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**ON THE MAXIMUM AND MINIMUM OF PARTIAL SUMS OF
RANDOM VARIABLES**

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Let X_1, X_2, \dots, X_n be independent identically distributed random variable with $S_k = X_1 + X_2 + \dots + X_k$ and $S_k^+ = \max [0, S_k]$. We shall derive formulas for the computation of $E \left[\min_{1 \leq k \leq n} S_k^+ \right]$, $E \left[\max_{1 \leq k \leq n} S_k \right]$, and $E \left[\min_{1 \leq k \leq n} S_k \right]$. The formulas are then applied to the case of standard normal random variables.

2. Notation, definitions, and preliminary lemmas. Let $x = (x_1, \dots, x_n)$ be a vector with real components and $-x = (-x_1, \dots, -x_n)$. In some instances we shall also assume that the components $x_i, i = 1, 2, \dots, n$ are rationally independent. (For rational $r_i, r_1x_1 + r_2x_2 + \dots + r_nx_n = 0$ if and only if each $r_i = 0$.) Let $x_{k+n} = x_k$, and $x(k) = (x_k, x_{k+1}, \dots, x_{k+n-1}), k = 1, 2, \dots, n$. Let $s_k = x_1 + x_2 + \dots + x_k$. Call the polygon connecting the points $(0, 0), (1, s_1), \dots, (k, s_k), \dots, (n, s_n)$ the sum polygon of the vector x , and the line connecting $(0, 0)$ with (n, s_n) the chord of the sum polygon. The sum polygon for the cyclically permuted vector $x(k)$ is defined the same way.

F. Spitzer proved in [2] (see especially page 325, lines 8-12) the following lemma.

LEMMA 1. *Let $x = (x_1, \dots, x_n)$ be a vector such that the components $x_i, i = 1, \dots, n$, are rationally independent. Consider the sum polygons of the n cyclic permutations of x and prescribe an integer r between 0 and $n - 1$. The sum polygon of exactly one of the cyclic permutations of x has the property that exactly r of its vertices lie strictly above its chord.*

We adopt the following notations and definitions. For any real a ,

$$(2.1) \quad a^+ = \max [0, a], \quad a^- = \min [0, a]$$

σ is the permutation on n symbols, so that

$$(2.2) \quad \sigma x = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} x = (x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}).$$

We write

$$\begin{aligned}
 s_k(\sigma x) &= x_{\sigma_1} + x_{\sigma_2} \cdots + x_{\sigma_k} \\
 (2.3) \quad S(\sigma x) &= \max_{1 \leq k \leq n} s_k^+(\sigma x) = \max_{1 \leq k \leq n} \left(\sum_{i=1}^k x_{\sigma_i} \right)^+ \\
 S^*(\sigma x) &= \min_{1 \leq k \leq n} s_k^+(\sigma x) = \min_{1 \leq k \leq n} \left(\sum_{i=1}^k x_{\sigma_i} \right)^+.
 \end{aligned}$$

τ is the permutation on n symbols represented as a product of cycles, including the one-cycles, and with each index contained in exactly one cycle. For example, with $n = 7$, we could have

$$(2.4) \quad \tau = (14)(2)(3756).$$

For each x and τ we define $T(\tau x)$ as follows. Suppose τ is the example given (2.4) above, then

$$(2.5) \quad T(\tau x) = (x_1 + x_4)^+ + x_2^+ + (x_3 + x_7 + x_6 + x_5)^+.$$

In formal notations we write

$$(2.6) \quad \tau = (\alpha_1(\tau))(\alpha_2(\tau)) \cdots (\alpha_{n(\tau)}(\tau))$$

where the $\alpha_i(\tau)$, $i = 1, 2, \dots, n(\tau)$, are disjoint sets of integers whose union is the set $[1, 2, \dots, n]$. Then $T(\tau x)$ is defined as

$$(2.7) \quad T(\tau x) = \sum_{i=1}^{n(\tau)} \left(\sum_{k \in \alpha_i(\tau)} x_k \right)^+.$$

If X_1, X_2, \dots, X_n are random variables and we write $X = (X_1, \dots, X_n)$ then it is understood that

$$\begin{aligned}
 S_k &= X_1 + X_2 + \cdots + X_k \\
 S_k^+ &= \max [0, S_k] \\
 (2.8) \quad S(\sigma X) &= \max_{1 \leq k \leq n} \left(\sum_{i=1}^k X_{\sigma_i} \right)^+ \\
 T(\tau X) &= \sum_{i=1}^{n(\tau)} \left(\sum_{k \in \alpha_i} X_k \right)^+.
 \end{aligned}$$

The following lemma is also due to F. Spitzer [2].

