AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR CERTAIN FUNCTION SPACES ON SL(2, C)

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The classical theorem of Paley-Wiener is concerned with characterizing Fourier transforms of $C^\infty$ functions of compact support on the real line. It states that an entire holomorphic function $F$ is the Fourier-Laplace transform of a $C^\infty$ function on the real line $\mathbb{R}$ with support in $|x| \leq R$ if and only if for given integer $m$, there exists a constant $C_m$ such that

$$|F(\xi + i\eta)| \leq C_m(1 + |\xi + i\eta|)^{-m} \exp R|\eta|, \quad \xi, \eta \in \mathbb{R}.$$  

The purpose of this paper is to prove an analogue of this theorem for certain convolution subalgebras of $C^\infty$ functions with compact support on the group $SL(2, C)$, by using Fourier transform involving elementary spherical functions of general type $\delta$.

These subalgebras have been defined on locally compact group by R. Godement [4], in order to study the spherical trace function, cf. also G. Warner [8]. On this special group mentioned, by use the differential equations satisfied by the spherical functions, we derive a parametrization of such functions. These are in turn utilized to prove the Paley-Wiener theorem.

The analogous question on symmetric space of noncompact type was considered by S. Helgason [5] and R. Gangolli [3]. L. Ehrenpreis and F. I. Mautner [2] studied the Fourier transform on the group $SL(2, \mathbb{R})$ in detail, and theorem of the same kind was proved there. Results of this sort involving spherical functions of general type $\delta$ on some other groups have also been investigated, see e.g. Y. Shimizu [7].

2. Preliminaries. Throughout this paper, let $G$ denote the complex semisimple Lie group $SL(2, C)$ and let $K$ denote the maximal compact subgroup consisting of all unitary matrices in $G$. A basis of the real Lie algebra $g_o$ of $G$ consists of

$$R_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad R_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{R_1, R_2, R_3\}$ also forms a basis of the Lie algebra $k_o$ of $K$. 617
Elements of \( g_0 \) are viewed as left invariant vector fields on \( G \), which generates the algebra \( \mathfrak{g}_0 \) of all left invariant differential operators on \( G \). Let \( a_{p_0} = \{ tS_3; t \in \mathbb{R} \} \). The root system for \( (g_0, a_{p_0}) \) consists of \( \{ \rho, -\rho \} \), where \( \rho(S_3) = 1 \), and each has multiplicity two. Let \( N_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( N_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \) and let \( \pi_0 \) be the subspace of \( g_0 \) spanned by \( \{ N_1, N_2 \} \), then \( \pi_0 \) is the root space for \( \rho \). Let \( N = \exp \pi_0 \) and \( A_p = \{ a_t = \exp tS_3; t \in \mathbb{R} \} \). Then \( g_0 = k_0 + a_{p_0} + \pi_0 \) and \( G = KA_p N \) (Iwasawa decomposition). It is also known that \( G = K A^+_p K, A^+_p = \{ a_t; t \geq 0 \} \). The Haar measure on \( G \) is normalized so that

\[
\int_G f(x) dx = \int_K \int_{A_p} \int_{N_0} f(ka,n)e^{i\pi} dk dt dn, \quad f \in C_c(G),
\]

where \( dk \) is the normalized Haar measure on \( K \), \( dt \) is the Lebesgue measure on \( \mathbb{R} \) and \( dn = d\xi d\xi_2 \) if \( n = \exp (\xi N_1 + \xi_2 N_2) \), is the Lebesgue measure on \( \mathbb{R}^2 \). Let \( k \in K \), we can write \( k = u_\varphi v_\theta u_\psi \) with \( u_\varphi = \exp \varphi R_3, v_\theta = \exp \theta R_2, \) and \( 0 \leq \varphi, \theta \leq 2\pi, 0 \leq \varphi_2 \leq 4\pi \). Then

\[
\int_K f(k) dk = \frac{1}{16\pi^2} \int_{\varphi_1 = 0}^{2\pi} \int_{\theta_1 = 0}^{\pi} \int_{\varphi_2 = 0}^{2\pi} f(u_\varphi v_\theta u_\psi) \sin \theta_2 d\varphi_2 d\theta_1 d\varphi_1,
\]

\( f \in C_c(K) \).

For each nonnegative integer or half integer \( s \), let \( D^s \) be the unique (up to equivalence) irreducible unitary representation of \( K \) on a \( 2s + 1 \) dimensional Hilbert space \( E_s \). We can choose a basis \( \{ v_-, v_-, \cdots, v_\} \) of \( E_s \) so that the matrix \( (D^s_{j,q}(k)) \), \( j, q = -s, -s + 1, \cdots, s \) has the following expression [see e.g. 6, p. 129].

\[
D^s_{j,q}(u_\varphi) = \delta_{j,q} e^{-i\varphi q}
\]

\[
D^s_{j,q}(v_\theta) = (-1)^{j-q}(\frac{(s+j)! (s-j)!}{(s+q)! (s-q)!})^{1/2}
\]

\[
\times \sum_{r=\max|j+q|}^{\min|s-j+q|} (-1)^r \left( s+q \right) \left( s-q \right) \left( s-j-r \right) \cos^{2s-j-q-2s} \frac{\theta}{2} \sin^{j-q+2r} \frac{\theta}{2}.
\]

