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Let S be a 3-convex subset of the plane. If $(\mathrm{cl}\, S \sim S) \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$ or if $(\mathrm{cl}\, S \sim S) \subseteq \mathrm{bdry}\,(\mathrm{cl}\, S)$, then S is expressible as a union of four or fewer convex sets. Otherwise, S is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let S be a subset of R^d. Then S is said to be 3-*convex* iff for every three distinct points in S, at least one of the segments determined by these points lies in S. Valentine [2] has proved that for S a closed, 3-convex subset of the plane, S is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set S to be closed. The following definitions and results obtained by Valentine will be useful.

For $S \subseteq R^d$, a point x in S is a *point of local convexity of* S iff there is some neighborhood U of x such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Let S be a closed, connected, 3-convex subset of the plane, and let Q denote the closure of the set of isolated lnc points of S. Valentine has proved that for S not convex, then $\mathrm{card}\, Q \geqq 1$, Q lies in the convex kernel of S, and $Q \subseteq \mathrm{bdry}\,(\mathrm{conv}\, Q)$. An *edge* of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$ is a closed segment (or ray) in $\mathrm{bdry}\,(\mathrm{conv}\, Q)$ whose endpoints are in Q. We define a *leaf* of S in the following manner: In case $\mathrm{card}\, Q \geqq 3$, let L be the line determined by an edge of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$, L_1, L_2 the corresponding open halfspaces. Then L supports $\mathrm{conv}\, Q$, and we may assume $\mathrm{conv}\, Q \subseteq \mathrm{cl}\,(L_1)$. We define $W = \mathrm{cl}\,(L_2 \cap S)$ to be a *leaf* of S. For $2 \geqq \mathrm{card}\, Q \geqq 1$, constructions used by Valentine may be employed to decompose S into two closed convex sets, and we define each of these convex sets to be a *leaf* of S.

By Valentine's results, every point of S is either in $\mathrm{conv}\, Q$ or in some leaf W of S (or both), and every leaf W is convex. Moreover, Valentine obtains his decomposition of S by showing that for any collection $\{s_i\}$ of disjoint edges of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$, with $\{W_i\}$ the corresponding collection of leaves, $\mathrm{conv}\, Q \cup (\bigcup W_i)$ is closed and convex.

Finally, we will use the following familiar definitions: For x, y in S, we say x *see* y *via* S iff the corresponding segment $[x, y]$ lies in S. A subset T of S is *visually independent via* S iff for every

x, y in T, x does not see y via S.

Throughout the paper, S will denote a 3-convex subset of the plane, Q the closure of the set of isolated lnc points of cl S.

2. Preliminary lemmas. The following sequence of lemmas will be useful in obtaining the desired representation theorems. We begin with an easy result.

LEMMA 1. *Cl S is 3-convex.*

Proof. Let x, y, z be distinct points in cl S and select disjoint sequences (x_i), (y_i), (z_i) in S converging to x, y, z respectively. For each i, one of the corresponding segments is in S, and for one pair, say x and y, infinitely many of the segments $[x_i, y_i]$ lie in S. Since these segments converge to $[x, y]$, $[x, y]$ lies in cl S.

The remaining lemmas are technical in nature. Lemmas 2, 3, and 4 reveal various pleasant features of int (cl S) $\sim S$, while 5 and 6 are concerned with lnc points of cl S.

LEMMA 2. *If $p \in$ int (cl S) $\sim$ ker (cl S) $\neq \emptyset$, then $p \in S$.*

Proof. Since $p \notin$ ker (cl S), there is some point x in cl S for which $[x, p] \not\subseteq$ cl S. Moreover, x may be chosen in S (for if p saw every member of S via cl S, then p would see every member of cl S via cl S and p would lie in ker (cl S)).

There is a convex neighborhood N of p, no point of which sees x via cl S, with $N \subseteq$ int (cl S). For any s, t distinct points in $N \cap S$, necessarily $[s, t] \subseteq S$ by the 3-convexity of S, so $N \cap S$ is convex. Since $N \subseteq$ int (cl S), p is interior to some triangle conv $\{w, y, z\}$ with vertices belonging to $N \cap S$. Then since $N \cap S$ is convex, conv $\{w, y, z\} \subseteq S$, and $p \in S$. In fact, $p \in$ int S.