LEMMA 2. *Let X_1, X_2, \dots, X_n be independent identically distributed random variables. Then,*

$$\begin{aligned}
 (2.9) \quad E \left[\max_{1 \leq k \leq n} S_k^+ \right] &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma X)] \\
 &= \frac{1}{n!} \sum_{\tau} E[T(\tau X)] = \sum_{k=1}^n \frac{1}{k} E[S_k^+],
 \end{aligned}$$

where it is understood that the second sum is over equivalence classes

of τ with the definition that if $\tau = (\alpha_1(\tau))(\alpha_2(\tau)) \cdots (\alpha_{n(\tau)}(\tau))$ and $\tau' = (\alpha'_1(\tau'))(\alpha'_2(\tau')) \cdots (\alpha'_{n(\tau')}(\tau'))$ then $\tau \sim \tau'$ if $n(\tau) = n(\tau')$ and for each i there is exactly one j and for each j there is exactly one i for which $\alpha_i(\tau)$ is some cyclic permutation of $\alpha'_j(\tau')$.

F. Spitzer proved Lemma 2 by using Lemma 1 and the following fundamental principle. Let $X = (X_1, \dots, X_n)$ be an n -dimensional vector valued random variable, and let $\mu(x) = \mu(x_1, \dots, x_n)$ be its probability measure (defined on Euclidean n -space E_n). Suppose that X has the property that $\mu(x) = \mu(gx)$ for every element g of a group G of order h of transformations of E_n into itself. Let $f(x) = f(x_1, \dots, x_n)$ be a μ -integrable complex valued function on E_n . Then the expected value of $f(X)$ is

$$(2.10) \quad Ef(X) = \int f(x)d\mu(x) = \int \bar{f}(x)d\mu(x)$$

where

$$\bar{f}(x) = \frac{1}{h} \sum_{g \in G} f(gx) .$$

To make use of this fundamental principle and Lemmas 1 and 2 in the proof of our main results we now define the permutation μ (not the measure μ above) as follows. (It is similar but not identical to the permutation τ defined above.) Let μ be represented as a product of cycles, including the one cycle, with each index $1, \dots, n$ contained in exactly one cycle, and beginning always with a one cycle. For example, with $n = 8$, we could have

$$(2.11) \quad \mu = (2)(14)(8)(3756) .$$

For each x and μ we define $U(\mu x)$ as follows. Suppose μ is the example given in (2.11) above, then

$$(2.12) \quad U(\mu x) = (x_1 + x_4)^+ + (x_8)^+ + (x_3 + x_7 + x_5 + x_6)^+ .$$

Observe that x_2 , the single element in the beginning one cycle, is not included in the sum. In formal notation, let

$$(2.13) \quad \mu = (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu))$$

where the $\beta_i(\mu)$, $i = 1, \dots, n(\mu)$, are disjoint sets of integers whose union is the set $[1, 2, \dots, n]$, and $\beta_i(\mu)$ consists of exactly one of the integers $1, \dots, n$. Then,

$$(2.14) \quad U(\mu x) = \sum_{i=2}^{n(\mu)} \left(\sum_{k \in \beta_i(\mu)} x_k \right)^+ .$$

We next define for each fixed vector x a map $\mu_x(\sigma)$ as follows. Given

$$(2.15) \quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

consider the sum polygon through the points $(1, x_{i_1}), (2, x_{i_1} + x_{i_2}), \dots, (k, x_{i_1} + \dots + x_{i_k}), \dots, (n, s_n)$. Define the highest concave majorant of the sum polygon as that unique concave polygon which goes through $(1, x_{i_1})$ and (n, s_n) in such a way that all its vertices are also vertices of the sum polygon and that it always lies below or coincides with the sum polygon. Suppose now that the highest concave majorant constructed for the permutation σ has the vertices $(1, x_{i_1}), (k_1, x_{i_1} + \dots + x_{i_{k_1}}), \dots, (k_\nu, x_{i_1} + \dots + x_{i_{k_\nu}}), (n, s_n)$ where $1 < k_1 < \dots < k_\nu < n$. Then we define

$$(2.16) \quad \mu_x(\sigma) = (i_1)(i_2, \dots, i_{k_1})(i_{k_1+1}, \dots, i_{k_2}) \cdots (i_{k_\nu+1}, \dots, i_n).$$