The infinitesimal form for \( D^s \) has

\[
D^s(R_3)v_j = \frac{1}{2}(s+j)v_{j-1} - \frac{1}{2}(s-j)v_{j+1}
\]

\[
D^s(R_2)v_j = \frac{1}{2}(s+j)v_{j-1} + \frac{1}{2}(s-j)v_{j+1}
\]

\[
D^s(R_3)v_j = -ijv_j.
\]

Hence \( D^s(R_3^2 + R_2^2 + R_3^2) = -s(s+1)I \).
Let \( M = \{u_\theta = \exp \theta S_5 : \theta \in \mathbb{R}\} \). Then \( M \) is the centralizer of \( A \) in \( K \), also it is a maximal torus in \( K \). The set \( \hat{M} \) of all characters of \( M \) is parametrized by half integers, i.e., for each \( p \) with \( 2p \) an integer, \( u_{\theta} \to e^{-i\theta p} \) gives a character of \( M \). Let \( p \in \hat{M} \), and let \( E^p = \{f \in L^2(K) : f(ku_\theta) = e^{ip\theta} f(k), \ k \in K \text{ and } u_\theta \in M\} \), with \( \|f\|^2 = \int_K |f(k)|^2 \, dk \). Let \( \lambda \) be a complex number and given \( x \in G \), define \( U^{p,\lambda} \) by the prescription
\[
(U^{p,\lambda}(x)f)(k) = \exp \left( -(i\lambda + 1)p(H(x^{-1}k))f(k(x^{-1}k)) \right), \quad f \in E^p
\]
where \( x = a(x) \cdot \exp H(x) \cdot u(x) \) is the Iwasawa decomposition for \( x \). Then \( U^{p,\lambda} \) defines a continuous representation of \( G \) on the Banach space \( E^p \), and every TCI Banach representation of \( G \) is equivalent to a subquotient of \( U^{p,\lambda} \) for some \( p, \lambda \). The restriction of \( U^{p,\lambda} \) to \( K \) is just the unitary representation of \( K \) induced from the character \( u_{\theta} \to e^{ip\theta} \) of \( M \), hence \( D^p \) occurs in \( U^{p,\lambda} \) exactly once if and only if \( s = |p| + q \) for some nonnegative integer \( q \).

\( U^{p,\lambda} \) is unitary if \( \lambda \) is real, which constitutes the principal series representation induced from the characters of the group \( MA_pN \).

Define
\[
U^{p,\lambda}(f) = \int_G f(x)U^{p,\lambda}(x)dx, \quad f \in C_c^\infty(G).
\]
Then \( U^{p,\lambda}(f) \) is of trace class and we have the inversion formula
\[
f(x) = \frac{1}{4\pi^2} \sum_{p \geq 0} \int_{\mathbb{R}} (p^2 + \lambda^2) \text{Trace} \left( U^{p,\lambda}(x^{-1})U^{p,\lambda}(f) \right) d\lambda
\]
where \( Z \) is the set of all integers and \( d\lambda \) is the usual Euclidean measure.

3. The spherical functions. Let \( C^\infty_c(G) \) be the algebra of all \( C^\infty \) functions with compact support on \( G \), with multiplication defined by convolution. The subalgebra \( I_s(G) \) is formed by those functions \( f \) in \( C^\infty_c(G) \) satisfying \( f(kxk^{-1}) = f(x) \) for \( x \in G, k \in K \). Define \( \chi_s(k) = (2k+1) \text{Trace} (D^s(k)), k \in K \) and \( D^s \in \hat{K} \). Let \( C_s(G) = \{f \in C^\infty_c(G) : f^*\chi_s = f = \chi_s^* f \} \) and \( I_s(G) = I_s(G) \cap C_s(G) \). \( I_s(G) \) is a subalgebra of \( C^\infty_c(G) \) and the mapping \( f \to f^*\chi_s, f^*(x) = \int_K f(kxk^{-1})dk \), is the projection of \( C^\infty_c(G) \) onto \( I_s(G) \).

**DEFINITION.** Let \( D^s \in \hat{K} \). By a spherical function \( \Phi \) on \( G \) of type \( s \) we mean a quasi-bounded continuous function on \( G \) such that (i) \( \Phi(kxk^{-1}) = \Phi(x), x \in G \) and \( k \in K \); (ii) \( \Phi^*\chi_s = \Phi \); (iii) the map \( f \to \int_G f(x)\Phi(x)dx \) is a nonzero homomorphism of the algebra \( I_s(G) \) onto...
the complex numbers $\mathbb{C}$.

Spherical functions of type $s$ relates naturally to the TCI Banach representations of $G$. Suppose $U$ is a TCI Banach representation of $G$ on a space $E$ such that $D^s$ occurs in the restriction of $U$ to $K$. Let $U(\chi_s) = \int_K U(k)\chi_s(k)dk$ and $E(s) = U(\chi_s)E$. The $s$-spherical function $\Psi^U_s$ of $U$ on $G$ is defined by $\Psi^U_s(x) = U(\chi_s)U(x)U(\chi_s)$. Since $D^s$ occurs in $U$ exactly once, choose a basis for $E(s)$ so that $U(k) = D^s(k)$ on $E(s)$. Then clearly $\Psi^U_s(k) = D^s(k)\Psi^U_s(x)$. Let $\Psi^U_s(x) = \int_K \Psi^U_s(kxk^{-1})dk$. Then $\Psi^U_s(x) = D^s(k)\Psi^U_s(x)$, $x \in G$, $k \in K$, and we have $\Psi^U_s(x)$ is a scalar $\Phi^U_s(x)$ times identity operator. We recall the following facts, [cf. 8, Ch. 6].

