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THE RANGE OF A CONTRACTIVE PROJECTION ON AN L_p-SPACE

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Suppose (X, Σ, μ) is a measure space, $1 \leqq p < \infty$ and $p \neq 2$. Let $L_p = L_p(X, \Sigma, \mu)$ be the usual space of equivalence classes of Σ-measurable functions f such that $|f|^p$ is integrable. A contractive projection on L_p is a linear operator $P: L_p \to L_p$ such that $P^2 = P$ and $\|P\| \leqq 1$. In this paper we give a complete description of such contractive projections in terms of conditional expectation operators. We also show that a closed subspace M of L_p is the range of a contractive projection if and only if M is isometrically isomorphic to another L_p-space. Another sufficient condition shows, in particular, that every closed vector sublattice of an L_p-space is the range of a positive contractive projection.

Most of our results are known. The case of finite μ was treated, for $p = 1$, by Douglas [2] and for $1 < p < \infty$ by Ando [1] who showed how to reduce this case to that of $p = 1$. These authors obtained our necessary and sufficient condition. Grothendieck [4] considered $p = 1$ and general μ and showed that the range of a contractive projection on L_1 is isometrically isomorphic to another L_1-space. Wulbert [11] showed that a positive contractive projection on L_1 which is also L_∞ contractive is a conditional expectation, and pointed out that his proofs applied for $p > 1$. Tzafriri [10] showed that for general μ the range of a contractive projection on L_p is isometrically isomorphic to another L_p-space. In [5] we gave an outline, based on Tzafriri's, of another proof of this fact.

We obtain complete generalizations of the Douglas-Ando results to the case of an arbitrary measure μ. We have chosen to give our proofs in detail. It seems easier not to reduce the case $p > 1$ to the case $p = 1$. The proofs for $p > 1$ often use duality arguments which are just not available for $p = 1$. By giving such proofs, generalizations to reflexive Banach function spaces may be possible. Some such generalizations have been tried by Rao [8] but his reduction from arbitrary norms to the L_1 case is faulty and his Theorem 2.7 is false in general (see Remark 4.4). Duplissey [3] considers Banach function spaces but requires $\|Pf\|_\infty \leqq \|f\|_\infty$ as well as P contractive. We also avoid reducing to the case of finite measures. This device turns out to be unnecessary, and needlessly complicated.

We have deliberately omitted the cases $0 < p < 1$, except in the appendix, and the case $p = 2$. A contractive projection on Hilbert space is an orthogonal projection and every closed subspace is the range of a unique one. For $0 < p < 1$ the arguments for $p = 1$ will work or can be modified to work. We no longer have a norm, however, and it seemed best to ignore this case.

We have included a section in which we discuss the proof of the famous theorem that if $1 \leqq p < \infty$, a Banach space is an L_p-space, if and only if it is an $\mathscr{L}_{p,\lambda}$ for all $\lambda > 1$, if and only if it contains an increasing set of finite dimensional subspaces whose union is dense and each of which is isometrically isomorphic to a finite dimensional l_p-space of appropriate dimension. This result is a combination of work of Zippin [12] and of Lindenstrauss and Pelczynski [7]. We discussed the real case in [5]. There seems to some value in going over the results again here because both [5] and [7] really consider only the real case. The extensions to the complex case are technically more difficult than is admitted in [7]. Also we have had many questions about some of the details omitted in [5].

In our final appendix we have given two technical results used by Ando [1] and Tzafriri [10]. Our proofs seem a little easier and Ando's result has been generalized to arbitrary measure spaces.

1. Notation and definitions. We consider complex L_p-spaces throughout. Our proofs are valid, with obvious modifications in the real case too. We use, for complex z, the version of the signum function, $\operatorname{sgn} z$ defined by

$$\operatorname{sgn} z = \begin{cases} z/|z| & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 . \end{cases}$$

We modify some standard vector lattice terminology to apply in the complex case. A *closed vector sublattice* of L_p is a closed subspace M such that if $f \in M$, $\operatorname{Re} f \in M$, and if $f \in M$ and f is real-valued, $f^+ = f \vee 0 \in M$.

If $f \in L_p$ write $S(f) = \{x \in X: f(x) \neq 0\}$ and call $S(f)$ the *support* of f. This only determines the support of f to a set of μ-measure zero. However, this will either not matter, or we will want all possible determinations for the support of f. If $M \subset L_p$, the *polar* of M, $M^\perp$, is defined by

$$M^\perp = \{g \in L_p: |g| \wedge |m| = 0 (m \in M)\} .$$

(By $|g| \wedge |m| = 0$ we mean μ-almost everywhere of course.) If $M = M^{\perp\perp}$ we call M a *band* (or *polar subspace*). If M is a band $L_p = M \oplus M^\perp$, and the, natural, *band projection* J_M of L_p onto M is given, for positive $h \in L_p$, by

$$J_M h = \sup\{g \in M: 0 \leqq g \leqq h\}\ .$$

If $f \in L_p$, and $M = f^{\perp\perp}$, we write J_f for the band projection on $f^{\perp\perp}$ and note that, if $0 \leqq h \in L_p$

$$J_f h = \sup\{h \wedge n|f|: n = 1, 2, \cdots\}\ ,$$

(indeed, by dominated convergence, $h \wedge n|f| \rightarrow J_f h$ in L_p-norm) while for any $h \in L_p$, $J_f h = \chi_{S(f)} h$. The following lemma is easy to prove.

LEMMA 1.1. *If M is a subspace of $L_p(X, \Sigma, \mu)$, $h \in L_p$, and J is the band projection on $M^{\perp\perp}$, then there is a sequence (f_n) in M such that $Jh = \lim \chi_S(_{f_n})h$.*

Proof. Choose a sequence (f_n) in M such that

$$\|\chi_{S(f_n)}h\|_p \longrightarrow \sup\{\|\chi_{S(f)}h\|_p: f \in M\}\ .$$

We omit the remaining details.

REMARK 1.2. This lemma can be strengthened, in case M is closed, to say that for each $h \in L_p$ there exists $f \in M$ such that $Jh = J_f h = \chi_{S(f)} h$. This depends essentially on the fact that the set of supports of functions whose equivalence classes are in M is closed under countable union. This is proved by Ando [1, Lemma 3] for finite μ, and we give a rather easier alternative proof in our appendix.

2. **Preliminary results.** In this section the cases $p = 1$, and $1 < p < \infty$, $p \neq 2$, are treated separately. Our first lemma is based on an argument of Douglas [2, p. 452].

LEMMA 2.1. *Let P be a contractive projection on $L_1(X, \Sigma, \mu)$ and suppose $f \in \mathscr{R}(P)$; then*

(i) $PJ_f = J_f P J_f$;

(ii) $P(h \operatorname{sgn} f) = |P(h \operatorname{sgn} f)| \operatorname{sgn} f\ (0 \leqq h \in L_1)$;

(iii) $\|P(h \operatorname{sgn} f)\| = \|J_f h\|\ (0 \leqq h \in L_1)$.

Proof. Suppose $0 \leqq h \leqq |f|$, then

$$\begin{aligned} \|f\| - \|h \operatorname{sgn} f\| &= \|f - h \operatorname{sgn} f\| \\ &\geqq \|P(f - h \operatorname{sgn} f)\| \\ &= \|f - P(h \operatorname{sgn} f)\| \\ &\geqq \|f\| - \|P(h \operatorname{sgn} f)\| \\ &\geqq \|f\| - \|h \operatorname{sgn} f\|\ . \end{aligned}$$

This gives equality throughout so (iii) is valid for $0 \leqq h \leqq |f|$. In addition we have $0 \leqq |f - P(h \operatorname{sgn} f)| = |f| - |P(h \operatorname{sgn} f)|$ μ-almost everywhere, and (ii) also follows for $0 \leqq h \leqq |f|$. We extend immediately to $h \in L_1$ such that $0 \leqq h \leqq n|f|$ for some n, and since linear combinations of such h are dense in $f^{\perp\perp}$ we have (ii) and (iii) for $0 \leqq h \in f^{\perp\perp}$. If $h \in L_1$ and $h \geqq 0$, $(J_f h) \operatorname{sgn} f = h \operatorname{sgn} f$ so (ii) and (iii) are proved.

For (i) take $g \in L_1$ and put $h = (\operatorname{Re}(g \operatorname{sgn} \bar{f}))^+ \operatorname{sgn} f$, by (ii) $Ph \in f^{\perp\perp}$ so $Ph = J_f Ph$. We conclude easily that

$$P(J_f g) = P((g \operatorname{sgn} \bar{f}) \operatorname{sgn} f) = J_f P J_f g$$

and (i) is proved.