Now observe that

$$(2.17) \quad \begin{aligned} S^*(\sigma x) &= \min_{1 \leq k \leq n} \left(\sum_{j=1}^k x_{i_j} \right)^+ = [s_n - U(\mu_x(\sigma)x)]^+ \\ &= S(\sigma(-x)) + s_n - U(\mu_x(\sigma)x) \end{aligned}$$

or

$$(2.18) \quad -U(\mu_x(\sigma)x) = S^*(\sigma x) - S(\sigma(-x)) - s_n.$$

Going in the other direction, we define for each fixed vector x with rationally independent components, a map $\sigma_x(\mu)$ as follows. Suppose that

$$(2.19) \quad \begin{aligned} \mu &= (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu)) \\ &= (j_1)(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu)). \end{aligned}$$

(See the paragraph following equation (2.12) for the definition of μ , and notice in particular that the leading cycle $\beta_1(\mu)$ must be a one cycle.) Rewrite each set $\beta_2(\mu), \beta_3(\mu), \dots, \beta_{n(\mu)}(\mu)$ as follows. Suppose $\beta_k(\mu) = (3, 6, 9, 2)$. Consider the sum polygon of the vector (x_3, x_6, x_9, x_2) . By Lemma 1 there is a unique cyclic permutation of (x_3, x_6, x_9, x_2) such that all of the vertices of the sum polygon lie strictly above its chord. Suppose that unique permutation is (x_9, x_2, x_3, x_6) then rewrite $\beta_k(\mu)$ as $(9, 2, 3, 6)$ and call it $\beta'_k(\mu)$. Define μ' as

$$(2.20) \quad \begin{aligned} \mu' &= (\beta_1(\mu))(\beta'_2(\mu)) \cdots (\beta'_{n(\mu)}(\mu)) \\ &= (j_1)(\beta'_2(\mu)) \cdots (\beta'_{n(\mu)}(\mu)). \end{aligned}$$

Observe that

$$(2.21) \quad U(\mu x) = U(\mu' x).$$

Now consider the vector

$$(2.22) \quad y = (y_2, y_3, \dots, y_n) = \left(\sum_{k \in \beta_2'(\mu)} x_k, \sum_{k \in \beta_3'(\mu)} x_k, \dots, \sum_{k \in \beta_n'(\mu)} x_k \right).$$

There is a unique permutation of the components of y , say

$$(2.23) \quad y' = (y'_2, y'_3, \dots, y'_n) = \left(\sum_{k \in \beta_2''(\mu)} x_k, \sum_{k \in \beta_3''(\mu)} x_k, \dots, \sum_{k \in \beta_n''(\mu)} x_k \right)$$

so that

$$(2.24) \quad \frac{\sum_{k \in \beta_2'(\mu)} x_k}{\sum_{k \in \beta_2'(\mu)} 1} < \frac{\sum_{k \in \beta_3'(\mu)} x_k}{\sum_{k \in \beta_3'(\mu)} 1} < \dots < \frac{\sum_{k \in \beta_n'(\mu)} x_k}{\sum_{k \in \beta_n'(\mu)} 1}.$$

Now define

$$(2.25) \quad \mu'' = (j_1)(\beta_2''(\mu))(\beta_3''(\mu)) \dots (\beta_n''(\mu)).$$

