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By its definition a conditional expectation is the Radon-Nikodym derivative of a finite signed measure. In this paper an inversion formula is given for recapturing $E(Y|X)$ as an inverse Fourier transform of the function $E(e^{i(u,X)}Y)$, $u \in R^n$, where X is an random vector and Y is a random variable satisfying some regularity conditions.

1. Introduction. Let $(\Omega, \mathfrak{B}, P)$ be a probability space and let X be a k-dimensional random vector on $(\Omega, \mathfrak{B}, P)$, i.e., a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^k, \mathfrak{B}^k)$ where $\mathfrak{B}^k$ is the σ-algebra of Borel sets in the k-dimensional Euclidean space R^k. Assume that the probability distribution X is absolutely continuous with respect to the Lebesgue measure m_L on $(R^k, \mathfrak{B}^k)$. For a real valued random variable Y on $(\Omega, \mathfrak{B}, P)$ with $E(|Y|) < \infty$ let $E(Y|X)$ be the conditional expectation of Y given X which is given as a function on the value space R^k of X. For $u \in R^k$ let $(u, X) = \sum_{j=1}^k u_j X_j$. In this paper we show that if $E[e^{i(u,X)}Y]$ is a m_L-integrable function of u on R^k then a version of $E(Y|X)$ is given by

$$(1.1) \qquad E(Y|X)(\xi) = \left(\frac{dP_X}{dm_L}(\xi)\right)^{-1} \frac{1}{(2\pi)^k} \int_{R^k} e^{-i(u,\xi)} E[e^{i(u,X)}Y] m_L(du)$$

for $\xi \in R^k$ assuming that $(dP_X/dm_L)(\xi) > 0$ for a.e. ξ in $(R^k, \mathfrak{B}^k, m_L)$. (Our conditional expectation $E(Y|X)$ given as a function on R^k rather than one on Ω is the "conditional expectation in the wide sense" in the terminology of [2].)

In preparation for (1.1) which is given in Theorem 2 in §3 we show in Theorem 1 in §3 that if the characteristic function (i.e., the Fourier transform) φ of a finite measure Φ on $(R^k, \mathfrak{B}^k)$ is m_L-integrable on R^k then Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ and a version of the Radon-Nikodym derivative of Φ with respect to m_L is given by the inverse Fourier transform of φ. We base this result on the Lévy-Haviland Theorem for the inversion of Fourier transforms of finite measures on $(R^k, \mathfrak{B}^k)$.

The substance of Propositions 1, 2, and 3 in §2 concerning conditional probabilities, conditional expectations and regular conditional distributions given as functions on the value space of X is well-known. We included them here in order to state them in a convenient form.

This research is an attempt at justifying a calculus of Wiener integral originated by M. D. Donsker. Its applications to conditional function space integrals will appear in a subsequent paper.

2. Integration of conditional expectations. Throughout §2 we write $(\Omega, \mathfrak{B}, P)$ for a probability space and X and Y for two measurable transformations of $(\Omega, \mathfrak{B})$ into two arbitrary measurable spaces $(S, \mathfrak{F})$ and $(T, \mathfrak{G})$ respectively unless further specified. We write P_X and P_Y for the probability measures on $(S, \mathfrak{F})$ and $(T, \mathfrak{G})$ determined by X and Y respectively, i.e.,

$$P_X(F) = P(X^{-1}(F)) \qquad \text{for} \quad F \in \mathfrak{F} \tag{2.1}$$

and similarly for P_Y.