**PROPOSITION 3.1.** (i) $\Phi^U_s(x)$ is a spherical function of type $s$ and every spherical function of type $s$ is of this form.

(ii) Let $\kappa_U$ be the infinitesimal character of $U$ defined on the center $Z$ of the algebra $\mathfrak{g}$, then $D\Phi^U_s = \kappa_U(D)\Phi^U_s$ and $D\Psi^U_s = \kappa_U(D)\Psi^U_s$, $D \in 3$.

Consider the Banach representation $U^{p,1}$ with $s = |p| + q$ for some nonnegative integer $q$, let $\Psi^{p,1}_s$ and $\Phi^{p,1}_s$ be the $s$-spherical function and the spherical function of type $s$ respectively of the TCI Banach representation of $G$ which occurs in $U^{p,1}$ and has $D^s$ occurs in it. Let $E^p(s) = U^{p,1}(\chi_s)E^p$, then $\{D^p_{j,-p}: j = -s, -s + 1, \ldots, s\}$ forms a basis for $E^p(s)$. Now

$$
\Psi^{p,1}_s(x) \cdot D^p_{j,-p} = U^{p,1}(\chi_s)U^{p,1}(x)U^{p,1}(\chi_s)D^p_{j,-p} \\
= (2s + 1) \sum_{l=-s}^s \int_K \exp \left( -(i\lambda + 1)\rho(H(x^{-1}k)) \right) \\
\times D^p_{j,-p}(\kappa(x^{-1}k))dk \cdot D^p_{j,-p}.
$$

But $\Phi^{p,1}_s(x) = 1/(2s + 1) \text{Trace}(\Psi^{p,1}_s(x)) = 1/(2s + 1) \text{Trace}(\Psi^{p,1}_s(x))$, so

$$
\Phi^{p,1}_s(x) = \int_K \exp \left( -(i\lambda + 1)\rho(H(x^{-1}k)) \right)D^p_{j,-p}(\kappa^{-1}k)dk.
$$

Using this formula and the above proposition, we will set up a differential equation which enables us to get a complete parametrization of the spherical functions of type $s$.

**LEMMA 3.2.** $\Phi^{p,1}_s(x) = \Phi^{p,-1}_s(x^{-1})$.

**Proof.** It suffices to show that

$$
\int_G f(x)\Phi^{p,-1}_s(x^{-1})dx = \int_G f(x)\Phi^{p,1}_s(x^{-1})dx
$$
for all $f \in C^\infty_c(G)$. Since $\Phi^\pm_\lambda(k \times k^{-1}) = \Phi^\pm_\lambda(x)$, $x \in G$, $k \in K$ and $\Phi^\xi_\lambda \chi_s = \Phi^\xi_\lambda$, we only need to consider those $f$ in $I_{\xi,s}(G)$. Thus let $f \in I_{\xi,s}(G)$, by (10)

$$
\int_G f(x)\Phi^\pm_\lambda(x)dx = \int_G f(x)^{-1})\Phi^\pm_\lambda(x)^{-1}dx \\
= \int_K \int_{A_p} \int_N f(n^{-1}a^{-1}_i k^{-1})e^{-(i\lambda + 1)D^\pm_\lambda(k)e^2dkdt dn} \\
= \int_K \int_{A_p} \int_N f(kn_x)e^{(i\lambda + 1)D^\pm_\lambda(k^{-1})dkdt dn} \\
= \int_K \int_{A_p} \int_N f(kn_x)e^{(i\lambda + 1)D^\pm_\lambda(k^{-1})dkdt dn}.
$$

But $D^\pm_\lambda(k^{-1}) = D^\pm_\lambda(k)$ by (6), hence the lemma.

Let $w_1 = S_1^2 + S_2^2 + S_3^2 - R_1^2 - R_2^2 - R_3^2$ and $w_2 = R_1S_1 + R_2S_2 + R_3S_3$. Then $\{w_1, w_2\}$ generates the center $\mathfrak{g}$. It is easy to see that $S_1 = R_2 - N_s$, $S_2 = N_1 - R_s$ and $N_1R_1 = R_1N_1 - S_s$, $N_2R_2 = R_2N_2 - S_s$, substitute into $w_1$, $w_2$ we get

(11) $w_1 = S_3^2 + 2S_3 - R_3^2 + N_1^2 + N_2^2 - 2(R_1N_1 + R_2N_2)$

(12) $w_2 = R_1S_1 + R_2 - R_1N_2 + R_2N_1$.

Use the formula for $\Phi^\pm_\lambda(x)$ in the above lemma, a direct computation gives us

(13) $w_1\Phi^\pm_\lambda(1) = p^2 - \lambda^2 - 1$, $w_2\Phi^\pm_\lambda(1) = p\lambda$.