Observe that

$$(2.26) \quad U(\mu''x) = U(\mu'x) = U(\mu x).$$

Suppose μ'' can be written as

$$(2.27) \quad \mu'' = (j_1)(j_2, \dots, j_{h_1})(j_{h_1+1}, \dots, j_{h_2}) \dots (j_{h_v+1}, \dots, j_n)$$

we then define

$$(2.28) \quad \sigma_x(\mu) = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$$

observe that with respect to the sum polygon through the points $(1, x_{j_1}), (2, x_{j_1} + x_{j_2}), \dots, (k, x_{j_1} + \dots + x_{j_k}), \dots, (n, s_n)$ the vertices of the highest concave majorant are at the points $(1, x_{j_1}), (h_1, x_{j_1} + \dots + x_{j_{h_1}}), (h_2, x_{j_1} + \dots + x_{j_{h_2}}), \dots, (h_v, x_{j_1} + \dots + x_{j_{h_v}}), (n, s_n)$. Thus

$$(2.29) \quad \mu_x(\sigma_x(\mu)) = \mu''.$$

Now use equations (2.17), (2.29), and (2.26) to obtain

$$(2.30) \quad \begin{aligned} S^*(\sigma_x(\mu)x) &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu_x(\sigma_x(\mu))x) \\ &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu''x) \\ &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu x) \end{aligned}$$

which is equivalent to

$$(2.31) \quad -U(\mu x) = S^*(\sigma_x(\mu)x) - S(\sigma_x(\mu)(-x)) - s_n.$$

Now consider the sets $[-U(\mu x)]$ and $[S^*(\sigma x) - S(\sigma(-x)) - s_n]$ generated by letting σ and τ run through all permutations. We have shown above that for vectors x with rationally independent components, the maps $\sigma_x(\mu)$ and $\mu_x(\sigma)$ define a one-to-one map of the set $[-U(\mu x)]$ onto the set $[S^*(\sigma x) - S(\sigma(-x)) - s_n]$. Furthermore, since the set x of vectors in E_n with rationally independent component is dense in E_n and the functions $S(\sigma x)$ and $U(\mu x)$ are continuous functions of x , we have proved the following lemma.

LEMMA 3. *For an arbitrary fixed vector $x = (x_1, \dots, x_n)$ the sets $[-U(\mu x)]$ and $[S^*(\sigma x) - S(\sigma(-x)) - s_n]$ which are generated by letting σ and τ run through all permutations are identical sets.*

3. The main theorems.

THEOREM 1. *Let X_1, X_2, \dots, X_n be independent, identically distributed random variables.*

$$\begin{aligned} (3.1) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] &= E(S_n) + \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) \\ &= E(S_n) + \frac{1}{n} E(S_n^+) - \sum_{k=1}^n \frac{1}{k} E(S_k). \end{aligned}$$

Proof. Using the fundamental principle (2.10) we may write

$$(3.2) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] = \frac{1}{n!} \sum_{\sigma} E[S^*(\sigma X)]$$

and applying (2.17) to the term on the right we have

$$\begin{aligned} (3.3) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X)) + S_n - U(\mu_x(\sigma)X)] \\ &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X))] + E(S_n) \\ &\quad - \frac{1}{n!} \sum_{\sigma} E[U(\mu_x(\sigma)X)]. \end{aligned}$$

Now apply Lemma 3 to obtain

$$(3.4) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] = \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X))] + E(S_n) - \frac{1}{n!} \sum_{\mu} E[U(\mu X)]$$

where it is understood that the second sum is over equivalence classes of μ with the definition that if

$$\mu = (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_n(\mu)(\mu))$$

and

$$\mu' = (\beta'_1(\mu'))(\beta'_2(\mu')) \cdots (\beta'_{n(\mu')}(\mu'))$$

then $\mu \sim \mu'$ if $n(\mu) = n(\mu')$, $\beta_i(\mu) = \beta'_i(\mu')$ and for each i there is exactly one j and for each j there is exactly one i for which $\beta_i(\mu)$ is some cycle permutation of $\beta'_j(\mu')$.

Let us consider the last term in equation (3.4). Define $X' = (X_2, \dots, X_n)$ and keep in mind that the random variables X_1, X_2, \dots, X_n are independent and identically distributed. It is then clear from our definition of $T(\tau x)$ and $U(\mu x)$ that

$$(3.5) \quad \frac{1}{n!} \sum_{\mu} E[U(\mu X)] = \frac{1}{n!} \sum_{\tau} E[T(\tau X')].$$

Now apply Lemma 2 to equations (3.4) and (3.5) to obtain

$$(3.6) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] = \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] + E(S_n) - \sum_{k=1}^{n-1} \frac{1}{k} E[S_k^+]$$

which is the first part of equation (3.1). Then, using the fact that $E[(-S_k)^+] - E(S_k^+) = -E(S_k)$, we obtain the second part of equation (3.1) as follows.