DEFINITION 1. For $G \in \mathfrak{G}$ fixed, the conditional probability of Y being in G given X, written $P(Y \in G \mid X)$, is defined to be any real valued $\mathfrak{F}$-measurable and P_X-integrable function ψ on S such that

$$P(Y^{-1}(G) \cap X^{-1}(F)) = \int_F \psi(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} .$$

From the Radon-Nikodym Theorem follows that such a function ψ always exists and is determined uniquely up to a null set of $(S, \mathfrak{F}, P_X)$. We shall use $P(Y \in G \mid X)$ to mean either the class of all such functions ψ or a particular member in it depending on the context. Thus

$$P(Y^{-1}(G) \cap X^{-1}(F)) = \int_F P(Y \in G \mid X)(\xi) P_X(d\xi) \ \text{for} \quad F \in \mathfrak{F} . \tag{2.2}$$

DEFINITION 2. Let Z be a real valued random variable on $(\Omega, \mathfrak{B}, P)$ with $E(|Z|) < \infty$. The conditional expectation of Z given X, written $E(Z \mid X)$, is defined to be any real valued $\mathfrak{F}$-measurable and P_X-integrable function ψ on S such that

$$\int_{X^{-1}(F)} Z(\omega) P(d\omega) = \int_F \psi(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} .$$

The same remark as the one following Definition 1 holds here too and we have

$$\int_{X^{-1}(F)} Z(\omega) P(d\omega) = \int_F E(Z \mid X)(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} . \tag{2.3}$$

DEFINITION 3. By the regular conditional distribution of Y given X, written $P(Y \mid X)$, we mean a real valued function ψ on $\mathfrak{G} \times S$ such that

1° for every $G \in \mathfrak{G}$, $\psi(G, \cdot)$ is a version of $P(Y \in G \mid X)$

2° for every $\xi \in S$, $\psi(\cdot, \xi)$ is a probability measure on $(T, \mathfrak{G})$.

Thus when we need to indicate the arguments of $P(Y|X)$, we write $P(Y|X)(G, \xi)$ for $(G, \xi) \in \mathfrak{G} \times S$. It is known that a function ψ satisfying the conditions 1° and 2° of Definition 3 always exists whenever the value space $(T, \mathfrak{G})$ of Y is a Borel space and in particular when $(T, \mathfrak{G}) = (R^k, \mathfrak{B}^k)$. For a proof of this statement see [1]. The proof of Proposition 1 below which relates the regular conditional distribution to the conditional expectation is parallel to the proof of the corresponding theorem in which these two are given as functions on Ω rather than as functions on the value space S of X. (See for instance Proposition 4.28 in [1].) We give the proof here for the sake of completeness.

PROPOSITION 1. *Let f be a measurable transformation of $(T, \mathfrak{G})$ into $(R^1, \mathfrak{B}^1)$ and $f \in L_1(T, \mathfrak{G}, P_Y)$. If $P(Y|X)$ exists then*

$$(2.4) \qquad E(f \circ Y|X)(\xi) = \int_T f(\eta)P(Y|X)(d\eta, \xi) \quad \textit{for a.e. } \xi \in (S, \mathfrak{F}, P_X) .$$

Proof. Consider the case where $f = \chi_G$ for some $G \in \mathfrak{G}$. Then for every $F \in \mathfrak{F}$ we have by (2.3)

$$(2.5) \qquad \begin{aligned} \int_F E(f \circ Y|X)(\xi)P_X(d\xi) &= \int_{X^{-1}(F)} \chi_G(Y(\omega))P(d\omega) \\ &= P(Y^{-1}(G) \cap X^{-1}(F)) . \end{aligned}$$

On the other hand, by 2° and then 1° of Definition 3,

$$\begin{aligned} \int_T f(\eta)P(Y|X)(d\eta, \xi) &= \int_T \chi_G(\eta)P(Y|X)(d\eta, \xi) \\ &= P(Y|X)(G, \xi) = P(Y \in G|X)(\xi) \end{aligned}$$

so that by (2.2) we have

$$(2.6) \qquad \int_F \left\{\int_T f(\eta)P(Y|X)(d\eta, \xi)\right\}P_X(d\xi) = P(Y^{-1}(G) \cap X^{-1}(F)) .$$

Thus the left side of (2.5) is equal to that of (2.6) for every $F \in \mathfrak{F}$ so that (2.4) holds in this case.