Now, $\Phi^\pm_\lambda = 1/(2s + 1) \text{Trace}(\Psi^\pm_\lambda)$, and for $x \in G$, we can write $x = k_1a_1k_2$, $k_1, k_2 \in K$, $a_1 \in A_\pm$, so $\Psi^\pm_\lambda(x) = \Psi^\pm_\lambda(k_1a_1k_2) = D^\pm_\lambda(k_1)\Psi^\pm_\lambda(a_1)D^\lambda(k_2)$. Then this function determined by the restriction of $\Psi^\pm_\lambda$ to $A_\pm$. Let $t \neq 0$, define $\text{Ad}(a_i^{-1})X = a_i^{-1}Xa_i$, $X \in g_\pm$; then we have

(14) $\text{Ad}(a_i^{-1})R_i = \cosh t \cdot R_i - \sinh t \cdot S_i$.

By substitution, we get
\[ w_1 = S_s + 2 \coth t \cdot S_s + \coth^2 t \cdot (R_s + R_s') \]
\[ + \csch^2 t \cdot \Ad (a_i^-) (R_s + R_s') \]
\[ - 2 \coth t \csch t \cdot ((\Ad (a_i^-) R_s) R_s' - (R_s + R_s' + R_s)) \]
\[ w_2 = S_s R_s + \coth t \cdot R_s - \csch t \cdot ((\Ad (a_i^-) R_s) R_s' - (R_s + R_s' + R_s)) \]

Hence for \( t > 0 \), apply \( w_1, w_2 \) on \( \Psi^{p, q}(a_i) \), we get

\[ \frac{d^2}{dt^2} \Psi^{p, q}(a_i) + 2 \coth t \frac{d}{dt} \Psi^{p, q}(a_i) \]
\[ + (\coth^2 t - \csch^2 t) D^s(R_s) \Psi^{p, q}(a_i) \]
\[ + \coth t \csch t (X \Psi^{p, q}(a_i) Y + Y \Psi^{p, q}(a_i) X) \]
\[ + s(s + 1) \Psi^{p, q}(a_i) = (\lambda^2 + \lambda_2^2 - 1) \Psi^{p, q}(a_i). \]

where \( X = D^s(R_s) - i D^s(R_s), Y = -D^s(R_s) - i D^s(R_s). \) Since \( u_i a_i = a_i u_i, \) \( u_i \in M, a_i \in A, \) by (5) we see that \( \Psi^{p, q}(a_i) \) is a diagonal matrix, so let \( \Psi^{p, q}_{i,j} \) be the \( j \)th diagonal element, \( j = -s, -s + 1, \ldots, s, \) we see from (18) and (6)

\[ -i j \frac{d}{dt} \Psi^{p, q}_{i,j}(a_i) - ij \coth t \Psi^{p, q}_{i,j}(a_i) \]
\[ - \frac{i}{2} \csch t (s - j)(s + j + 1) \Psi^{p, q}_{i,j+1}(a_i) \]
\[ - (s - j)(s - j + 1) \Psi^{p, q}_{i,j-1}(a_i) = \lambda \Psi^{p, q}_{i,j}(a_i). \]

Hence for \( j = s, s - 1, s - 2, \ldots, -s + 1, \) we get

\[ (s + j)(s - j + 1) \csch t \Psi^{p, q}_{i,j+1}(a_i) \]
\[ = 2j \frac{d}{dt} \Psi^{p, q}_{i,j}(a_i) + 2j \coth t \Psi^{p, q}_{i,j}(a_i) \]
\[ - 2i \lambda \Psi^{p, q}_{i,j}(a_i) + (s - j)(s + j + 1) \csch t \Psi^{p, q}_{i,j+1}(a_i). \]

Therefore, \( \Psi^{p, q}(a_i) \) is determined by knowing \( \Psi^{p, q}(a_i), \) \( t > 0. \) Consider the \( s \)th diagonal element of (17) + 2i \coth t \cdot (18), we find that \( \Psi^{p, q}(a_i) \) satisfies the following differential equation

\[ \varphi''(t) + 2(1 + s) \coth t \varphi(t) \]
\[ + ((s + 1)^2 - \lambda^2 - \lambda_2^2 - 2i \lambda \coth t) \varphi(t) = 0. \]
This is a differential equations with regular singular point at $t = 0$. The initial equation $f(z) = z(z + 1 + s)$, so we have $z_1 = 0$ and $z_2 = -(1 + s)$ as roots for $f(z) = 0$. From the general theory of such differential equation [e.g. 1, Ch. 4] we have

**Proposition 3.3.** Two linearly independent solutions of (21) can be represented in the following form

(22) \[ \varphi_1(t) = t^{i \lambda t} U_1(t) = U_1(t) \]
(23) \[ \varphi_2(t) = t^{i \lambda t} U_2(t) + \alpha \varphi_1(t) \ln t \]

here $U_1$ and $U_2$ are analytic on $[0, \infty)$ with $U_1(0) = U_2(0) = 1$ and $\alpha$ is some constant.

**Corollary 1.** The function $\Psi_{1; i}^*(a_i) = \varphi_1(t)$.

*Proof.* The only solutions of (21) which are bounded at $t = 0$ are constant multiples of $\varphi_1(t)$ and we know that $\Psi_{1; i}^*(1) = 1$.