$$\begin{aligned} & E(S_n) + \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) \\ &= E(S_n) + \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] - \sum_{k=1}^n \frac{1}{k} E(S_k^+) + \frac{1}{n} E(S_n^+) \\ &= E(S_n) + \sum_{k=1}^n \frac{1}{k} \{E(-S_k)^+ - E(S_k^+)\} + \frac{1}{n} E(S_n^+) \\ &= E(S_n) + \frac{1}{n} E(S_n^+) - \sum_{k=1}^n \frac{1}{k} E(S_k). \end{aligned}$$

THEOREM 2.

$$(3.7) \quad \begin{aligned} E\left[\max_{1 \leq k \leq n} S_k\right] &= \sum_{k=1}^{n-1} \frac{1}{k} E[(-S_k)^+] + E(S_n) \\ &= E(S_n) + E\left[\max_{1 \leq k \leq n-1} (-S_k)^+\right]. \end{aligned}$$

Proof.

$$(3.8) \quad \max_{1 \leq k \leq n} S_k = \max_{1 \leq k \leq n} S_k^+ + \max_{1 \leq k \leq n} S_k^-, \text{ and } \max_{1 \leq k \leq n} S_k^- = -\min_{1 \leq k \leq n} (-S_k)^+.$$

Therefore,

$$(3.9) \quad E\left[\max_{1 \leq k \leq n} S_k\right] = E\left[\max_{1 \leq k \leq n} S_k^+\right] - E\left[\min_{1 \leq k \leq n} (-S_k)^+\right].$$

From Theorem 1, we have

$$(3.10) \quad \begin{aligned} E\left[\min_{1 \leq k \leq n} (-S_k)^+\right] \\ = E(-S_n) + \sum_{k=1}^n \frac{1}{k} E[(S_k)^+] - \sum_{k=1}^{n-1} \frac{1}{k} E[(-S_k)^+]. \end{aligned}$$

From Lemma 2, we have

$$(3.11) \quad E\left[\max_{1 \leq k \leq n} S_k^+\right] = \sum_{k=1}^n \frac{1}{k} E[S_k^+].$$

Therefore,

$$(3.12) \quad \begin{aligned} E\left[\max_{1 \leq k \leq n} S_k\right] \\ = \sum_{k=1}^n \frac{1}{k} E[S_k^+] - E(-S_n) - \sum_{k=1}^n \frac{1}{k} E[(S_k)^+] + \sum_{k=1}^{n-1} \frac{1}{k} E[(-S_k)^+] \\ = \sum_{k=1}^{n-1} \frac{1}{k} E[(-S_k)^+] - E(-S_n) \\ = E(S_n) + \sum_{k=1}^{n-1} \frac{1}{k} E[(-S_k)^+]. \end{aligned}$$

THEOREM 3.

$$(3.13) \quad E\left[\min_{1 \leq k \leq n} S_k\right] = E(S_n) - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) = E(S_n) - E\left[\max_{1 \leq k \leq n-1} S_k^+\right].$$

Proof.

$$(3.14) \quad \min_{1 \leq k \leq n} S_k = \min_{1 \leq k \leq n} S_k^+ + \min_{1 \leq k \leq n} S_k^-, \text{ and } \min_{1 \leq k \leq n} S_k^- = -\max_{1 \leq k \leq n} (-S_k)^+.$$

Therefore,

$$(3.15) \quad \begin{aligned} E\left[\min_{1 \leq k \leq n} S_k\right] &= E\left[\min_{1 \leq k \leq n} S_k^+\right] - E\left[\max_{1 \leq k \leq n} (-S_k)^+\right] \\ &= E(S_n) + \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) \\ &\quad - \sum_{k=1}^n \frac{1}{k} E[(-S_k)^+] \\ &= E(S_n) - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) = E(S_n) - E\left[\max_{1 \leq k \leq n-1} S_k^+\right]. \end{aligned}$$

4. Application to standard normal random variables. If $X_1,$

X_2, \dots, X_n are standard normal random variables, then

$$(4.1) \quad E\left[\max_{1 \leq k \leq n} S_k^+\right] = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^n \frac{1}{\sqrt{k}}$$

$$(4.2) \quad E\left[\min_{1 \leq k \leq n} S_k^+\right] = \frac{1}{\sqrt{2\pi n}}$$

$$(4.3) \quad E\left[\max_{1 \leq k \leq n} S_k\right] = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}}$$

$$(4.4) \quad E\left[\min_{1 \leq k \leq n} S_k\right] = -\frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}}.$$