Now that (2.4) holds when f is the characteristic function of a member of $\mathfrak{G}$ we can follow the usual procedure in integration theory to show that (2.4) holds for nonnegative simple functions on T, nonnegative $\mathfrak{G}$-measurable function on T and finally real valued $\mathfrak{G}$-measurable functions on T. Since $f \in L_1(T, \mathfrak{G}, P_Y)$, both sides of (2.4) always exist and are finite. In passing from nonnegative simple functions on T to nonnegative $\mathfrak{G}$-measurable function on T we use the Monotone Convergence Theorem for the conditional expectation which states that if $\{Z_n, n = 1, 2, \cdots\} \subset L_1(\Omega, \mathfrak{B}, P)$ and $Z_n(\omega) \uparrow Z_0(\omega)$ for a.e.

$\omega \in (\Omega, \mathfrak{B}, P)$ then $E(Z_n|X)(\xi) \uparrow E(Z_0|X)(\xi)$ for a.e. $\xi \in (S, \mathfrak{F}, P_X)$ and which can be proved readily.

PROPOSITION 2. *Let $\sigma(\mathfrak{F} \times \mathfrak{G})$ be the σ-algebra of subsets of $S \times T$ generated by the semialgebra $\mathfrak{F} \times \mathfrak{G}$ and let $P_{[X,Y]}$ be the probability measure on $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$ determined by the measurable transformation $[X, Y]$ of $(\Omega, \mathfrak{B})$ into $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$. Let f be a measurable transformation of $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$ into $(R^1, \mathfrak{B}^1)$. If $P(Y|X)$ exists then*

$$(2.7)\quad \begin{aligned} E(f\circ[X, Y]) &= \int_{S\times T} f(\xi, \eta)P_{[X,Y]}(d(\xi, \eta)) \\ &= \int_S \left\{\int_T f(\xi, \eta)P(Y|X)(d\eta, \xi)\right\} P_X(d\xi) \end{aligned}$$

in the sense that the existence of any member in (2.7) implies that of the other and the equality of all.

Proof. The first equality in (2.7) is standard. Let us prove the second. Consider the case where

$$f(\xi, \eta) = \chi_{F\times G}(\xi, \eta) = \chi_F(\xi)\chi_G(\eta) \quad \text{for} \quad (\xi, \eta) \in S \times T$$

where $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$. Then by 2° and 1° of Definition 3 and by (2.2)

$$\begin{aligned} &\int_S \left\{\int_T f(\xi, \eta)P(Y|X)(d\eta, \xi)\right\} P_X(d\xi) \\ &\quad = \int_S \chi_F(\xi)\left\{\int_T \chi_G(\eta)P(Y|X)(d\eta, \xi)\right\} P_X(d\xi) \\ &\quad = \int_S \chi_F(\xi)P(Y|X)(G, \xi)P_X(d\xi) \\ &\quad = \int_F P(Y \in G|X)(\xi)P_X(d\xi) \\ &\quad = P(Y^{-1}(G) \cap X^{-1}(F)) \end{aligned}$$

while

$$\begin{aligned} &\int_{S\times T} f(\xi, \eta)P_{[X,Y]}(d(\xi, \eta)) \\ &\quad = \int_{S\times T} \chi_{F\times G}(\xi, \eta)P_{[X,Y]}(d(\xi, \eta)) \\ &\quad = P_{[X,Y]}(F \times G) \\ &\quad = P\{\omega \in \Omega;\ X(\omega) \in F \text{ and } Y(\omega) \in G\} \\ &\quad = P(Y^{-1}(G) \cap X^{-1}(F)) \end{aligned}$$

so that the second equality in (2.7) holds for this particular case.

We then proceed as in the proof of the Fubini Theorem to an arbitrary real valued $\sigma(\mathfrak{F} \times \mathfrak{S})$-measurable function f on $S \times T$ to complete the proof.