Suppose $1 < p < \infty$; then identify the dual of $L_p(X, \Sigma, \mu)$ with $L_q(X, \Sigma, \mu)$ in the usual way $(1/p + 1/q = 1)$. Let P be a contractive projection on L_p. The conjugate operator P^* is defined uniquely on L_q by the equation

$$\int Pf \cdot g d\mu = \int f \cdot P^* g d\mu \quad (f \in L_p, g \in L_q) .$$

Clearly P^* is a contractive projection on L_q.

LEMMA 2.2. [1, Lemma 1]. *Suppose $1 < p < \infty$ and let P be a contractive projection on $L_p(X, \Sigma, \mu)$, then $f \in \mathscr{R}(P)$ if and only if $|f|^{p-1} \operatorname{sgn} \bar{f} \in \mathscr{R}(P^*)$.*

Proof. Suppose $f \in \mathscr{R}(P)$; by Hölder's inequality

$$\begin{aligned} \|f\|_p^p = \int |f|^p d\mu &= \int Pf \cdot |f|^{p-1} \operatorname{sgn} \bar{f} d\mu \\ &= \int f \cdot P^*(|f|^{p-1} \operatorname{sgn} \bar{f}) d\mu \\ &\leqq \|f\|_p \|P^*(|f|^{p-1} \operatorname{sgn} \bar{f})\|_q \\ &\leqq \|f\|_p \| |f|^{p-1} \operatorname{sgn} \bar{f}\|_q \\ &= \|f\|_p \|f\|_p^{p/q} \\ &= \|f\|_p^p . \end{aligned}$$

The conditions for equality in Hölder's inequality lead to

$$P^*(|f|^{p-1} \operatorname{sgn} \bar{f}) = |f|^{p-1} \operatorname{sgn} \bar{f}$$

as required. This proves necessity. Sufficiency follows dually.

We next generalize an argument in Ando's Theorem 1 [1].

LEMMA 2.3. *Suppose $1 < p < \infty$, $p \neq 2$; and let P be a contractive projection on $L_p(X, \Sigma, \mu)$; if $f \in \mathscr{R}(P)$ then,*

(i) $|f| \operatorname{sgn} g \in \mathscr{R}(P) \quad (g \in \mathscr{R}(P))$,

(ii) $PJ_f = J_f P$,

(iii) $P(h \operatorname{sgn} f) = |P(h \operatorname{sgn} f)| \operatorname{sgn} f \quad (0 \leqq h \in L_p)$.

Proof. (i) Suppose first that $p > 2$, let $\lambda \in R$, $0 < |\lambda| < 1$, and let $g \in \mathscr{R}(P)$. By Lemma 2.2,

$$g_\lambda = \lambda^{-1}(|f + \lambda g|^{p-1} \operatorname{sgn} \overline{(f + \lambda g)} - |f|^{p-1} \operatorname{sgn} \bar{f}) \in \mathscr{R}(P^*) .$$

Since $p > 2$,

$$\begin{aligned} g_\lambda &= \lambda^{-1}[(|f + \lambda g|^{p-2} - |f|^{p-2})\overline{(f + \lambda g)} + |f|^{p-2} \cdot \lambda \bar{g}] \\ &= \lambda^{-1}[(|f + \lambda g|^{p-2} - |f|^{p-2})\overline{(f + \lambda g)}] + |f|^{p-2} \bar{g} . \end{aligned}$$

Recall, that for real λ and complex $w, z, d/d\lambda\, |w + \lambda z|\,|_\lambda = \operatorname{Re}[z \operatorname{sgn} \overline{(w + \lambda z)}]$, provided $w + \lambda z \neq 0$. It follows that as $\lambda \to 0$,

$$g_\lambda \longrightarrow (p - 2)|f|^{p-3} \operatorname{Re}(g \operatorname{sgn} \bar{f}) \cdot \bar{f} + |f|^{p-2} \bar{g}$$

at all points of X where $f \neq 0$.

If $2|\lambda g| < |f|$ we have $|f|/2 < |f + \theta \lambda g| < 2|f|$ if $0 < \theta < 1$; and, by the mean value theorem there exists θ, $0 < \theta < 1$ such that

$$\begin{aligned} |g_\lambda| &\leqq (p - 2)|f + \theta\lambda g|^{p-3}|\operatorname{Re}(g \operatorname{sgn} (\overline{f + \theta\lambda g}))|\,|f + \lambda g| + |f|^{p-2}|g| \\ &\leqq (p - 2)2^{|p-3|}|f|^{p-3}|g|\,2\,|f| + |f|^{p-2}|g| \\ &\leqq ((p - 2)2^p + 1)|f|^{p-2}|g| \in L_q . \end{aligned}$$

If $2|\lambda g| \geqq |f|$, $|f + \lambda g| \leqq 3|\lambda g|$ and

$$\begin{aligned} |g_\lambda| &\leqq \lambda^{-1}[(3|\lambda g|)^{p-1} + (2|\lambda g|)^{p-1}] \\ &= (3^{p-1} + 2^{p-1})|g|^{p-1}|\lambda|^{p-2} \\ &\leqq (3^{p-1} + 2^{p-1})|g|^{p-1} \in L_q . \end{aligned}$$

The penultimate line above shows that $g_\lambda \to 0 (\lambda \to 0)$ if $f = 0$.

This shows that g_λ converges to

$$g_0 = (p - 2)|f|^{p-2} \operatorname{sgn} \bar{f} \operatorname{Re}(g \operatorname{sgn} \bar{f}) + |f|^{p-2} \bar{g} ,$$

pointwise almost everywhere on X and that the convergence is dominated by an element of L_q. Hence $\|g_\lambda - g_0\|_q \to 0$ and $g_0 \in \mathscr{R}(P^*)$ because $\mathscr{R}(P^*)$ is closed.

By the same argument, applied to $-ig$, we have, using $\operatorname{Re} - iz = \operatorname{Im} z$,

$$k_0 = (p - 2)|f|^{p-2} \operatorname{sgn} \bar{f} \operatorname{Im}(g \operatorname{sgn} \bar{f}) + i|f|^{p-2} \bar{g} \in \mathscr{R}(P^*) .$$

Now,

$$\begin{aligned} g_0 - ik_0 &= (p - 2)|f|^{p-2} \operatorname{sgn} \bar{f} \cdot \overline{(g \operatorname{sgn} \bar{f})} + 2|f|^{p-2} \bar{g} \\ &= (p - 2)|f|^{p-2} \operatorname{sgn} \bar{f} \cdot \bar{g} \cdot \operatorname{sgn} f + 2|f|^{p-2} \bar{g} \\ &= p|f|^{p-2} \cdot \bar{g} \in \mathscr{R}(P^*) . \end{aligned}$$

(Note that this last is valid in the real case too.)

Using Lemma 2.2 again, we conclude that $||f|^{p-2}\bar{g}|^{q-1}\operatorname{sgn}\overline{|f|^{p-2}\bar{g}} = |f|^{1-(q-1)}|g|^{q-1}\operatorname{sgn} g \in \mathscr{R}(P)$. Set

$$k_n = |f|^{1-(q-1)^n}|g|^{(q-1)^n}\operatorname{sgn} g \qquad (n = 1, 2 \cdots) .$$

We have just shown that $k_1 \in \mathscr{R}(P)$ and the same method, applied inductively, gives $k_n \in \mathscr{R}(P)$ for all n. Since $0 < q - 1 < 1$,

$$|k_n| \leqq \max\{|f|, |g|\} \leqq |f| + |g| \in L_p ,$$

so (k_n) is dominated in L_p. Since $k_n \to |f|\operatorname{sgn} g$ μ-almost everywhere on X, we have $\|k_n - |f|\operatorname{sgn} g\|_p \to 0$ and since $\mathscr{R}(P)$ is closed $|f|\operatorname{sgn} g \in \mathscr{R}(P)$ which proves (i) for $p > 2$.

Suppose $1 < p < 2$; as we have already stated P^* is a contractive projection on L_q, and $q > 2$. By Lemma 2.2, $f_1 = |f|^{p-1}\operatorname{sgn}\bar{f}$ and $g_1 = |g|^{p-1}\operatorname{sgn}\bar{g}$ are in $\mathscr{R}(P^*)$. By our proof above $|f_1|\operatorname{sgn} g_1 = |f|^{p-1}\operatorname{sgn}\bar{g} \in \mathscr{R}(P^*)$, and, by Lemma 2.2 again, $|f|\operatorname{sgn} g \in \mathscr{R}(P)$.

This completes the proof of (i).

