose boundary values are square integrable on the unit circle, and obtain another category $\mathscr{C}'$ then $\mathscr{C}'$ certainly has an initial object; namely the triple $(M_z, H^2, 1)$. This will be given as a corollary in the latter part of this paper.

We now make the following definition and show that the kernel function map k defined as below connects $\mathscr{C}$ and the harmonic analysis of the two-torus.

DEFINITION 5.5. The map k named the kernel function map is the map which assigns to a member (T, H, e) of $\mathscr{C}$ its kernel function.

Now we prove the following properties of k related to the operations $\oplus$ and $\otimes$ of $\mathscr{C}$.

THEOREM 5.6. *If K_1 and K_2 are the kernel functions for (T_1, H_1, e_1) and (T_2, H_2, e_2) respectively, then the kernel functions for $(T_1 \oplus T_2, H_1 \oplus H_2, e_1 \oplus e_2)$ and $(T_1 \hat{\otimes} T_2, H_1 \hat{\otimes} H_2, e_1 \otimes e_2)$ are $K_1 + K_2$ and $K_1 * K_2$ respectively where $K_1 * K_2 (z, w) = \int_0^{2\pi} \int_0^{2\pi} K(ze^{-i\theta}, we^{-i\psi}) K_2(e^{i\theta}, e^{i\psi})\, (d\theta/2\pi)(d\psi/2\pi)$ and where $d\theta/2\pi$ and $d\psi/2\pi$ denote the normalized Haar measure of the circle.*

Proof. Let $C^1_{m,n}$ and $C^2_{m,n}$ be the coefficients of $\bar{z}^m w^n$ in a Taylor expansion of the kernel functions of the triples $(T_1 \oplus T_2, H_1 \oplus H_2, e_1 \oplus e_2)$ and $(T_1 \hat{\otimes} T_2, \hat{H}_1 \otimes H_2, e_1 \otimes e_2)$ respectively.

Then,

$$\begin{aligned} C^1_{m,n} &= ((T_1 \oplus T_2)^n (e_1 \oplus e_2), (T_1 \oplus T_2)^m (e_1 \oplus e_2))_{H_1 \oplus H_2} \\ &= (T_1^n \oplus T_2^n (e_1 \oplus e_2), T_1^m \oplus T_2^m (e_1 \oplus e_2))_{H_1 \oplus H_2} \\ &= (T_1^n e_1, T_1^m e_1)_{H_1} + (T_2^n e_2, T_2^m e_2)_{H_2} \end{aligned}$$

and

$$\begin{aligned} C^2_{m,n} &= ((T_1 \hat{\otimes} T_2)^n (e_1 \otimes e_2), (T_1 \hat{\otimes} T_2)^m (e_1 \otimes e_2))_{H_1 \hat{\otimes} H_2} \\ &= (T_1^n \otimes T_2^n (e_1 \otimes e_2), T_1^m \otimes T_2^m (e_1 \otimes e_2))_{H_1 \otimes H_2} \\ &= (T_1^n e_1, T_1^m e_1)_{H_1} (T_2^n e_2, T_2^m e_2)_{H_2} \end{aligned}$$

and hence $\sum_{m,n=0}^{\infty} C^1_{m,n} z^{-m} w^n = K_1(z, w) + K_2(z, w)$. We also observe that since T_1 and T_2 are proper contractions K_1 and $K_2 \in A_{s,s}$ for some $s > 1$ and hence

$$\begin{aligned}\sum_{m,n=0}^{\infty} C^2_{m,n} z^{-m} w^n &= \int_0^{2\pi}\int_0^{2\pi} \sum_{m,n=0}^{\infty} (T_1^n e_1, T_1^m e_1)\bar{z}^m e^{im\theta} w^n e^{-in\psi} \\ &\quad \times \sum_{m,n=0}^{\infty} (T_2^n e_2, T_2^m e_2) e^{-im\theta} e^{in\psi} d\theta/2\pi d\psi/2\pi \\ &= \int_0^{2\pi}\int_0^{2\pi} K(ze^{-i\theta}, we^{-i\psi}) K(e^{i\theta}, e^{i\psi}) d\theta/2\pi d\psi/2\pi\ .\end{aligned}$$

DEFINITION 5.7. The equivalence relation $\simeq$ defined on $\mathscr{C}$ is such that $(T, H, e) \simeq (T_1, H_1, e_1)$ iff there is a unitary map $\phi: 0(e) \to 0(e_1)$ so that $\phi(e) = e_1$ and $\phi \cdot T = T_1 \cdot \phi$.

Now let us consider the category $\mathscr{C}$ with the equivalence relation $\simeq$ and denote this new object by $\underset{\simeq}{\mathscr{C}}$. Then the operations $\oplus$ and $\otimes$ are defined on $\underset{\simeq}{\mathscr{C}}$ and we have the following corollary.

THEOREM 5.8. *The class $\underset{\simeq}{\mathscr{C}}$ is isomorphic to a subset of the group algebra of the torus and so it is a cancellative abelian semigroup under the operation $\oplus$. It also has no divisors of zero under the operation $\otimes$.*

We remark that the set of a.f.p.t.'s is closed under pointwise multiplication. However, the set of a.f.p.d.t.'s is not closed under pointwise multiplication. (See counterexample 3.)