Let $\varphi_1(t) = \sum_{j=0}^{\infty} c_j t^j$. We will compute the coefficients $c_j$ more explicitly. Since $\lim_{t \to 0} t \coth t = 1$, we get

(24) \[ \cotht = \frac{1}{t} + \sum_{j=0}^{\infty} a_j t^j \]

with $g(t) = \sum_{j=0}^{\infty} a_j t^j$ analytic at $t = 0$. Substitute $\varphi_1(t)$ into (21), we get

(25) \[ 2(1 + s)c_1 - 2i \lambda c_0 = 0 \]

and the recursion formula, $j = 2, 3, \ldots$

(26) \[ j(j + 1 + 2s)c_j = [p^2 - \lambda^2 - (s + 1)^2] c_{j-2} - 2(1 + s) \sum_{r=1}^{j-1} rc_r c_{j-1-r} \]
\[ + 2i \lambda \sum_{r=0}^{j-1} c_r a_{j-2-r} . \]

**Corollary 2.** Two spherical functions $\Phi_{1, i}^{p_1, \lambda_1}$ and $\Phi_{1, i}^{p_2, \lambda_2}$ of type $s$ are equal if and only if $(p_1, \lambda_1) = \pm (p_2, \lambda_2)$ or $(p_1, \lambda_2) = \pm i(\lambda_1, -\lambda_2)$.

*Proof.* From earlier discussion, it suffices to consider the functions $\Psi_{1; i}^{p_1, \lambda_1}(a_i)$ and $\Psi_{1; i}^{p_2, \lambda_2}(a_i)$, hence their corresponding coefficients derived from (25) and (26). Clearly then it is equivalent to have $p_1 \lambda_1 = p_2 \lambda_2$ and $p_1^2 - \lambda_1^2 = p_2^2 - \lambda_2^2$ and the corollary follows.

**Proposition 3.4.** $\Phi_{i}^{p, \lambda}$ is bounded if $\lambda = \sigma + ib$ with $\sigma, b \in \mathbb{R}$ and $|b| \leq 1$. 
Proof. Let \( x \in G \) and write \( x = k_1a_1k_2 \) with \( t \geq 0 \). Then

\[
\Phi_{x^{-1}}(w^{-1}) = \Phi_{x^{-1}}((k_1a_1k_2)^{-1}) = \Phi_{x^{-1}}((k_2k_1a_1)^{-1})
\]

\[
= \int_k \exp \left( -(i\lambda + 1)\rho(H(a_1,k))D_{-p,-p}(k^{-1}k_1k_2k(a_1,k))dk \right).
\]

Now, write \( k = u_\psi v_\theta u_\varphi \), then \( a_1k = (u_\psi v_\theta, u_\varphi) a_1n, n \in N \),

\[
e^t = e^t \cos^2 \frac{\theta}{2} + e^{-t} \sin^2 \frac{\theta}{2}
\]

\[
\cos \frac{\theta'}{2} = e^{(t-t')/2} \cos \frac{\theta}{2}, \quad \sin \frac{\theta'}{2} = e^{-(t+t')/2} \sin \frac{\theta}{2}, \quad 0 \leq \theta' \leq \pi.
\]

Thus by (4) and (5) we get

\[
\Phi_{x^{-1}}(w^{-1}) = \frac{1}{2} \int_0^\infty \exp \left( -(i\lambda + 1)t'\right)D_{-p,-p}(v_\psi^{-1}k_1k_2v_\psi) \sin \theta d\theta.
\]

If \( t = 0 \), then \( t' = 0 \) and the integral (29) bounds by 1. If \( t > 0 \), by (28) with change of variable gives

\[
\Phi_{x^{-1}}(w^{-1}) = \frac{1}{2 \sinh t} \sum_{j=0}^s \int_{-t}^t e^{-it'}D_{-p,-p}(v_\psi^{-1})D_{j,0}(k_1k_2)D_{j,-1}(v_\theta)dt'.
\]

and

\[
|\Phi_{x^{-1}}(w^{-1})| \leq \frac{1}{2 \sinh t} \int_{-t}^t e^{it'}dt = \frac{\sinh t}{b \sinh t} \leq 1.
\]

4. The analogue of Paley-Wiener theorem. Let

\[
B_\ast = \{(p, \lambda); p = -s, -s + 1, \ldots, s; \lambda \in C\}.
\]

For each pair \((p, \lambda) \in B_\ast\), there corresponds a spherical functions \( \Phi_{p,\lambda} \) of type \( s \). Let \( f \in I_\ast(G) \), the Fourier-Laplace transform \( \hat{f} \) of \( f \) is a function defined on \( B_\ast \) by

\[
\hat{f}(p, \lambda) = \int_G f(x) \Phi_{p,\lambda}(x) dx.
\]

Given \( f \in I_\ast(G) \). Let \( B_f = \{a_t \in A_p; f(ka_t) \neq 0 \text{ for some } k \in K\} \). We say that \( f \) has support in the ball of radius \( R \) if \( \sup \{|t|; a_t \in B_f\} \leq R \). Clearly \( f \) has compact support if and only if there exists an \( R \) which is finite. For each \( D^r \in K \), define

\[
F_f(a_t) = e^t \int_K \int_N f(ka_tn)D^r(k^{-1})dkdn.
\]

This gives a map of \( A_p \) to the space of linear operators \( L(E_s) \) on \( E_s \). It is easy to see that \( F_f = F_{f*} \), \( f \in I_\ast(G) \) and \( f* = f*\chi_s \).
Lemma 4.1. Let $n \in \mathbb{N}$, $a_i \in A$, and write $a_n = k_1 a_i k_2$ for some $k_1, k_2 \in K$. Then $| t_1 | \geq | t |$.