COROLLARY. *If $p \in$ cl $S \sim S$, then either $p \in$ bdry (cl S) or $p \in$ ker (cl S) (or both).*

LEMMA 3. *Let $T \neq \emptyset$ be the set of points p of cl $S \sim S$ for which $p \notin$ bdry (cl S). Then every connected component of T is either an isolated point of cl $S \sim S$ or an interval. Moreover, there can be at most one isolated point, and all components of T lie on a common line.*

Proof. If T is a singleton point, the result is immediate, so assume that T contains at least two distinct points x, y. Let $L(x, y)$ denote the line determined by these points. It is clear that not both x and y can be isolated in $\mathrm{cl}\, S \sim S$, for otherwise, since $x, y \in \mathrm{int}\,(\mathrm{cl}\, S)$, it would be easy to select three points of S on $L(x, y)$ visually independent via S.

Again using the 3-convexity of S, $L(x, y) \cap S$ has at most two components, and $L(x, y) \cap T \subseteq \ker(\mathrm{cl}\, S)$ has at most three components. By an earlier argument, at most one component of $L(x, y) \cap T$ is an isolated point, and clearly each component is either an isolated point or an interval.

To complete the proof, it suffices to show that $T \subseteq L(x, y)$. Let $t \in \mathrm{int}\,(\mathrm{cl}\, S) \sim L(x, y)$ to show $t \notin T$. Since $L(x, y) \cap T$ contains at most one isolated point, $L(x, y) \cap T$ contains at least one interval $(r, s) \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, and we may choose some point u in S for which (u, t) cuts (r, s). Then select a convex neighborhood N of t, $N \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, so that for every q in N, (u, q) cuts (r, s). By techniques similar to those used in the proof of Lemma 2, $N \cap S$ is convex and $t \in S$. Hence $t \notin T$ and $T \subseteq L(x, y)$.

LEMMA 4. *If* $\mathrm{cl}\, S \sim S$ *contains an interval* (r, s) *disjoint from* $\mathrm{bdry}\,(\mathrm{cl}\, S)$, *then every lnc point of* $\mathrm{cl}\, S$ *lies on* $L(r, s)$.

Proof. Assume that for some lnc point t of $\mathrm{cl}\, S$, $t \notin L(r, s)$. As in the proof of Lemma 3, choose a point u and a neighborhood N of t so that u sees no point of $N \cap S$ via S. Since t is an lnc point of $\mathrm{cl}\, S$, N contains points v, w in S which are visually independent via S. Hence u, v, w are visually independent via S, a contradiction, and t must lie on $L(r, s)$.

LEMMA 5. *If p is in* $\ker(\mathrm{cl}\, S)$ *and q, r are in Q, then* $q \notin (p, r)$ (where p, q, r are distinct points).

Proof. Assume, on the contrary, that the points are collinear, with $p < q < r$. Let L denote the line containing p, q, r, L_1, L_2 the corresponding open halfspaces. Since $p \in \ker(\mathrm{cl}\, S)$ and $\mathrm{cl}\, S$ is not convex, there must be some point x of $\mathrm{cl}\, S$ not on L, say in L_1. Our hypothesis implies that $\mathrm{cl}\, S$ is connected, so by [2], Corollary 1, $r \in \ker(\mathrm{cl}\, S)$, and the triangle $\mathrm{conv}\,\{p, x, r\}$ has its boundary in $\mathrm{cl}\, S$. It is easy to see that the closed, 3-convex set $\mathrm{cl}\, S$ is simply connected, so $\mathrm{conv}\,\{p, x, r\} \subseteq \mathrm{cl}\, S$. Thus since q is an lnc point for $\mathrm{cl}\, S$, there must be some point y of $\mathrm{cl}\, S$ in L_2, $\mathrm{conv}\,\{p, y, r\} \subseteq \mathrm{cl}\, S$, and q cannot be an lnc point for $\mathrm{cl}\, S$, clearly impossible. Our assumption is false, and $q \notin (p, r)$.