For (ii) we have by (i), that $|f|\operatorname{sgn} Pk \in \mathscr{R}(P)$ $(k \in L_p)$. By (i) again,

$$J_f Pk = |Pk|\operatorname{sgn}(|f|\operatorname{sgn} Pk) \in \mathscr{R}(P) .$$

Thus $J_f P = PJ_f P$. Further, since P^* is a contractive projection on L_q, and $|f|^{p-1}\operatorname{sgn}\bar{f} \in \mathscr{R}(P^*)$ we have $J_g P^* = P^* J_g P^*$ with

$$g = |f|^{p-1}\operatorname{sgn}\bar{f} .$$

In addition $J_g = J_f^*$, since J_g and J_f are each multiplication by the same characteristic function. We conclude

$$J_f P = PJ_f P = (P^* J_f^* P^*)^* = (P^* J_g P^*)^* = (J_g P^*)^* = PJ_f ,$$

which is (ii).

(iii) The proof is like the proof of Lemma 2.1(ii). Suppose $0 \leqq h \leqq |f|$. By (i), $|f|\operatorname{sgn} P(h\operatorname{sgn} f) \in \mathscr{R}(P)$, so by Lemma 2.2,

$$|f|^{p-1}\operatorname{sgn}\overline{P(h\operatorname{sgn} f)} \in \mathscr{R}(P^*) .$$

Hence,

$$\begin{aligned}\int |P(h\operatorname{sgn} f)|\,|f|^{p-1}d\mu &= \int P(h\operatorname{sgn} f)\cdot|f|^{p-1}\operatorname{sgn}\overline{P(h\operatorname{sgn} f)}d\mu \\ &= \int h\operatorname{sgn} f\cdot|f|^{p-1}\operatorname{sgn}\overline{P(h\operatorname{sgn} f)}d\mu \\ &\leqq \int h|f|^{p-1}d\mu .\end{aligned}$$

Also $0 \leqq |f - h\operatorname{sgn} f| = |f| - h \leqq |f|$.

Hence,

$$\begin{aligned}
\|f\|_p^p &= \int |P(|f| \operatorname{sgn} f)|\,|f|^{p-1} d\mu \\
&= \int |P(h \operatorname{sgn} f) + P((|f| - h) \operatorname{sgn} f)|\,|f|^{p-1} d\mu \\
&\leqq \int |P(h \operatorname{sgn} f)|\,|f|^{p-1} d\mu + \int |P((|f| - h) \operatorname{sgn} f|\,|f|^{p-1} d\mu \\
&\leqq \int h |f|^{p-1} d\mu + \int (|f| - h)|f|^{p-1} d\mu \\
&= \|f\|_p^p .
\end{aligned}$$

We have equality at each stage and hence, (μ-almost everywhere),

$$|f| = |P(|f| \operatorname{sgn} f)| = |P(h \operatorname{sgn} f)| + |f - P(h \operatorname{sgn} f)| .$$

This proves (iii) for $0 \leqq h \leqq |f|$. The extension to $0 \leqq h \in L_p$ is the same as in the proof of Lemma 2.1(ii) and (iii) so we are done.

3. Contractive projections and conditional expectations. In this section we describe the contractive projections on $L_p(X, \Sigma, \mu)$ ($1 \leqq p < \infty$, $p \neq 2$) in terms of conditional expectation.

We first need the necessary σ-subring.

LEMMA 3.1. *Suppose $1 \leqq p < \infty$, $p \neq 2$, and let P be a contractive projection on $L_p(X, \Sigma, \mu)$. Define Σ_0 to be the set of supports of all functions whose equivalence classes are in $\mathscr{R}(P)$; then*

(i) $PJ_g f = J_g f \quad (f, g \in \mathscr{R}(P))$;

(ii) *Σ_0 is a σ-subring of Σ.*

Proof. (i) By Lemma 2.3(ii), (i) is valid if $p \neq 1$. We give a proof that uses only the identity $J_g P J_g = PJ_g$ valid for $1 \leqq p < \infty$, $p \neq 2$ (Lemma 2.1(i) or 2.3(ii) weakened). Since $f - J_g f \in g^{\perp}$ and $J_g f - PJ_g f \in g^{\perp\perp}$, we have

$$\begin{aligned}
\|P(f - J_g f)\|^p &= \|f - PJ_g f\|^p \\
&= \|f - J_g f\|^p + \|J_g f - PJ_g f\|^p \\
&\geqq \|P(f - J_g f)\|^p + \|J_g f - PJ_g f\|^p .
\end{aligned}$$

Thus $PJ_g f = J_g f$ which is (i).

(ii) By (i), $S(f) \sim S(g) = S(f - J_g f) = S(P(f - J_g f)) \in \Sigma_0$. Thus Σ_0 is closed under differences. If (f_n) is a sequence of nonzero elements in $\mathscr{R}(P)$ such that $S(f_n) \cap S(f_m) = \varnothing (m \neq n)$ then

$$f = \Sigma 2^{-n} \|f_n\|^{-1} f_n \in \mathscr{R}(P)$$

and $S(f) = \bigcup S(f_n)$. This proves (ii).

COROLLARY 3.2. *Let P be a contractive projection on $L_p(X, \Sigma, \mu)$ $(1 \leqq p < \infty, p \neq 2)$. If $h \in \mathscr{R}(P)^{\perp\perp}$ there exists $f \in \mathscr{R}(P)$ such that $h \in f^{\perp\perp}$.*

Proof. By Lemma 1.1 there is a sequence (f_n) in $\mathscr{R}(P)$ such that $h = \lim_{n\to\infty} \chi_{S(f_n)} h$. Choose $f \in \mathscr{R}(P)$ such that $S(f) = \bigcup S(f_n)$, then $h \in f^{\perp\perp}$.

Observe now that if $f \in L_p$ the measure $|f|^p\mu$ restricted to any σ-subring, Σ_0, of Σ, is finite. By the Radon-Nikodym theorem we may define the *conditional expectation operator*, $\mathscr{E}_f = \mathscr{E}(\Sigma_0, |f|^p)$, for the measure $|f|^p\mu$ relative to Σ_0. $\mathscr{E}_f$ is uniquely determined by the equation

$$\int_A h|f|^p d\mu = \int_A (\mathscr{E}_f h)|f|^p d\mu \quad (A \in \Sigma_0)$$

for $h \in L_1(X, \Sigma, |f|^p d\mu)$, and the condition that $\mathscr{E}_f h$ is Σ_0-measurable.

LEMMA 3.3. *Suppose $1 \leqq p < \infty$, $p \neq 2$; let P be a contractive projection on $L_p(X, \Sigma, \mu)$ and let Σ_0 be the σ-subring of Σ, consisting of supports of functions in $\mathscr{R}(P)$. If $M_f = f^{-1}J_f\mathscr{R}(P) = \{f^{-1}J_f g: g \in \mathscr{R}(P)\}$ then $M_f = L_p(S(f), \Sigma_0|S(f), |f|^p\mu)$ where $\Sigma_0|S(f) = \{A \in \Sigma_0: A \subset S(f)\}$ and we make the obvious identification of functions on $S(f)$ and functions on X which vanish off $S(f)$. In addition the map $h \to f^{-1}h$ is an isometric isomorphism between $J_f\mathscr{R}(P)$ and $L_p(S(f), \Sigma_0|S(f), |f|^p\mu)$.*

Proof. Observe that $|f|^p\mu$ is finite on $S(f)$, and that the isometry claim is obviously true. If $A \in \Sigma_0|S(f)$ then $A = S(g)$ for some $g \in \mathscr{R}(P)$. By Lemmas 2.1 and 3.1 (if $p = 1$) or 2.3 (if $p > 1$) we have $J_g f = PJ_g f$ so that $\chi_A = f^{-1}J_g f \in M_f$. Let h be a simple function with respect to $\Sigma_0|S(f)$. Then $h \in M_f$ and $hf \in \mathscr{R}(P)$. In addition

$$\int_{S(f)} |h|^p \cdot |f|^p d\mu = \int_X |hf|^p d\mu .$$

We conclude that

$$M_f \supset L_p(S(f), \Sigma_0|S(f), |f|^p\mu) .$$

Conversely, let $h \in M_f$, then $h \in L_p(S(f), \Sigma|S(f), |f|^p\mu)$ and it is enough to show that h is Σ_0-measurable. Let $g = (\operatorname{Re} h)^+$, then $gf \in L_p(X, \Sigma, \mu)$. By Lemma 2.1(ii) or 2.3(iii)

$$P(gf) = P(|gf| \operatorname{sgn} f) = |P(|gf| \operatorname{sgn} f)| \operatorname{sgn} f$$

so $f^{-1}P(gf) = |f|^{-1}|P(|gf| \operatorname{sgn} f)| \in M_f$. It follows that

$$\operatorname{Re} h = f^{-1}P((\operatorname{Re} h)^+ f) - f^{-1}P((\operatorname{Re} h)^- f) \in M_f .$$

Since each of these functions is nonnegative it is sufficient to consider $0 \leqq h \in M_f$. Suppose $\alpha > 0$ and put $k = h \vee \alpha\chi_{S(f)}$. Arguing as above, we have $f^{-1}P(kf) \geqq h$ and $f^{-1}P(kf) \geqq \alpha\chi_{S(f)}$ so that $f^{-1}P(kf) \geqq k \geqq 0$. Since P is contractive we have

$$\begin{aligned} \|kf\|^p \geqq \|P(kf)\|^p &= \|P(kf) - kf + kf\|^p \\ &\geqq \|P(kf) - kf\|^p + \|kf|^p . \end{aligned}$$

This gives $P(kf) = kf$, so that $k \in M_f$. This shows, incidently, that M_f is a lattice. For our purpose, however, we have

$$\begin{aligned} \{t \in S(f): h(t) > \alpha\} &= \{t \in S(f): (k - \alpha\chi_{S(f)})(t) \neq 0\} \\ &= S(kf - \alpha f) \in \Sigma_0 . \end{aligned}$$

Thus M_f consists of Σ_0-measurable functions and we are done.