Proof. Let $n = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in C$ and $k_j = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix}$ with $| \alpha_j |^2 + | \beta_j |^2 = 1$, $j = 1, 2$. Equating the corresponding matrix coefficients from $a_n k_2^{-1}$ yields $e^t + (1 + | z |^2) e^{-t} = e^t + e^{-t}$, i.e., $2 \cosh t_1 + e^t | z |^2$. Thus $| t_1 | \geq | t |$.

Proposition 4.2. Let $f \in I_c, s(G)$ have support in the ball of radius $R$, then $F^*_f$ is $C^\infty$ with support in $\{ a_i : | t | \leq R \}$.

Proof. Suppose $F^*_f(a_i) \neq 0$, then $f(k a_i, n) \neq 0$ for some $k \in K$, $n \in \mathbb{N}$. Now $k a_i, n = k a_i k_2$ for some $k_1, k_2 \in K$ and $a_i \in A$. Thus $a_n = k^{-1} k_1 a_i k_2$ and $f(k a_i, a_i) = f(k a_i, k_2) = f(k a_i, n) \neq 0$. By the above lemma and the assumption we get $| t | \leq | t_1 | \leq R$. Differentiability is clear.

Proposition 4.3. The map $f \rightarrow F^*_f$ is a one-to-one algebra homomorphism of $I_c, s(G)$ into $C^\infty_c(A, L(E))$.

Proof. Let $f, g \in I_c, s(t)$, use Fubini’s theorem repeatedly

$$
F^*_{f \star}(a_i) = e^t \int_{A} \int_{N} \int_{K} (f \star g)(k a_i, n) D^t(k^{-i}) d \kappa d \nu d \pi
$$

$$
= e^t \int_{A} \int_{N} \int_{K} \int_{A} \int_{N} f(k a_i, n, k^{-i}) g(k a_i, n) d \kappa d \nu d \pi
$$

$$
\times e^{t_1} D^t(k^{-i}) d k d t_1 d \kappa d \nu d d \pi
$$

$$
= \int_{A} \int_{N} \int_{K} \int_{A} \int_{N} f(k a_i, a_i^{-i} n) g(k a_i, n) e^{t_1} \cdot e^{t_1} D^t(k^{-i})
$$

$$
\times D^t(k^{-i}) d k d t_1 d \kappa d \nu d d \pi
$$

$$
= \int_{A} \int_{N} F^*_f(a_i, a_i^{-i}) F^*_s(a_i) d t_1 = F^*_f \ast F^*_s(a_i).
$$

The linearity is trivial, hence it is algebra homomorphism. As for one-to-one, given $f \in I_c, s(G)$ and $F^*_f \equiv 0$, to show $f \equiv 0$. Note first that $F^*_f(a_i) D^t(u_\alpha) = D^t(u_\alpha) F^*_f(a_i)$, hence $F^*_f(a_i)$ is a diagonal matrix. From (10) and Lemma 3.2, we see that if $F^*_f(a_i)$ is the $p$th diagonal element of $F^*_f(a_i)$,

$$
\int_{A} F^*_{f,p}(a_i) e^{-t_1} d t = \int_{\beta} f(\alpha) \Phi_{\beta,1}^{-1}(\alpha) d \alpha.
$$
If $F'_\lambda = 0$, then $F'_\lambda \rho = 0$ for all $p$, hence $\hat{f}(p, \lambda) = 0$ for all $p, \lambda$. Thus $U^{p,i}(f) = 0$ for all $p, \lambda$. But the set $\{U^{p,i}\}$ forms a complete set of representations on $G$, thus we get $f = 0$.

**COROLLARY.** $I_{c,*}(G)$ is commutative.

For each nonnegative real number $R$, let $H_s(R)$ be the set of functions $g$ defined on $B_s$ satisfying (i) $g$ is entire holomorphic in $\lambda$; (ii) $g(p, \lambda) = g(-p, -\lambda)$, $(p, \lambda) \in B_s$; (iii) $g(p, \lambda) = g(i\lambda, -ip)$ if both $(p, \lambda)$ and $(i\lambda, -ip)$ are in $B_s$; (iv) given a positive integer $m$, there exists a constant $C_m$ such that $|g(p, \lambda)| \leq C_m(1 + |\lambda|)^{-m} \exp R|\eta|$, $\lambda = \xi + i\eta \in R + iR$. Let $H_s$ be the union of all the $H_s(R)$.

Given $f$ in $I_{c,*}(G)$, by Corollary 2 of Proposition 3.3 we see the function $\hat{f}$ defined in (30) satisfies conditions (i), (ii), (iii) of the definition of $H_s$. By (32), $\hat{f}(p, \lambda)$ is just the usual Fourier transform of the function $F'_\lambda \rho$ on the real line, which is $C^\infty$ with compact support, hence $\hat{f}$ is holomorphic in $\lambda$. If $f$ has support in the ball of radius $R$, so is $F'_\lambda$, hence the classical Paley-Wiener theorem asserts that $\hat{f} \in H_s(R)$. Thus we have a linear map $f \mapsto \hat{f}$ of $I_{c,*}(G)$ into $H_s(R)$ such that if $f$ has support in the ball of radius $R$, we get $\hat{f} \in H_s(R)$. We want to show that this map is also onto now.