COROLLARY. *No three members of Q are collinear.*

LEMMA 6. *If* $p \in \operatorname{conv} Q$, $q \in Q$, $q \neq p$, *and* W_1, W_2 *are leaves of* $\operatorname{cl} S$ *containing* q, *then* W_1, W_2 *are in opposite closed halfspaces determined by* $L(p, q)$.

Proof. Clearly the hypothesis implies that $\operatorname{cl} S$ is connected and that card $Q \geqq 2$. If card $Q = 2$, the result is an immediate consequence of an argument used by Valentine (Case 2, Theorem 3 of [2]), so we may assume that card $Q \geqq 3$. Let r lie on the edge of bdry (conv Q) which defines W_1, $r \neq q$. If $r \in L(p, q) \equiv L$, then by the definition of W_1, it is obvious that W_1 is in one of the closed halfspaces determined by L, say $\operatorname{cl} L_1$. Otherwise, without loss of generality, assume that r is in the open halfspace L_1. Clearly p and W_1 are separated by $L(r, q)$. Now if any point x of W_1 lay in L_2, then q would lie interior to the triangle $\operatorname{conv}\{p, x, r\} \subseteqq \operatorname{cl} S$, and q could not be an lnc point for $\operatorname{cl} S$, a contradiction. Hence W_1 lies in $\operatorname{cl} L_1$ in either case.

Since $W_1 \cup \operatorname{conv} Q$ is convex (by Valentine's results) and q is an lnc point for $\operatorname{cl} S$, W_2 necessarily contains points in L_2, and $W_2 \subseteqq \operatorname{cl} L_2$, finishing the proof.

3. Decomposition theorems. With the preliminary lemmas behind us, we begin to investigate conditions under which S may be represented as a union of four or fewer convex sets, dealing primarily with the case for $(\operatorname{cl} S \sim S) \subseteqq \operatorname{int}(\operatorname{cl} S)$.

The first theorem, allowing us to restrict attention to the case for $\operatorname{cl} S = \operatorname{cl}(\operatorname{int} S)$, will be helpful later.

THEOREM 1. *If* $\operatorname{cl} S \neq \operatorname{cl}(\operatorname{int} S)$, *then* S *is a union of two or fewer convex sets.*

Proof. Without loss of generality, assume S is connected, for otherwise the result is trivial. Let $x \in S \sim \operatorname{cl}(\operatorname{int} S) \neq \emptyset$, and let N be a convex neighborhood of x disjoint from $\operatorname{int} S$. Since S is connected, x is not an isolated point of S, and it is clear that $N \cap S$ contains at least one segment.

We examine the maximal segments of $N \cap S$ (i.e., the segments which are not proper subsets of segments in $N \cap S$). It is easy to show that $N \cap S$ has at most two maximal segments, for otherwise, the 3-convexity of S together with the simple connectedness of $\operatorname{cl} S$ would yield an open region in $\operatorname{cl} S \cap N$. Since by Lemma 3 the points of $\operatorname{int}(\operatorname{cl} S) \sim S$ are collinear, this would imply that $N \cap S$ has interior points, clearly impossible by our choice of N.

In case $N \cap S$ has exactly two maximal segments, an argument similar to the one above may be used to show that any point of S

lies on one of the corresponding lines, and S is a union of two segments (possibly infinite). If $N \cap S$ has just one segment, let K_1 denote a maximal convex subset of S containing it, and let $K_2 \equiv \text{conv}(S \sim K_1)$. Again using the facts that N contains no interior points of $\text{cl}\, S$ and $\text{cl}\, S$ is simply connected, it is not hard to show that $K_2 \subseteqq S$, and $S = K_1 \cup K_2$, completing the proof.

Theorems 2 and 3 show that a decomposition is possible when $(\text{cl}\, S \sim S) \subseteqq \text{int}(\text{cl}\, S)$. There are two cases to consider, depending on the cardinality of Q.