PROPOSITION 3. *Let Z be a real valued random variable on $(\Omega, \mathfrak{B}, P)$ with $E(|Z|) < \infty$ and let g be a measurable transformation of $(S, \mathfrak{F})$ into $(R^1, \mathfrak{B}^1)$. Then*

$$E[(g \circ X)Z] = \int_S g(\xi)E(Z \mid X)(\xi)P_X(d\xi) \tag{2.8}$$

in the sense that the existence of one side implies that of the other and the equality of the two.

Proof. Let us define a set function Φ on $\mathfrak{B}$ by

$$\Phi(B) = \int_B Z(\omega)P(d\omega) \qquad \text{for} \quad B \in \mathfrak{B}\,.$$

Since $E(|Z|) < \infty$, Φ is a finite signed measure on $(\Omega, \mathfrak{B})$ which is absolutely continuous with respect to P and has Z as its Radon-Nikodym derivative with respect to P. Thus for the real valued random variables $g \circ X$ and Z on $(\Omega, \mathfrak{B}, P)$ we have

$$E[(g \circ X)Z] = \int_\Omega g(X(\omega))Z(\omega)P(d\omega) = \int_\Omega g(X(\omega))\Phi(d\omega)$$

in the sense that the existence of one member implies that of the others and the equality of all. Then, to prove (2.8) it suffices to show that

$$\int_\Omega g(X(\omega))\Phi(d)\omega = \int_S g(\xi)E(Z \mid X)(\xi)P_X(d\xi) \tag{2.9}$$

in the sense that the existence of one side implies that of the other and the equality of the two.

Let us consider the case where $g = \chi_F$ for some $F \in \mathfrak{F}$. Then

$$\begin{aligned}\int_\Omega g(X(\omega))\Phi(d\omega) &= \int_\Omega \chi_F(X(\omega))\Phi(d\omega) = \int_{X^{-1}(F)} \Phi(d\omega)\\ &= \int_{X^{-1}(F)} Z(\omega)P(d\omega) = \int_F E(Z \mid X)(\xi)P_X(d\xi)\\ &= \int_S g(\xi)E(Z \mid X)(\xi)P_X(d\xi)\end{aligned}$$

by (2.3) so that (2.9) holds. Following the standard procedure in integration theory we proceed from this particular case to nonnegative simple functions on S, nonnegative $\mathfrak{F}$-measurable functions on S and finally real valued $\mathfrak{F}$-measurable functions on S to complete the proof.

3. Inversion of conditional expectations. It is well-known that if the characteristic function φ of a distribution function F on R^1 is m_L-integrable on R^1, then F is absolutely continuous and

$$F'(\xi) = \frac{1}{2\pi}\int_{R^1} e^{-i(\xi,\eta)}\varphi(\eta)m_L(d\eta) \quad \text{for a.e. } \xi \in (R^1, \mathfrak{B}^1, m_L) .$$

Let Φ be a finite measure on $(R^k, \mathfrak{B}^k)$ and let φ be its characteristic function, i.e.,

$$\text{(3.1)} \qquad \varphi(\eta) = \int_{R^k} e^{i(\xi,\eta)}\Phi(d\xi) \qquad \text{for} \quad \eta \in R^k$$

where $(\xi, \eta) = \sum_{j=1}^k \xi_j\eta_j$. According to the Lévy-Haviland Inversion Theorem (see [3] and [4])

$$\text{(3.2)} \qquad \begin{aligned} &\int_{R^k} \prod_{j=1}^k \tilde{\chi}_{a_j,b_j}(\xi_j)\Phi(d\xi) \\ &\quad = \lim_{h\to\infty} \frac{1}{(2\pi)^k}\int_{C_h}\prod_{j=1}^k \frac{e^{-ib_j\eta_j} - e^{-ia_j\eta_j}}{-i\eta_j}\varphi(\eta)m_L(d\eta) \end{aligned}$$