THEOREM 3.4. *Suppose $1 \leqq p < \infty$, $p \neq 2$ and that P is a contractive projection on $L_p(X, \Sigma, \mu)$. If $f \in \mathscr{R}(P)$ and $h \in f^{\perp\perp}$ then*

$$Ph = f\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) .$$

Proof. Since $f^{-1}Ph \in M_f$ we know $f^{-1}Ph$ is Σ_0-measurable. Thus we have only to show

$$\int_A f^{-1}Ph|f|^p d\mu = \int_A hf^{-1}\cdot|f|^p d\mu \quad (A \in \Sigma_0) .$$

Choose $g \in \mathscr{R}(P)$ such that $A = S(g)$. By Lemma 3.1(i), $u = J_g f \in \mathscr{R}(P)$.

Suppose $p = 1$ and $0 \leqq k \in L_1$. By Lemma 2.1(ii) and (iii),

$$\begin{aligned} \int_A k \operatorname{sgn} f\cdot f^{-1}|f|d\mu = \int_{A\cap S(f)} kd\mu &= \|J_u k\| = \|P(k \operatorname{sgn} u)\| \\ &= \|\,|P(J_g k \operatorname{sgn} f)|\operatorname{sgn} f\| \\ &= \int_A f^{-1}P(J_g k \operatorname{sgn} f)\cdot|f|d\mu . \end{aligned}$$

Putting $v = f - u = f - J_g f \in \mathscr{R}(P)$, we have, by Lemma 2.1(i),

$$P(k \operatorname{sgn} f) = J_u P(J_u k \operatorname{sgn} f) + J_v P(J_v k \operatorname{sgn} f) .$$

Hence

$$\int_A f^{-1}P(J_g k \operatorname{sgn} f)\cdot|f|d\mu = \int_A f^{-1}P(k \operatorname{sgn} f)\cdot|f|d\mu .$$

We conclude that

$$\int_A hf^{-1}\cdot|f|d\mu = \int_A f^{-1}Ph\cdot|f|d\mu$$

for all $h \in f^{\perp\perp}$ and all $A \in \Sigma_0$ so we are finished for $p = 1$.

If $p > 1$ we have $PJ_g = J_gP$ by Lemma 2.3(ii) and $|f|^{p-1}\operatorname{sgn}\bar{f} \in \mathscr{R}(P^*)$ by Lemma 2.2. Hence,

$$\begin{aligned}\int_A hf^{-1}\cdot|f|^p d\mu &= \int_X J_gh\cdot|f|^{p-1}\operatorname{sgn}\bar{f}d\mu \\ &= \int_X J_gh\cdot P^*(|f|^{p-1}\operatorname{sgn}\bar{f})d\mu \\ &= \int_X PJ_gh\cdot|f|^{p-1}\operatorname{sgn}\bar{f}d\mu \\ &= \int_X J_gPh\cdot f^{-1}|f|^p d\mu \\ &= \int_A f^{-1}Ph\cdot|f|^p d\mu \quad (A \in \Sigma_0)\,.\end{aligned}$$

Thus

$$Ph = f^{-1}\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) \quad (h \in f^{\perp\perp})$$

as claimed.

Our theorem has useful consequences.

THEOREM 3.5. *Suppose* $1 \leqq p < \infty$, $p \neq 2$, *let* P *be a contractive projection on* $L_p(X, \Sigma, \mu)$ *and let* J *be the band projection on* $\mathscr{R}(P)^{\perp\perp}$; *then* PJ *is the unique contractive projection on* L_p *which satisfies* $\mathscr{R}(PJ) = \mathscr{R}(P)$ *and* $PJ\mathscr{R}(P)^{\perp} = \{0\}$. *If* $p \neq 1$, $P = PJ$ *so* P *is uniquely determined by its range. If* $p = 1$, *and* A *is a linear contraction on* L_1 *which satisfies* $PA = A$ *and* $AJ = 0$, *then* $PJ + A$ *is a contractive projection on* L_1 *with the same range as* P.

Proof. Let Q be a contractive projection on L_p such that $\mathscr{R}(Q) = \mathscr{R}(P)$ and $Q\mathscr{R}(P)^{\perp} = \{0\}$. Then $Q = QJ$ and if $h \in L_p$ there exists, by Corollary 3.2, $f \in \mathscr{R}(P) = \mathscr{R}(Q)$ such that $Jh = J_fh$. By Theorem 3.4, $Qh = QJh = f^{-1}\mathscr{E}(\Sigma_0, |f|^p)(Jh\cdot f^{-1}) = PJh$. Thus $Q = PJ$. (It is clear that PJ satisfies the stated conditions.)

If $p \neq 1$ take h, f as above and put $u = Ph - PJh = Ph - PJ_fh = Ph - J_fPh$, by Lemma 2.3(ii). Since band projections commute and $u \in \mathscr{R}(P) \cap f^{\perp}$, $J_uh = J_uJh = J_uJ_fh = 0$. By Lemma 2.3(ii) again,

$$u = J_uu = J_uPh - J_uPJ_fh = PJ_uh - J_uJ_fPh = 0 - 0 = 0\,.$$

Hence $P = PJ$ as required.

If $p = 1$, $PA = A$, and $AJ = 0$, we have $AP = AJP = 0$ and $A^2 =$

$APA = 0$. Also $(PJ + A)^2 = PJPJ + PJA + APJ + A^2 = PPJ + PJPA + 0 + 0 = PJ + A$. Thus $PJ + A$ is a projection. Observe that

$$\begin{aligned}\mathscr{R}(PJ + A) &= \mathscr{R}(PJ + PA) \subset \mathscr{R}(P) = \mathscr{R}(PJP + AP)\\ &= \mathscr{R}((PJ + A)P) \subset \mathscr{R}(PJ + A)\,.\end{aligned}$$

It remains to show that if A is contractive, $PJ + A$ is contractive. If $h \in L_1$,

$$\begin{aligned}\|(PJ + A)h\|_1 &= \|PJh + A(h - Jh)\|_1\\ &\leqq \|PJh\|_1 + \|A(h - Jh)\|_1\\ &\leqq \|Jh\|_1 + \|h - Jh\|_1\\ &= \|Jh + h - Jh\|_1\\ &= \|h\|_1\,.\end{aligned}$$

4. **Contractive projections and isometric isomorphisms.** In this section we prove the equivalence of various conditions on a subspace of L_p so that it is the range of a contractive projection.

Let $\mathscr{S}(X, \Sigma)$ denote the set of Σ-measurable functions h such that $S(h)$ is σ-finite. By a *multiplication operator* on $\mathscr{S}(X, \Sigma)$ we mean a map $h \to kh$ defined for functions h in some subset of $\mathscr{S}(X, \Sigma)$ and some fixed Σ-measurable function k. If k satisfies $|k| = 1$ on $S(k)$ we will call k a *unitary multiplication.*

A multiplication operator on $\mathscr{S}(X, \Sigma)$ preserves equality almost everywhere and hence induces a multiplication operator on each $L_p(X, \Sigma, \mu)$ into $\mathscr{S}(X, \Sigma)$ modulo null functions ($1 \leqq p < \infty$). Further, k_1 and k_2 will induce the same such multiplication operator on L_p if k_1 and k_2 agree locally almost everywhere.