6. Some examples and counterexamples.

EXAMPLES.

1. Consider $(M_z, H^2, 1)$. Then $(M_z, H^2, 1)$ is a cyclic triple and its kernel function K is given by

$$K(z, w) = \sum_{m,n=0}^{\infty} (z^n, z^m)\bar{z}^m w^n = (1 - \bar{z}w)^{-1}\ .$$

If q is an inner function then $k(M_z, H^2, 1) = k(M_z, H^2, q)$.

2. The triple $(V, L^2\,[0, 1], 1)$, where V is the Volterra operator on the Hilbert space of square integrable real-valued functions on the closed interval $[0, 1]$ with Lebesgue measure, is also a cyclic triple. Its kernel function K is given by

$$\begin{aligned}K(z, w) &= \sum_{m,n=0}^{\infty} (V^n(1), V^m(1)) z^{-m} w^n \\ &= \int_0^1 e^x(\bar{z} + w)dx = \frac{e^{\bar{z}+w} - 1}{\bar{z} + w}\ .\end{aligned}$$

3. The kernel function of the triple $(M_\alpha, \mathbb{C}, 1)$ is the function $1\sqrt{(1-\alpha z)^{-1}}(1-\alpha w)^{-1}$. This is a special case of an example to be given in the latter part of this paper.

Counterexamples.

1. This counterexample shows that in general H_K is not a space of functions. We claim that the following holds.

Let $(M_z^\wedge, H_K, 1)$ be a canonical triple with

$$K(z, w) = \sum_{n=0}^{\infty} e^{2(-e^n+1)}\bar{z}^n w^n .$$

Let $M(X)$ denote the space of functions (measurable or otherwise) for a Borel field (X, A, μ) finite valued except for a set of measure zero, and let it be given the topology of pointwise convergence. Then the identity map i of the polynomials contained in H to the polynomials contained in $M(X)$ if injective is not continuous. This is seen as follows.

First the space $M(X)$ is complete. Second the sequence of functions $\{f_n\}_n$ where $f_n(z) = e^{e^n-1}z^n$ is a complete orthonormal sequence in H_K. Hence the sequence of polynomials $\{p_n\}_{n\in N}$ given by $p_n(z) = \sum_{i=0}^n 2^{-i}e^{e^i-1}z^i$ converges to a member h of H_K. Now let X be any subset of the complex plane $\mathbb{C}$. Since i is injective $X \neq \{0\}$. It suffices to show that the sequence $p_n(z)_n$ does not converge for any $z \in \mathbb{C}$, $z \neq 0$. Let $z = re^{i\theta}$. Then $|p_n(z) - p_{n-1}(z)| = e^{e^n-1}(r/2)^n = e^{e^n-1+n\log r/2}$, which tends to ∞ if $r \neq 0$. Hence the sequence $\{p_n(z)\}_n$ is divergent unless $z = 0$.

2. If (T, H, e) is a cyclic triple then the subspace $\{p(T)e \mid p \in \mathscr{P}\}$ can be given the structure of an abelian ring if we define $p(T)e\cdot q(T)e = pq(T)e$. However, this operation is not continuous and does not extend to all of H. Take for (T, H, e) the triple $(M_z, H^2, 1)$. Consider the convergent sequence $\{p_n\}_n$ given by $p_n(z) = \sum_{m=1}^n z^m/m^{3/5}$. The limit is of course in H^2. However, a simple calculation shows that p_n^2 does not converge in H^2. We leave the calculation to the reader. (To see how it relates to $\otimes$ not being a sum in $\mathscr{C}$ take for j_1: $(T, H, e) \to (T \hat{\otimes} T, H \hat{\otimes} H, e \otimes e)$ the map $f(T)e \mapsto f(T)e \otimes e$ and for j_2: $(T, H, e) \to (T \hat{\otimes} T, H \hat{\otimes} H, e \otimes e)$ the map $g(T)e \to e \otimes g(T)e$ and let (T_3, H_3, e_3) be the triple (T, H, e) with ϕ_{13} and ϕ_{23} as the identity map $(f(T)e \mapsto f(T)e)$ then there is no map from ϕ: $(T \hat{\otimes} T, H \hat{\otimes} H, e \otimes e) \to (T, H, e)$ so that $\phi\circ j_1 = \phi_{13}$ and $\phi \circ j_2 = \phi_{23}$.)

3. We show now that every a.f.p.t. is not an a.f.p.d.t. as well as the pointwise product of two a.f.p.d.t.'s is not necessarily an a.f.p.d.t. The kernel function of the triple $(1, \mathbb{C}, 1)$ is the function

$K(z, w) = (1 - \bar{z})^{-1}(1 - w)^{-1}$. This is an a.f.p.d.t. Its square the function $K_1(z, w) = (1 - \bar{z})^{-2}(1 - w)^{-2}$ is obviously an a.f.p.t. But it is not an a.f.p.d.t. as H_K is one dimensional and the a.f.p.d.t. corresponding to a triple $(M_\alpha, \mathbb{C}, \beta)$ is the function $\beta^2(1 - \bar{\alpha}\bar{z})^{-1}(1 - \alpha w)^{-1}$.

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