In the inversion formula (8), when $f \in I_{c,*}(G)$, it is easy to see that $\text{Trace} (U^{p,i}(x^{-1}) U^{p,i}(f)) = (2s + 1)\hat{f}(p, \lambda) \Phi^{p,i}(x^{-1})$ for $p = -s, -s + 1, \ldots, s$; and $U^{p,i}(f) = 0$ otherwise. Thus we have

$$f(x) = \frac{2s + 1}{4\pi^2} \sum_{p=-s}^{s} (p^2 + \lambda^2)\hat{f}(p, \lambda) \Phi^{p,i}(x^{-1})d\lambda.$$  

**LEMMA 4.4.** Let $g \in H_s(R)$ and define

$$f_t(x) = \sum_{p=-s}^{s} \int_{\lambda=\infty}^{\lambda=-\infty} (p^2 + \lambda^2)g(p, \lambda) \Phi^{p,i}(x^{-1})d\lambda.$$  

Then $f_t \in I_{c,*}(G)$ and $f_t$ has support in the ball of radius $R$.

**Proof.** Since $g(p, \lambda)$ decreases rapidly at infinity on $\lambda$ and $\Phi^{p,i}$ is $C^\infty$ and bounded when $\lambda$ is real, the integral converges absolutely and defines a $C^\infty$ function on $G$. By the property of $\Phi^{p,i}$, it is clear that $f_t(k \times k^{-1}) = f_t(x)$, $k \in K$, $x \in G$ and $f_t \ast \chi_s = f_t$. It remains to show that $f_t$ has support in the ball of radius $R$. Thus let $x = k a_t$ with $k \in K$ and $t \neq 0$. Since $a_t \in B_{f_t}$ if and only if $a_{-t} \in B_{f_t}$, may assume that $t > 0$. Using the expression and notation in Proposition 3.4, we get

$$f_t(k a_t) = \frac{1}{2 \sinh t} \sum_{p, \rho} D^{p,i}_{\rho, \lambda}(k) \int_{\lambda=-\infty}^{\lambda=\infty} \int_{t'=t}^{t'} (p^2 + \lambda^2)g(p, \lambda)e^{-2\lambda t'} \times D^{\rho,p}_{\lambda, \lambda}(v^{t'})D^{\rho,-p}_{\lambda, -\lambda}(v_{t'})dt'd\lambda.$$
For each $p, j = -s, -s + 1, \ldots, s$, define
\begin{equation}
(36) f_{p,j}(t) = \int_{-\infty}^{\infty} \int_{t'=t}^{t} (p^2 + \lambda^2)g(p, \lambda)e^{-it'\lambda}D_{p,j}(v_{\theta'})D_{j,-p}(v_{\theta'})dt'd\lambda.
\end{equation}

Let $t > R$, to show $f_i(k, a_i) = 0$, it suffices to show that $\sum_{p=-s}^{s} f_{p,j}(t) = 0$ for all $j$. Let
\begin{equation}
(37) h_s(t') = \int_{-\infty}^{\infty} (p^2 + \lambda^2)g(p, \lambda)e^{-it'\lambda}d\lambda.
\end{equation}

By the classical Paley-Wiener theorem, $h_s(t') = 0$ if $t' > R$. Thus
\begin{equation}
(38) f_{p,j}(t) = \int_{-\infty}^{\infty} h_s(t')D_{p,j}(v_{\theta'})D_{j,-p}(v_{\theta'})dt'.
\end{equation}

Put $x_1 = e^{it'}, x_2 = -e^{-it}$, then by (5), (28) we get
\begin{equation}
(39) D_{p,j}(v_{\theta'})D_{j,-p}(v_{\theta'}) = \frac{(-1)^{s+j}e^{-it'}}{(s+j)!(s-j)!(2\sin h t)^{2s}}e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}} [x_1x_2]^r [x_1^{2s} + x_2^{2s}] .
\end{equation}

The above expression is just the linear combination of terms
\[ e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}} [x_1x_2]^r (x_1^{2s} + x_2^{2s}) \]
with coefficients as functions of $t$, and $r_1, r_2 \geq 0, r_1 + r_2 \leq 2s$. Pick one of these terms and consider the two integrals
\begin{equation}
(40) \sum_{p=-s}^{s} \int_{-\infty}^{\infty} h_s(t')e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}} [x_1x_2]^r [(x_1^{2s} + x_2^{2s})]dt' = \frac{\min\{s, r_1 - s\}}{\max\{s, r_1 + r_2 - s\}} \int_{-\infty}^{\infty} h_s(t') \frac{(r_1 + r_2)!r_1!}{(r_1 + r_2 - s + p)! (r_1 - s - p)!}e^{(r_2 + p)t'}dt' = 2\pi \int_{p=s-r_1}^{s} \frac{(r_1 + r_2)!r_1!}{(r_1 + r_2 - s + p)! (r_1 - s - p)!} \times [p^2 + (-i(r_2 + p))]g(p, -i(r_2 + p))
\end{equation}
\begin{equation}
\times (r_1 + r_2)! r_1! \times x_1x_2 \times (r_1 + r_2 - s + p)! (r_1 - s - p)!)
\end{equation}
\begin{equation}
(41) \sum_{p=-s}^{s} \int_{-\infty}^{\infty} h_s(t')e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}} [x_1x_2]^r [(x_1^{2s} + x_2^{2s})]dt' = 2\pi \int_{p=s-r_1}^{s} \frac{(r_1 + r_2)!r_1!}{(r_1 - s + p)! (r_1 + r_2 - s + p)!} \times (r_1 + r_2)! r_1! \times x_1x_2 \times (r_1 + r_2 - s + p)! (r_1 - s - p)!)
\end{equation}
\begin{equation}
\times (r_1 + r_2)! r_1! \times x_1x_2 \times (r_1 + r_2 - s + p)! (r_1 - s - p)!)
\end{equation}
By changing the index and the fact that
\[ g(p, i(r_2 - p)) = g(p - r_2, -ip) , \]
we get the sum of (40) and (41) is zero. Now the lemma is clear.