Suppose that $\mathscr{K}$ is a set of Σ-measurable functions such that if $k_1, k_2 \in \mathscr{K}$ and $k_1 \neq k_2$, $\mu(S(k_1) \cap S(k_2)) = 0$. If $f \in \mathscr{S}(X, \Sigma)$ then, because $S(f)$ has σ-finite measure, $S(f)$ meets at most countably many $S(k)$, with $k \in \mathscr{K}$, in a set of positive measure. Enumerate these as (k_n), then there is a unique set $N \in \Sigma$ such that, $N \subset S(f)$ and each $t \in S(f) \sim N$ lies in at most one set $S(k_n)$. (In fact $N = \bigcup_{1 \leqq n < m < \infty} (S(k_n) \cap S(k_m))$.) On $S(f) \sim N$ the series $\sum_{n=1}^{\infty} f(t)k_n(t)$ has at most one nonzero term. Thus $\mathscr{K}$ determines a map $U_{\mathscr{K}}: \mathscr{S}(X, \Sigma) \to \mathscr{S}(X, \Sigma)$ by taking, for f as above, $U_{\mathscr{K}}f(t) = \sum_{n=1}^{\infty} f(t)k_n(t)$ for $t \in S(f) \sim N$ and $U_{\mathscr{K}}f(t) = 0$ elsewhere. We call $U_{\mathscr{K}}$ the *direct sum* of the (disjoint) multiplication operators induced by the elements of $\mathscr{K}$. If $U_{\mathscr{K}}$ maps L_p to $L_p (1 \leqq p < \infty)$ it is not hard to check that the net of finite sums of the multiplication operators in $\mathscr{K}$ is strongly convergent to $U_{\mathscr{K}}$.

We can now state our theorem. The equivalence of (i) and (ii) generalizes [1, Theorem 4] and extends [10, Theorem 6].

THEOREM 4.1. *Suppose* $1 \leqq p < \infty$ *and* $p \neq 2$ *and let* M *be a subspace of* $L_p(X, \Sigma, \mu)$. *The following conditions on* M *are equivalent.*

(i) M *is the range of a contractive projection on* L_p.

(ii) *There is a measure space* (Ω, Ξ, λ) *such that* M *is isometrically isomorphic to* $L_p(\Omega, \Xi, \lambda)$.

(iii) *There is a direct sum of unitary multiplication operators* $U: L_p(X, \Sigma, \mu) \rightarrow L_p(X, \Sigma, \mu)$ *such that* U *is an isometry and* UM *is a closed vector sublattice of* $L_p(X, \Sigma, \mu)$.

Furthermore, in (ii) *we can always choose* $\Omega = X$, Ξ *a* σ*-subring of* Σ, λ *absolutely continuous with respect to* μ, *and the isometry a direct sum of multiplication operators.*

If μ *is* σ*-finite the direct sums of multiplication operators can be taken to be ordinary multiplications.*

Proof. Assume (i). By Zorn's lemma there is a maximal subset $\mathscr{K}$ of M consisting of functions $f \in M$, such that $\mu(S(f_1) \cap S(f_2)) = 0$ if $f_1 \neq f_2$. If $g \in M$, $S(g)$ is σ-finite and there is countable subset $\{f_n\}$ of $\mathscr{K}$ such that if $f \in \mathscr{K} \sim \{f_n\}$, $\mu(S(f) \cap S(g)) = 0$. By Lemma 3.1, Σ_0 is a σ-ring so, there exists $h \in M$ such that $S(h) = S(g) \sim \bigcup S(f_n)$ and by maximality of $\mathscr{K}$, $h = 0$. Define a measure λ on Σ_0 by $\lambda A = \sum_{f \in \mathscr{K}} \int_A |f|^p d\mu$. This definition is meaningful since A has σ-finite μ-measure and at most countably many of the integrals are nonzero. For $f \in \mathscr{K}$ define f^{-1} by

$$f^{-1}(t) = \begin{cases} 1/f(t) & t \in S(f) \\ 0 & t \notin S(f) \,, \end{cases}$$

and let V be the direct sum of the multiplications $f^{-1}(f \in \mathscr{K})$. By Lemma 3.3 $J_f h \rightarrow f^{-1}h (h \in M)$ is an isometric isomorphism of $J_f M$ with $L_p(S(f), \Sigma_0 | S(f), |f|^p \mu)$. It is routine to check that V is an isometric isomorphism of M with $L_p(X, \Sigma_0, \lambda)$. ($M$ is the direct sum of its subspaces $J_f M (f \in \mathscr{K})$ and similarly for the L_p-spaces.)

It μ is σ-finite $\mathscr{K}$ will be countable, say $\mathscr{K} = \{f_n\}$ and we can find $f \in M$ such that $S(f) = \bigcup S(f_n)$. Then Σ_0 consists entirely of subsets of $S(f)$ and sets of measure zero so that $M_f = L_p(X, \Sigma_0, |f|^p \mu)$, $J_f M = M$, and V can be multiplication by f^{-1}.

Assume (ii) and let $T: L_p(\Omega, \Xi, \lambda) \rightarrow L_p(X, \Sigma, \mu)$ be a linear isometry with range M. Suppose $a, b \in L_p(\Omega, \Xi, \lambda)$ and $|a| \wedge |b| = 0$, we claim that $|Ta| \wedge |Tb| = 0$. This is essentially proved by Lamperti [6]. Since $|a| \wedge |b| = 0$, $||a + b||^p + ||a - b||^p = 2||a||^p + 2||b||^p$. Since T is an isometry, $|| Ta + Tb||^p + || Ta - Tb||^p = 2|| Ta||^p + 2|| Tb||^p$. Since $p \neq 2$, the equality condition for Clarkson's inequality [6, Corollary 2.1] shows that $|Ta| \wedge |Tb| = 0$.

Take a maximal subset of Ξ consisting of sets of nonzero finite

λ-measure which intersect pairwise in sets of λ-measure zero and let $\mathscr{K}$ be the corresponding set of characteristic functions. Let $a \in \mathscr{K}$ and suppose $B \in \Xi$ and $B \subset S(a)$. Write $b = \chi_B$, then $T(a - b)$, Tb are disjoint in M so we have $Tb = |Tb| \operatorname{sgn} Ta$. This extends to nonnegative simple functions b in $a^{\perp\perp}$ and then to all nonnegative $b \in a^{\perp\perp}$. Define $U: L_p(X, \Sigma, \mu) \to L_p(X, \Sigma, \mu)$ to be the direct sum of the unitary multiplications $\operatorname{sgn} \overline{Ta} (a \in \mathscr{K})$. It is easy to see that U is an isometry of M such that UT is positive and hence $UM = UT\, L_p(\Omega, \Xi, \lambda)$ is a closed vector sublattice of $L_p(X, \Sigma, \mu)$ (compare the proof in Lemma 3.3 where we showed that functions in M_f were Σ_0-measurable).

Assume (iii) and let Σ_0 be the set of supports of functions (whose equivalence classes are) in M. Then Σ_0 is a σ-subring of Σ. (If (f_n) is a sequence in M, $S(f_n) = S(Uf_n) = S(|Uf_n|)$ so

$$\bigcup S(f_n) = S(U^{-1}\Sigma 2^{-n} \|f_n\|^{-1} |Uf_n|)\ .$$

If $f, g \in M$, $J_g = J_{Ug}$; $J_g|Uf| = \lim |Uf| \wedge n|Ug| \in UM$ and $S(f) \sim S(g) = S(U^{-1}(|Uf| - J_g|Uf|))$.) Let $f, g \in UM$ and suppose f is real, $g \geqq 0$ and $f \in g^{\perp\perp}$, then $\{t \in X: (f/g)(t) > \alpha\} = S((f - \alpha g)^+) \in \Sigma_0$. Thus f/g is Σ_0-measurable. This extends to all $f \in UM \cap g^{\perp\perp}$ and hence $J_g f/g$ is Σ_0-measurable if $f, g \in UM$ and $g \geqq 0$. This now extends to all $f, g \in UM$ and, since $U^{-1}J_g f/U^{-1}g = J_g f/g$, we have f/g, Σ_0-measurable for $f, g \in M$ and $f \in g^{\perp\perp}$. It follows that M is the set of all elements in $L_p(X, \Sigma, \mu)$ which can be written in the form hf with h, Σ_0-measurable and $f \in M$. (If $h = \chi_{S(g)}$ with $g \in M$, $hf = J_g f = U^{-1}J_{Ug}Uf \in U^{-1}(UM) = M$.)