for any $a_j, b_j \in R^1$, $a_j < b_j$, $j = 1, 2, \cdots, k$, where

$$\text{(3.3)} \qquad C_h = (-h, h) \times \cdots \times (-h, h) \subset R^k \quad \text{with} \quad h > 0 ,$$

and the modified characteristic function $\tilde{\chi}_{a_j,b_j}$ is defined by

$$\text{(3.4)} \qquad \tilde{\chi}_{a_j,b_j}(\eta_j) = \begin{cases} 1 & \text{for} \quad \eta_j \in (a_j, b_j) \\ 0 & \text{for} \quad \eta_j \in [a_j, b_j]^c \\ \dfrac{1}{2} & \text{for} \quad \eta_j = a_j \text{ and for } \eta_j = b_j . \end{cases}$$

From (3.2) we derive the following:

THEOREM 1. *If the characteristic function φ of a finite measure Φ on $(R^k, \mathfrak{B}^k)$ is m_L-integrable on R^k, then Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ and a version of the Radon-Nikodym derivative of Φ with respect to m_L is given by*

$$\text{(3.5)} \qquad \frac{d\Phi}{dm_L}(\xi) = \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(\xi,\eta)}\varphi(\eta)m_L(d\eta) \qquad \text{for} \quad \xi \in R^k .$$

Proof. Since the jth factor of the product in the integrand on the right side of (3.2) is a bounded continuous function of $\eta_j \in R^1$, if we assume the m_L-integrability of φ on R^k then the integrand on the right side of (3.2) is m_L-integrable on R^k so that (3.2) reduces to

$$\begin{aligned}\int_{R^k}\prod_{j=1}^{k}\tilde{X}_{a_j,b_j}(\xi_j)\Phi(d\xi) \\ = \frac{1}{(2\pi)^k}\int_{R^k}\prod_{j=1}^{k}\frac{e^{-ib_j\eta_j}-e^{-ia_j\eta_j}}{-i\eta_j}\varphi(\eta)m_L(d\eta)\ .\end{aligned} \tag{3.6}$$

To show that Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ let $A\in\mathfrak{B}^k$ and $m_L(A)=0$. We proceed to show that $\Phi(A)=0$. Let $\mathfrak{A}$ be the algebra of subsets of R^k which are unions of finitely many disjoint half open and half closed intervals $(a_1, b_1]\times\cdots\times(a_k, b_k]$ in R^k. Then the σ-algebra of subsets of R^k generated by $\mathfrak{A}$ is precisely our $\mathfrak{B}^k$. Let $\varepsilon>0$ be arbitrarily given. Since Φ is a finite measure on $\mathfrak{B}^k$ and since $m_L(A)$ is finite (in fact equal to zero), $(\Phi+m_L)(A)$ is finite so that there exists some $B\in\mathfrak{A}$ such that

$$(\Phi+m_L)(A\varDelta B)<\varepsilon \tag{3.7}$$

where $A\varDelta B$ is the symmetric difference between A and B. Now (3.7) implies that $\Phi(A\varDelta B)<\varepsilon$ so that

$$\Phi(A)\leqq\Phi(B)+\Phi(A\varDelta B)<\Phi(B)+\varepsilon\ . \tag{3.8}$$

It also implies that $m_L(A\varDelta B)<\varepsilon$ so that in view of $m_L(A)=0$ we have

$$m_L(B)<\varepsilon\ .$$

Since B is the union of finitely many, say m, disjoint half open half closed intervals, there exist m open intervals $B^{(n)}$, $n=1, 2, \cdots, m$, such that

$$B\subset\bigcup_{n=1}^{m}B^{(n)}\quad\text{and}\quad m_L(B)<\sum_{n=1}^{m}m_L(B^{(n)})<\varepsilon\ . \tag{3.9}$$