Combine the above discussion, we get the following analogue of Paley-Wiener theorem.

**PROPOSITION 4.5.** The Fourier transform \( \hat{f} \) defined in (30) is a one-to-one algebra homomorphism of \( I_{s, s}(G) \) onto \( H_s \). A function \( f \) in \( I_{s, s}(G) \) has support in the ball of radius \( R \) if and only if \( \hat{f} \) is in \( H_s(R) \).

Let \( L^i(G) \) be the closure of \( I^i(G) \) in \( L'(G) \). Given \( f \in L^i(G) \), by Proposition 3.4, the integral

\[ (42) \quad \hat{f}(p, \lambda) = \int_G f(x) \Phi^i(x) dx \]
is defined for \( (p, \lambda) \in B_s \) with \( \lambda = \xi + i\eta, |\eta| \leq 1 \). Then we have the following analogue of Riemann Lebesgue lemma.

**PROPOSITION 4.6.** Let \( f \in L^i(G) \) and define \( \hat{f} \) as in (42), then
\[ \lim_{|\xi| \to \infty} \hat{f}(p, \xi + i\eta) = 0 \]
uniformly for \( |\eta| \leq 1 \).

**Proof.** Given \( \varepsilon > 0 \), choose \( g \) in \( I^s(G) \) such that \( \| f - g \|_1 < \varepsilon/2 \).

But then we have

\[ (43) \quad |\hat{f}(p, \lambda) - \hat{g}(p, \lambda)| \leq \int_G |f(x) - g(x)| dx < \varepsilon/2 . \]

Choose \( R, C \) such that

\[ (44) \quad |\hat{g}(p, \lambda)| \leq C(1 + |\lambda|)^{-1} \exp R |\eta| \leq C(1 + |\lambda|)^{-1} \exp R \]
since \( |\eta| \leq 1 \). Combine (43), (44) we get \( |\hat{f}(p, \lambda)| < \varepsilon \) when \( |\xi| \) is large enough.

Let \( B = \{(s, p, \lambda) : s \text{ is a nonnegative integer or half integer,} \ (p, \lambda) \in B_s \} \). Given \( f \in L(G) \) and \( (s, p, \lambda) \in B \), define

\[ (45) \quad \hat{f}(s, p, \lambda) = \int_G f(x) \Phi^i_s(x) dx . \]

It is clear that \( \hat{f}(s, p, \lambda) = \hat{f}_s(p, \lambda) \).

**LEMMA 4.7.** Let \( f \in L_s(G) \). Then \( f \) has support in the ball of radius \( R \) if and only if \( f \) has support in the ball of radius \( R \) for all \( s \).
Proof. By definition, \( f_s(x) = \int_{K} f(k^{-1}x)\chi_s(k)dk \). Thus if \( f \) has support in the ball of radius \( R \) and \( f_s(k, a_t) \neq 0 \) with \( k \in K, a_t \in A_t \), we have \( f(k^{-1}k, a_t) \neq 0 \) for some \( k \in K \) and therefore \( |t| \leq R \). The converse follows from the fact that \( \sum_s f_s \) converges to \( f \) absolutely, [8, vol. I, p. 264].

**Proposition 4.8.** The map \( f \mapsto \hat{f} \) defined in (45) is a one-to-one algebra homomorphism of \( L(G) \) into the algebra of all functions \( g \) on \( B \) satisfying (i) \( g(s, p, \lambda) \) is entire holomorphic in \( \lambda \), (ii) \( g(s, p, \lambda) = g(s, -p, -\lambda) \), \( (s, p, \lambda) \in B \), (iii) \( g(s, p, \lambda) = g(s, i\lambda, -ip) \) if both \( (s, p, \lambda) \) and \( (s, i\lambda, -ip) \) are in \( B \), (iv) there exists \( R > 0 \), for each given positive integer \( m \), there exists \( C_{m,s} \) such that

\[
|g(s, p, \lambda)| \leq C_{m,s}(1 + |\lambda|)^{-m} \exp R|\eta|, \xi + i\eta \in R + iR.
\]

Proof. This is clear by Proposition 4.6 and Lemma 4.7.

**Corollary.** Let \( f \in L'(G) \). Then \( \hat{f}(s, p, \lambda) \) is defined for \( \lambda = \xi + i\eta \), \( |\eta| \leq 1 \) and \( \lim_{t \to \pm \infty} \hat{f}(s, p, \xi + i\eta) = 0 \) for \( |\eta| \leq 1 \).

**References**


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