Let J be the band projection on $M^{\perp\perp}$, let $h \in L_p(X, \Sigma, \mu)$, choose $f \in M$ such that $Jh = J_f h$, (such an f exists by the arguments used in Corollary 3.2) and define

$$Ph = f\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1})\ .$$

Then $Ph \in M$ and this definition is independent of the choice of f in M such that $h \in f^{\perp\perp}$. To see this suppose $g \in M$ and $h \in g^{\perp\perp}$. Then h is zero outside $S(f) \cap S(g) \in \Sigma_0$ and so is $\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1})$, μ-almost everywhere. Let $B = S(f) \cap S(g)$, then $f_1 = \chi_B f \in M$ and

$$\int_A hf^{-1}|f|^p d\mu = \int_{A\cap B} hf^{-1}|f|^p d\mu = \int_A hf_1^{-1}|f_1|^p d\mu \quad (A \in \Sigma_0)\ ,$$

so that $f\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) = f_1\mathscr{E}(\Sigma_0, |f_1|^p)(hf_1^{-1})$. Thus we may assume $S(f) = S(g)$. Now

$$g^{-1}f\mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) \in L_1(X, \Sigma_0, |g|^p\mu)\ ,$$

so we have, for $A \in \Sigma_0$,

$$\int_A g^{-1} f \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) |g|^p d\mu$$
$$= \int_A g^{-1} f |f^{-1} g|^p \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) |f|^p d\mu .$$

Because $g^{-1}f$ and $f^{-1}g$ are Σ_0-measurable and the integrals are finite, the second integral is

$$\int_A g^{-1} f |f^{-1} g|^p h f^{-1} |f|^p d\mu = \int_A h g^{-1} |g|^p d\mu .$$

Thus

$$f \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) = g \mathscr{E}(\Sigma_0, |g|^p)(hg^{-1})$$

and our definition of Ph is unambiguous. If $h_1, h_2 \in L_p$ we can take $f \in M$ such that $Jh_1 = J_f h_1$ and $Jh_2 = J_f h_2$. Thus P is linear. Since $f^{-1}Ph = \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1})$ we see $P^2 = P$. Finally, if $p > 1$, write $u = \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1})$, we have

$$\|Ph\|_p^p = \int |u|^{p-1} \operatorname{sgn} \bar{u} \cdot \mathscr{E}(\Sigma_0, |f|^p)(hf^{-1}) |f|^p d\mu .$$

Since u is Σ_0-measurable, this is

$$\begin{aligned} \int |u|^{p-1} \operatorname{sgn} \bar{u} \cdot h f^{-1} |f|^p d\mu &= \int |Ph|^{p-1} \operatorname{sgn} \bar{f}\bar{u} \cdot h d\mu \\ &\leqq \| |Ph|^{p-1} \|_q \|h\|_p \\ &= \|Ph\|_p^{p/q} \|h\|_p . \end{aligned}$$

(We used Hölder's inequality and q for the conjugate index to p.) We conclude that $\|Ph\|_p \leqq \|h\|_p$.

Since $Ph = h (h \in M)$ we have shown that M is the range of the contractive projection P.

REMARK 4.2. The results (iii) implies (i) (with the same proof) and (i) is equivalent to (ii) are valid if $p = 2$; in fact (i) and (ii) are equivalent for any Hilbert space. If we assume the projection P, is positive as well as contractive the proof in Lemma 3.3 that M_f is a lattice shows $\mathscr{R}(P)$ is a sublattice of L_2 and Theorem 4.1 is valid for L_2 with the projection and the isometry both required to be positive and in (iii) M required to be a closed vector sublattice. We use this remark in our next result.

COROLLARY 4.3. *If M is a closed vector sublattice of L_p $(1 \leqq p < \infty)$ then M is the range of a positive contractive projection.*

Proof. Clearly M satisfies condition (iii) with $U = I$. In the definition of Ph we may always choose a positive $f \in M$ such that $h \in f^{\perp\perp}$. Positivity of P follows from positivity of conditional expectation.

REMARK 4.4. In the introduction we referred to Rao's paper [8] and claimed that its treatment of contractive projections contained errors. In particular, his Theorem II. 2.7 asserts that if M is the range of a contractive projection P on a Banach function space $L^\rho(\Sigma)$ there is, under suitable conditions, a unitary multiplication U such that UPU^{-1} is a positive contractive projection.

The conditions are all satisfied if M is the subspace of $l^2(3) = \boldsymbol{C}^3$ spanned by $(1, 1, 1)$ and $(1, 2, -3)$. Rao's theorem now claims the existence of a unitary multiplication, say by $u = (\lambda_1, \lambda_2, \lambda_3)$, such that uM is a vector sublattice of $\boldsymbol{C}^3$. This is impossible, as we show. First, uM contains the elements $(0, \lambda_2, -4\lambda_3)$, $(\lambda_1, 0, 5\lambda_3)$, and $(4\lambda_1, 5\lambda_2, 0)$. If $\mathrm{Re}\,\lambda_2\bar{\lambda}_3 = 0$ we have $\lambda_2\lambda_3^{-1} = \lambda_2\bar{\lambda}_3 = \pm i$ and uM contains $\mathrm{Im}\,(0, \lambda_2\bar{\lambda}_3, -4) = (0, \pm 1, 0)$; so that $(0, 1, 0) \in uM$, and $uM = \boldsymbol{C}^3$. If all $\mathrm{Re}\,\lambda_i\bar{\lambda}_j \neq 0$ $(i \neq j)$, then uM contains $\mathrm{Re}\,(0, 1, -4\lambda_3\bar{\lambda}_2)$ and $\mathrm{Re}\,(1, 0, 5\lambda_3\bar{\lambda}_1)$; hence, taking a multiple of their infimum, $(0, 0, 1) \in uM$ and again $uM = \boldsymbol{C}^3$.

Exactly the same counterexample vitiates the proof of Rao's Theorem II. 2.8 see p. 177 lines -15 to -11.

The error in both cases seems to be the reduction of the general case of $L^\rho(\Sigma)$ to the L_1 situation. Vital to this reduction, but invalid, is the assertion that if $L^\rho(\Sigma) \subset L^1(\Sigma, G)$ and $\|\cdot\|_{1,G} \leqq \rho(\cdot)$ then a contraction on $L^\rho(\Sigma)$ for the ρ-norm can be extended to the closure of $L^\rho(\Sigma)$ in $L^1(\Sigma, G)$ with the $1, G$-norm and that the extension is contractive for the $1, G$-norm.

5. The theorem of Lindenstrauss, Pelczynski, and Zippin. We begin by recalling some definitions.

If E, F are isomorphic Banach spaces, $d(E, F) = \inf\{\|L\|\,\|L^{-1}\|: L$ is a linear isomorphism between E and $F\}$.

A Banach space E is an $\mathscr{L}_{p,\lambda}$ space (for $1 \leqq p \leqq \infty$ and $\lambda \geqq 1$) if for each finite dimensional subspace F of E there is a finite dimensional subspace G of E such that $F \subset G$ and $d(G, l_p(\dim G)) \leqq \lambda$.

We shall say that a Banach space E is a *Z_p-space* (for $1 \leqq p \leqq \infty$) if there exists a set $\mathscr{Z}$ of finite dimensional subspaces of E such that:

(i) $\mathscr{Z}$ is upwards directed by set inclusion;

(ii) $\mathrm{cl} \cup \mathscr{Z} = E$;

(iii) each $F \in \mathscr{Z}$ is linearly isometric to $l_p(\dim F)$.

Our definitions apply, of course, over the real or complex number fields.

We now state the theorem of Lindenstrauss-Pelczynski-Zippin, [5], [7], [12].

THEOREM 5.1. *Let E be a Banach space and suppose $1 \leqq p < \infty$, then the following are equivalent.*

(1) *There is a measure (X, Σ, μ) such that E is isometrically isomorphic to $L_p(X, \Sigma, \mu)$.*

(2) *E is a Z_p space.*

(3) *E is an $\mathscr{L}_{p,\lambda}$-space for all $\lambda > 1$.*

As outlined in the introduction we discuss some details of the proof for the complex case.

Observe first that (3) is a trivial consequence of (1). Simply identify E with $L_p(X, \Sigma, \mu)$ and take for $\mathscr{X}$ the subspaces spanned by finite sets of (pth power)-integrable characteristic functions.

The proof that (3) *implies* (2). This result is certainly part of the folklore. It can be obtained quite efficiently as follows.

LEMMA 5.2. *Let $x_1, \cdots, x_n$ be n linearly independent elements of a normed space E then there exists $\varepsilon > 0$ such that if $y_i \in E$, and $\|x_i - y_i\| < \varepsilon (i = 1, 2, \cdots, n)$ then $\{y_1, \cdots, y_n\}$ is a linearly independent subset of E.*

Proof. (This is standard but our proof may be novel.) Let K denote the scalar field and S the unit sphere in K^n, $S = \{\lambda \subset K^n : \|\lambda\| = 1\}$. The map $g: S \times E^n \rightarrow E$ defined by $g((\lambda_1, \cdots, \lambda_n), (y_1, \cdots, y_n)) = \lambda_1 y_1 + \cdots + \lambda_n y_n$ is continuous. By linear independence, the compact set $S \times (x_1, \cdots, x_n)$ does not meet the closed set $g^{-1}(0)$. Hence there are open neighborhoods U_i of x_i, $i = 1, \cdots, n$, such that $(S \times U_1 \times \cdots \times U_n) \cap g^{-1}(0) = \emptyset$. If $y_i \in U_i (i = 1, \cdots, u)$ it follows that $\{y_1, \cdots, y_n\}$ is linearly independent.