Let each $B^{(n)}$ be given as

$$B^{(n)}=\zeta^{(n)}+C^{(n)} \tag{3.10}$$

where

$$\zeta^{(n)}\in R^k\ \text{and}\ C^{(n)}=(-h_1^{(n)}, h_1^{(n)})\times\cdots\times(-h_k^{(n)}, h_k^{(n)})\subset R^k\ . \tag{3.11}$$

In view of the openness of $C^{(n)}$ and the definition of $\tilde{\chi}_{a_j,b_j}$ by (3.4) we have from (3.6)

$$\begin{aligned}&\Phi(\zeta^{(n)}+C^{(n)}) \\ &\leqq\frac{1}{(2\pi)^k}\int_{R^k}\prod_{j=1}^{k}\frac{e^{-i(\zeta_j^{(n)}+h_j^{(n)})\eta_j}-e^{-i(\zeta_j^{(n)}-h_j^{(n)})\eta_j)}}{-i\eta_j} \\ &=\frac{m_L(C^{(n)})}{(2\pi)^k}\int_{R^k}\prod_{j=1}^{k}\frac{\sin\eta_jh_j^{(n)}}{\eta_jh_j^{(n)}}e^{-i\zeta_j^{(n)}\eta_j}\varphi(\eta)m_L(d\eta)\ .\end{aligned} \tag{3.12}$$

Since $|(\eta_j h_j^{(n)})^{-1} \sin \eta_j h_j^{(n)}| \leqq 1$ for $\eta_j \in R^1$ and since $m_L(C^{(n)}) = m_L(B^{(n)})$ we have

$$\Phi(B^{(n)}) = \Phi(\zeta^{(n)} + C^{(n)}) \leqq \frac{m_L(B^{(n)})}{(2\pi)^k} \int_{R^k} |\varphi(\eta)|\, m_L(d\eta) . \tag{3.13}$$

From (3.9) and (3.13) we obtain

$$\Phi(B) \leqq \sum_{n=1}^{m} \Phi(B^{(n)}) \leqq \frac{\varepsilon}{(2\pi)^k} \int_{R^k} |\varphi(\eta)|\, m_L(d\eta) . \tag{3.14}$$

Using (3.14) in (3.8) we have

$$\Phi(A) < \varepsilon\left\{\frac{1}{(2\pi)^k} \int_{R^k} |\varphi(\eta)|\, m_L(d\eta) + 1\right\} .$$

From the arbitrariness of $\varepsilon > 0$ we have $\Phi(A) = 0$. This proves the absolute continuity of Φ with respect to m_L on $(R^k, \mathfrak{B}^k)$.

To obtain the Radon-Nikodym derivative of Φ with respect to m_L on $(R^k, \mathfrak{B}^k)$, let us observe first that the absolute continuity of Φ with respect to m_L implies that the Φ measure of the boundary of the open interval $C^{(n)}$ in (3.11) is equal to zero. Thus in (3.12) the strict equality actually holds. If we apply this improved (3.12) to $\zeta + C_h$ where ζ is an arbitrary point in R^k and C_h is an open interval in R^k as given by (3.3) then we have

$$\begin{aligned} \int_{\zeta + C_h} \frac{d\Phi}{dm_L}(\xi) m_L(d\xi) &= \Phi(\zeta + C_h) \\ &= \frac{m_L(C_h)}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{\sin \eta_j h}{\eta_j h} e^{-i\zeta_j \eta_j} \varphi(\eta) m_L(d\eta) . \end{aligned} \tag{3.15}$$