Statement (4.1) was proved by J. A. McFadden and J. L. Lewis in [1]. They applied F. Spitzer's lemma (Lemma 2) to the fact that for standard normal random variables,

$$(4.5) \quad E(S_k^+) = \sqrt{\frac{k}{2\pi}}.$$

To obtain (4.2), (4.3), and (4.4), apply (4.5) and the fact that for standard normal random variables, $E(S_k^+) = E[(-S_k)^+]$, to Theorems 1, 2, and 3 respectively.

REFERENCES

1. J. A. McFadden and J. L. Lewis, *Multivariate normal integrals for highly correlated samples from a Wiener process*, J. of Applied Probability, **4** (1967), 303-312.
2. F. Spitzer, *A combinatorial lemma and its application to probability theory*, Trans. Amer. Math. Soc., **82**, 323-339.

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Vol. 52, No. 2

February, 1974

Harm Bart, <i>Spectral properties of locally holomorphic vector-valued functions</i>	321
J. Adrian (John) Bondy and Robert Louis Hemminger, <i>Reconstructing infinite graphs</i>	331
Bryan Edmund Cain and Richard J. Tondra, <i>Biholomorphic approximation of planar domains</i>	341
Richard Carey and Joel David Pincus, <i>Eigenvalues of seminormal operators, examples</i>	347
Tyrone Duncan, <i>Absolute continuity for abstract Wiener spaces</i>	359
Joe Wayne Fisher and Louis Halle Rowen, <i>An embedding of semiprime P.I.-rings</i>	369
Andrew S. Geue, <i>Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators</i>	377
Charles Lemuel Hagopian, <i>Locally homeomorphic λ connected plane continua</i>	403
Darald Joe Hartfiel, <i>A study of convex sets of stochastic matrices induced by probability vectors</i>	405
Yasunori Ishibashi, <i>Some remarks on high order derivations</i>	419
Donald Gordon James, <i>Orthogonal groups of dyadic unimodular quadratic forms. II</i>	425
Geoffrey Thomas Jones, <i>Projective pseudo-complemented semilattices</i>	443
Darrell Conley Kent, Kelly Denis McKennon, G. Richardson and M. Schroder, <i>Continuous convergence in $C(X)$</i>	457
J. J. Koliha, <i>Some convergence theorems in Banach algebras</i>	467
Tsang Hai Kuo, <i>Projections in the spaces of bounded linear operations</i>	475
George Berry Leeman, Jr., <i>A local estimate for typically real functions</i>	481
Andrew Guy Markoe, <i>A characterization of normal analytic spaces by the homological codimension of the structure sheaf</i>	485
Kunio Murasugi, <i>On the divisibility of knot groups</i>	491
John Phillips, <i>Perturbations of type I von Neumann algebras</i>	505
Billy E. Rhoades, <i>Commutants of some quasi-Hausdorff matrices</i>	513
David W. Roeder, <i>Category theory applied to Pontryagin duality</i>	519
Maxwell Alexander Rosenlicht, <i>The nonminimality of the differential closure</i>	529
Peter Michael Rosenthal, <i>On an inversion theorem for the general Mehler-Fock transform pair</i>	539
Alan Saleski, <i>Stopping times for Bernoulli automorphisms</i>	547
John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces. II</i>	553
Vashishtha Narayan Singh, <i>Reproducing kernels and operators with a cyclic vector. I</i>	567
Peggy Strait, <i>On the maximum and minimum of partial sums of random variables</i>	585
J. L. Brenner, <i>Maximal ideals in the near ring of polynomials modulo 2</i>	595
Ernst Gabor Straus, <i>Remark on the preceding paper: "Ideals in near rings of polynomials over a field"</i>	601
Masamichi Takesaki, <i>Faithful states on a C^*-algebra</i>	605
R. Michael Tanner, <i>Some content maximizing properties of the regular simplex</i>	611
Andrew Bao-hwa Wang, <i>An analogue of the Paley-Wiener theorem for certain function spaces on $SL(2, \mathbb{C})$</i>	617
James Juei-Chin Yeh, <i>Inversion of conditional expectations</i>	631