LEMMA 5.3. *Let E be a Z_p-space, then E is an $\mathscr{L}_{p,\lambda}$-space for every $\lambda > 1$.*

Proof. Let F be a finite dimensional subspace of E. Let $\{x_1, \cdots, x_n\}$ be a basis for F, such that $\|x_i\| = 1 (i = 1, \cdots, n)$. Let $x_1^*, \cdots, x_n^* \in E^*$ be such that $x_i^*(x_j) = \delta_{ij}$, and let $M = \sum_{i=1}^n \|x_i^*\|$. Choose $\varepsilon > 0$ such that $M\varepsilon < 1$ and $\|x_i - y_i\| < \varepsilon$ for $i = 1, \cdots, n$ implies that $\{y_1, \cdots, y_n\}$ is linearly independent. By the Z_p-hypothesis there is a finite dimensional subspace H of E and points $y_1, \cdots, y_n$ in H, such that H is isometrically isomorphic to $l_p(\dim H)$, and $\|x_i - y_i\| < \varepsilon (i = 1, \cdots, n)$. Then $\{y_1, \cdots, y_n\}$ is a linearly independent subset of

H. If

$$\sum_{i=1}^{n} \alpha_i y_i \in \bigcap_{i=1}^{n} \mathscr{N}(x_i^*) ,$$

then

$$\begin{aligned}\sum_{j=1}^{n} |\alpha_j| &= \sum_{j=1}^{n} \left| x_j^*\Big(\sum_{i=1}^{n} \alpha_i x_i\Big)\right| \\ &= \sum_{j=1}^{n} \left| x_j^*\Big(\sum_{i=1}^{n} \alpha_i (x_i - y_i)\Big)\right| \\ &\leqq \sum_{j=1}^{n} ||x_j^*|| \Big(\sum_{i=1}^{n} |\alpha_i| \varepsilon\Big) \\ &= M\varepsilon \sum_{i=1}^{n} |\alpha_i| .\end{aligned}$$

Since $M\varepsilon < 1$ we conclude that $\alpha_i = 0$ for each i. Thus we can extend $y_1, \cdots, y_n$ to a basis, say $y_1, \cdots, y_n, x_{n+1}, \cdots, x_p$, of H with the property that $\{x_{n+1}, \cdots, x_p\} \subset \bigcap_{i=1}^{n} \mathscr{N}(x_i^*)$.

Let G be the subspace of E spanned by $x_1, \cdots, x_n, x_{n+1}, \cdots, x_p$. Then $F \subset G$. If $y = \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=n+1}^{p} \alpha_i x_i \in H$ define $Ty = \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=n+1}^{p} \alpha_i x_i \in G$. We have

$$\begin{aligned}||y - Ty|| &= \left|\left| \sum_{i=1}^{n} \alpha_i (y_i - x_i)\right|\right| \leqq \varepsilon \sum_{i=1}^{n} |\alpha_i| \\ &= \varepsilon \sum_{j=1}^{n} |x_j^*(Ty)| \\ &\leqq M\varepsilon \, || Ty ||.\end{aligned}$$

This gives $(1 - M\varepsilon)|| Ty || \leqq ||y|| \leqq (1 + M\varepsilon)|| Ty ||(y \in H)$; so that T is an isomorphism between F and H such that $|| T|| \, || T^{-1}|| \leqq (1+M\varepsilon)/(1-M\varepsilon)$. If $\lambda > 1$ we can choose ε such that $(1+M\varepsilon)/(1-M\varepsilon) < \lambda$. Thus E is an $\mathscr{L}_{p,\lambda}$-space for all $\lambda > 1$.

The proof that (2) *implies* (1). Here the plan is first to embed E, isometrically, in an L_p-space, and then to use the theory of contractive projections of L_p-spaces.

This is carried out in detail for the real reparable case in [7] and for the real nonseparable case in [5]. The generalizations to cover the complex case are mostly obvious. For $1 < p < \infty$ our Theorem 4.1 is used. For $p = 1$, it follows as in the real case that E^* is a $\mathscr{P}_1$ space whence by the complex version of Grothendieck's theorem [9] E is an $L_1(\mu)$ space.

There is an aspect of the construction which needs a little elaboration. At one stage of the proof we have a complex vector space, say V, consisting of complex valued functions on a set U. V is a vector sublattice of the space of all complex functions on U. There is a seminorm π on V such that $\pi(f) \leqq \pi(g)$ whenever $|f| \leqq |g|$, and $\pi(f+g)^p = \pi(f)^p + \pi(g)^p$ whenever $|f| \wedge |g| = 0$. We then need to embed the quotient V/N, where $N = \{f \in V : \pi(f) = 0\}$, isometrically in a concrete, complex, L_p-space. For this, let V_R and N_R denote the spaces of real-valued functions in V and N respectively. The quotient V_R/N_R with the norm induced by π is then linearly and lattice isomorphic, and isometric, to a vector sublattice of real $L_p(X, \Sigma, \mu)$ just as in [7]. Let h_1 denote the composition of the quotient map $U_R \rightarrow V_R/N_R$ and the isometric isomorphism into real $L_p(X, \Sigma, \mu)$. Then h_1 is a linear and lattice homomorphism and $\|h_1 f\| = \pi(f) (f \in V_R)$. We construct the required embedding of V/N into complex $L_p(X, \Sigma, \mu)$ by defining

$$h(f+N) = h_1(\operatorname{Re} f) + i h_1(\operatorname{Im} f) .$$

Then h is clearly well defined. To verify that h is an isometry we need the next lemma.

LEMMA 5.4. *The map h constructed above satisfies $h|f| = |hf|$, $(f \in V)$.*

Proof. For any real θ $|f| \geqq \operatorname{Re}(e^{i\theta} f)$ so

$$h|f| = h_1|f| \geqq h_1(\operatorname{Re} e^{i\theta} f) = \operatorname{Re} h(e^{i\theta} f) = \operatorname{Re} e^{i\theta} hf .$$

Hence $h|f| \geqq |hf|$. For the converse, let ω be a complex nth root of unity and observe that for any complex z

$$\max \{\operatorname{Re} \omega^r z : r = 1, 2, \cdots, n\} \geqq \cos(\pi/n)|z| .$$

Hence,

$$\begin{aligned} \cos(\pi/n) h|f| &\leqq h(\sup\{(\operatorname{Re} \omega^r f) : r = 1, \cdots, n\}) \\ &= \sup\{\operatorname{Re} \omega^r hf : r = 1, \cdots, n\} \\ &\leqq |hf|. \end{aligned}$$

Letting $n \rightarrow \infty$ we have $h|f| = |hf|$ as required.

This completes our discussion of the proof of Theorem 5.1. We add a comment. It seems that a more elementary proof that a space which is an $\mathscr{L}_{p,\lambda}$-space for all $\lambda > 1$, is an $L^p(\mu)$ space, should be possible. Certainly the result should not depend on the entire theory of contractive projections for such spaces. Indeed if $p = 2$ the $\mathscr{L}_{2,\lambda}$ condition already implies the parallelogram law and this makes the space a Hilbert space. For $p \neq 2$ we can see that the Clarkson inequalities are valid and these with enough finite dimensional l_p-subspaces might give a more elementary proof.

6. Appendix. We prove two technical results used in [1], [10]. The first is also an extension of that in [1].

LEMMA 6.1. [1]. *Suppose* $0 < p < \infty$ *and let* M *be a closed subspace of* $L_p(X, \Sigma, \mu)$. *If* (f_n) *is a sequence in* M, *then there exists* $f \in M$ *such that* $S(f) = \bigcup_{n=1}^{\infty} S(f_n)$. *In particular if* μ *is finite or* M *is separable there exists* $f \in M$ *such that* $J_f = J_{M^{\perp\perp}}$; *that is,* f *is a function in* M *of maximum support.*

Proof. If $f, g \in L_p$ and α is a scalar, the zero sets $\{t \in X: (f + \alpha g)(t) = 0\}$ have disjoint intersection with $S(f) \cup S(g)$ for differing values of α. Since $S(f) \cup S(g)$ is σ-finite, $\mu(S(f) \cup S(g) \sim S(f + \alpha g)) = 0$ except, perhaps for countably many values of α.