Let $h \to 0$ on both sides of (3.15). On the one hand we have

$$\begin{aligned} \lim_{h\to 0} \frac{1}{m_L(C_h)} \int_{\zeta + C_h} \frac{d\Phi}{dm_L}(\xi) m_L(d\xi) \\ = \frac{d\Phi}{dm_L}(\zeta), \text{ for a.e. } \zeta \in (R^k, \mathfrak{B}^k, m_L) \end{aligned} \tag{3.16}$$

and on the other hand by the Dominated Convergence Theorem

$$\begin{aligned} \lim_{h\to 0} \frac{1}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{\sin \eta_j h}{\eta_j h} e^{-i\zeta_j \eta_j} \varphi(\eta) m_L(d\eta) \\ = \frac{1}{(2\pi)^k} \int_{R^k} e^{-(\zeta, \eta)} \varphi(\eta) m_L(d\eta) . \end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) in (3.15) we have

$$\frac{d\Phi}{dm_L}(\zeta) = \frac{1}{(2\pi)^k} \int_{R^k} e^{-(\zeta, \eta)} \varphi(\eta) m_L(d\eta) \text{ for a.e. } \zeta \in (R^k, \mathfrak{B}^k, m_L) .$$

This completes the proof of the theorem.

By means of Proposition 2 and Theorem 1 our inversion theorem for conditional expectation can be derived now.

THEOREM 2. *Let Y be a real valued random variable on a probability space $(\Omega, \mathfrak{B}, P)$ with $E(|Y|) < \infty$ and let X be a k-dimensional random vector i.e., a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^k, \mathfrak{B}^k)$. Assume that the probability distribution P_X of X is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$. If $E[e^{i(u,X)}Y]$ is a m_L-integrable function of u on R^k then a version of the conditional expectation of Y given X, $E(Y|X)$, is given by*

$$E(Y|X)(\xi)\frac{dP_X}{dm_L}(\xi) = \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(u,\xi)}E[e^{i(u,X)}Y]m_L(du) \tag{3.18}$$

for $\xi \in R^k$.

Proof. Since Y is a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^1, \mathfrak{B}^1)$ which is a Borel space, the regular conditional distribution of Y given X, $P(Y|X)$, exists. With fixed $u \in R^k$ consider a complex valued function f on $R^k \times R^1$ defined by

$$f(\xi, \eta) = e^{i(u,\xi)}\eta \qquad \text{for } \xi \in R^k \text{ and } \eta \in R^1 .$$

Applying (2.7) of Proposition 2 and (2.4) of Proposition 1 to the real and the imaginary parts of f we obtain

$$\begin{aligned} E[e^{i(u,X)}Y] &= \int_{R^k} e^{i(u,\xi)}\left\{\int_{R^1}\eta P(Y|X)(d\eta, \xi)\right\}P_X(d\xi) \\ & \int_{R^k} e^{i(u,\xi)}E(Y|X)(\xi)P_X(d\xi) . \end{aligned} \tag{3.19}$$

Consider a set function Φ defined on $\mathfrak{B}^k$ by

$$\Phi(F) = \int_F E(Y|X)(\xi)P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{B}^k . \tag{3.20}$$

Since $E(Y|X)$ is P_X integrable on R^k, Φ is a finite signed measure on $(R^k, \mathfrak{B}^k)$ which is absolutely continuous with respect to P_X on $(R^k, \mathfrak{B}^k)$ and has $E(Y|X)$ as its Radon-Nikodym derivative with respect to P_X. According to (3.19), $E[e^{i(u,X)}Y]$, $u \in R^k$, is the characteristic function of Φ. Under the hypothesis of the theorem, this characteristic function is m_L-integrable over R^k. Applying Theorem 1 to the positive and the negative part of Φ we obtain

$$
\begin{aligned}
(3.21)\qquad & \frac{d\Phi}{dm_L}(\xi) \\
& = \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(u,\xi)} E[e^{i(u,X)}Y] m_L(du) \text{ for a.e. } \xi \in (R^k, \mathfrak{B}^k, m_L) \,.
\end{aligned}
$$

Since by (3.20)

$$
\frac{d\Phi}{dm_L}(\xi) = E(Y|X)(\xi)\frac{dP_X}{dm_L}(\xi) \quad \text{for a.e. } \xi \in (R^k, \mathfrak{B}^k, m_L)
$$

we have (3.18).

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