Assume, as we may, that $\int |f_n|^p = 1$ for all n. We define, inductively, two sequences (α_n), (ε_n) of positive real numbers such that, if we write $g_n = \alpha_1 f_1 + \cdots + \alpha_n f_n$, $A_n = \{t \in X: |g_n(t)| \leqq \varepsilon_n\}$, and $B_n = \{t \in X: |\alpha_{n+1} f_{n+1}(t)| \geqq \varepsilon_n/2\}$, then

(i) $\alpha_{n+1} < 2^{-n/p}$ and $\varepsilon_{n+1} < \varepsilon_n/2$;

(ii) $\mu(S(g_n) \cup S(f_{n+1}) \sim S(g_{n+1})) = 0$;

(iii) $\displaystyle\int_{A_n \cup B_n} |f_i|^p d\mu < 2^{-n} \qquad (i = 1, 2, \cdots, n)$.

Start with $\alpha_1 = 1$. Suppose $\alpha_1, \cdots, \alpha_n; \varepsilon_1, \cdots, \varepsilon_{n-1}$ have been chosen. Note that $\mu(S(f_i) \sim S(g_n)) = 0 (i = 1, \cdots, n)$ so if $C_\varepsilon = \{t \in X: |g_n(t)| \leqq \varepsilon\}$, $\int_{C_\varepsilon} |f_i|^p d\mu \to 0 (\varepsilon \to 0+)$ for $i = 1, \cdots, n$. Also if

$$D_\eta = \{t \in X: |f_{n+1}(t) \geqq \eta\}, \int_{D_\eta} |f_i|^p d\mu \to 0 (\eta \to \infty) \text{ for } i = 1, \cdots, n .$$

Thus we choose ε_n such that $0 < \varepsilon_n < \varepsilon_{n-1}/2$, and $\int_{A_n} |f_i|^p d\mu < 2^{-n-1} (i = 1, 2, \cdots, n)$; then choose η such that $\int_{D_\eta} |f_i|^p d\mu < 2^{-n-1} (i = 1, 2, \cdots, n)$, and α_{n+1} such that $0 < \alpha_{n+1} < 2^{-n/p}$, (ii) is satisfied, and $\alpha_{n+1}\eta < \varepsilon_n/2$. Since $B_n \subset D_\eta$ we also have (iii) satisfied.

By (i) (g_n) converges in L_p to an element $f \in M$, and $S(f) \subset \bigcup S(f_n)$. Let $E = \limsup (A_n \cup B_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_n \cup B_n)$. Fix i and let $N > i$, then, by (iii)

$$\begin{aligned} \int_E |f_i|^p d\mu &\leqq \int_{\cup_N^\infty (A_n \cup B_n)} |f_i|^p d\mu \\ &\leqq \sum_N^\infty \int_{A_n \cup B_n} |f_i|^p d\mu \\ &\leqq \sum_N^\infty 2^{-n} \\ &= 2^{1-N} \longrightarrow 0 \quad (N \longrightarrow \infty) . \end{aligned}$$

Thus $\mu(E \cap S(f_i)) = 0$ for all i and $\mu(E \cap \bigcup S(f_n)) = 0$. We complete our proof by showing that $X \sim E \subset S(f)$. If $t \in X \sim E$ choose the smallest integer n such that $t \notin \bigcup_{k=n}^{\infty} (A_k \cup B_k)$, then $|g_n(t)| > \varepsilon_n$ and $|\alpha_k f_k(t)| < \varepsilon_{k-1}/2 < \varepsilon_n/2^{k-n} (k \geqq n+1)$. Hence

$$\begin{aligned} |g_k(t)| &\geqq |g_n(t)| - |\alpha_{n+1} f_{n+1}(t)| - \cdots - |\alpha_k f_k(t)| \\ &> |g_n(t)| - \varepsilon_n (2^{-1} + \cdots + 2^{-(k-n)}) \\ &> |g_n(t)| - \varepsilon_n \qquad (k > n)\,. \end{aligned}$$

Thus $|f(t)| = \lim_{k\to\infty} |g_k(t)| \geqq |g_n(t)| - \varepsilon_n > 0$, and we are done.

LEMMA 6.2. [10]. *Let M be a separable subspace of $L_p(X, \Sigma, \mu)$ $(p \geqq 1)$ and T a bounded linear operator on L_p. Then there is a σ-finite set $X_0 \in \Sigma$ and a σ-subring Σ_0 of Σ such that Σ_0 consists of subsets of X_0 and $L_p(X_0, \Sigma_0, \mu)$ is separable, T-invariant and contains M.*

Proof. The subspace $M + TM$ is separable, T-invariant and generates a separable vector sublattice M_1 of L_p. Inductively construct separable vector sublattices M_n such that $M_n + TM_n \subset M_{n+1}$. Then $\mathrm{cl} \cup M_n$ is a separable T-invariant closed vector sublattice of L_p. Writing $K_1 = \mathrm{cl} \cup M_n$ we have K_1 closed under all band projections J_x with $x \in K_1$. Let $\Sigma_1 = \{S(x) : x \in K_1\}$ then Σ_1 is a σ-subring of Σ and if $x, y \in K_1$ with $x \in y^{\perp\perp}$ then x/y is Σ_1-measurable. If (f_n) is dense in K_1, $f = \Sigma 2^{-n} \|f_n\|^{-1} |f_n| \in K_1$ and $\mu(S(x) \sim S(f)) = 0 (x \in K_1)$. Consider $L_p(S(f), \Sigma_1, \mu)$. It is easy to see that this is the closure of the vector sublattice spanned by K_1 and the functions $\chi_{f^{-1}(\alpha,\infty]}$ with α positive rational. Thus, writing $X_1 = S(f)$ we have

$$K_1 \subset L_p(X_1, \Sigma_1, \mu)$$

with $L_p(X_1, \Sigma_1, \mu)$ separable. Continue inductively, we obtain a sequence $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$ of σ-finite subsets of X and a sequence $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots$ of σ-subrings of Σ, such that each Σ_n consists of subsets of X_n, $L_p(X_n, \Sigma_n, \mu) + TL_p(X_n, \Sigma_n, \mu) \subset L_p(X_{n+1}, \Sigma_{n+1}, \mu)$ and each $L_p(X_n, \Sigma_n, \mu)$ is separable.

Let $K_0 = \mathrm{cl} \bigcup_{n=1}^{\infty} L_p(X_n, \Sigma_n, \mu)$. Then K_0 is a separable T-invariant closed vector sublattice of $L_p(X, \Sigma, \mu)$. Define $\Sigma_0 = \{S(f) : f \in K_0\}$ and find, as for K_1, $f \in K_0$ such that $\mu(S(x) \sim S(f)) = 0 (x \in K_0)$. It is routine to show that $K_0 = L_p(S(f), \Sigma_0, \mu)$. This proves our lemma with $X_0 = S(f)$.

Added in Proof (October 1974). In a manuscript, "A local characterization of complex Banach lattices with order continuous norm," submitted to Studia Math., the authors have given a necessary and sufficient condition for a complex Banach space to admit a lattice structure so that it is a complex Banach lattice with order continuous norm. The condition is automatically satisfied if the Banach space is an $\mathscr{L}_{p,\lambda}$ space for every $\lambda > 1$. This does provide an elementary proof that such spaces are L_p-spaces.

REFERENCES

1. T. Ando, *Contractive projections in L_p-spaces*, Pacific J. Math., **17** (1966), 391-405.

2. R. G. Douglas, *Contractive projections on an L_1-space*, Pacific J. Math., **15** (1965), 443-462.

3. C. V. Duplissey, *Contractive projections in abstract Banach function spaces*, Ph. D. Dissertation, University of Texas at Austin, 1971.

4. A. Grothendieck, *Une characterisation vectorielle metrique des espaces L^1*, Canad. J. Math., **7** (1955), 552-561.

5. H. Elton Lacey and S. J. Bernau, *Characterizations and classifications of some classical Banach spaces*, Advances in Math., **12** (1974), 367-401.

6. J. Lamperti, *On the isometries of certain spaces*, Pacific J. Math., **8** (1958), 459-466.

7. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math., **29** (1968), 275-326.

8. M. M. Rao, *Linear operations, tensor products, and contractive projections in function spaces*, Studia Math., **38** (1970), 131-186.

9. S. Sakai, *C*-Algebras and W*-Algebras*, Ergebnisse der Mathematik, Bd 60, Springer-Verlag, 1971.

10. L. Tzafriri, *Remarks on contractive projections in L_p-spaces*, Israel J. Math., **7** (1969), 9-15.

11. Daniel E. Wulbert, *A note on the characterization of conditional expectation operators*, Pacific J. Math., **34** (1970), 285-288.

12. M. Zippin, *On Bases in Banach Spaces*, Ph. D. thesis, Hebrew University, Jerusalem, 1968 (Hebrew).

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