THEOREM 2. *If* $(\text{cl}\, S \sim S) \cap \text{bdry}(\text{cl}\, S) = \emptyset$, *and* $\text{card}\, Q = n$ *for n an odd integer, $n > 1$, then S is expressible as a union of four or fewer convex sets.*

Proof. Clearly the hypothesis implies that $\text{cl}\, S = \text{cl}(\text{int}\, S)$. By the Corollary to Lemma 2, $\text{cl}\, S \sim S \subseteqq \ker(\text{cl}\, S)$, and by Lemma 3, every component of $\text{cl}\, S \sim S$ is either an isolated point or an interval. Since $\text{card}\, Q \geqq 3$ and (by the corollary to Lemma 5) no three members of Q can be collinear, Lemma 4 implies that $\text{cl}\, S \sim S$ cannot contain an interval. Hence $\text{cl}\, S \sim S$ consists of exactly one isolated point p in $\ker(\text{cl}\, S)$.

Select $q \in Q$ in the following manner: If $p \in \text{conv}\, Q$, choose $q \in Q$ so that the line $L(p, q)$ contains no other member of Q. (Clearly this is possible since card Q is odd and no three members of Q are collinear.) If $p \notin \text{conv}\, Q$, let $\{e_i : 1 \leqq i \leqq n\}$ denote the edges of $\text{conv}\, Q$, $\{E_i : 1 \leqq i \leqq n\}$ the corresponding lines, with $\text{conv}\, Q$ in the closed halfspace $\text{cl}(E_{i1})$ for each i. Then $p \in E_{i2}$ for exactly one i, for otherwise, if $p \in E_{12} \cap E_{22}$, then $\text{int conv}(\{p\} \cup e_1 \cup e_2)$ would contain an lnc point of $\text{cl}\, S$, clearly impossible since $\{p\} \cup e_1 \cup e_2 \subseteqq \ker(\text{cl}\, S)$ and $\text{conv}(\{p\} \cup e_1 \cup e_2) \subseteqq \text{cl}\, S$. Thus we may choose some $q \in Q$ so that $p \in \text{cl}\, E_{i1}$ for each edge e_i containing q. Then (p, q) contains points of $\text{conv}\, Q$. Since all points of $L(p, q) \cap \text{conv}\, Q$ are on the open ray at p emanating through q, Lemma 5 implies that $L(p, q)$ contains no other members of Q (and in fact p cannot lie on any line E_i).

To review, in either case we have chosen q in Q so that $L(p, q)$ contains no other member of Q and (p, q) contains points of $\text{conv}\, Q$. Letting L_1, L_2 denote distinct open halfspaces determined by $L = L(p, q)$, define $A \equiv \text{cl}(S \cap L_1)$, $B \equiv \text{cl}(S \cap L_2)$. If W_1, W_2 are leaves of $\text{cl}\, S$ containing q, then by Lemma 6, W_1 and W_2 are in opposite closed halfspaces determined by L, say $W_1 \subseteqq \text{cl}\, L_1$, $W_2 \subseteqq \text{cl}\, L_2$.

Let R_1, R_2 denote opposite closed rays at p, $R_1 \cup R_2 = L$, labeled so that $q \in R_2$. Each of $R_1 \cap S$, $R_2 \cap S$ is an interval by the 3-convexity of S. Points of $R_1 \cap S$ necessarily lie in $A \cap B$, for otherwise

R_1 would contain an lnc point of $\mathrm{cl}\, S$, clearly impossible. If there are any points of $R_2 \cap S$ not in $A \cap B$, without loss of generality we may assume such points lie in W_1 and hence in $A \sim B$. Then $R_2 \cap S \subseteqq A$.

By Case 4 in Theorems 2 and 3 of [2], $\mathrm{cl}\,(S \sim W_2)$ is a union of two closed convex sets C_1, C_2, selected as in Valentine's proof. Since $A = \mathrm{cl}\,[\mathrm{cl}\,(S \sim W_2) \cap L_1]$, A is the union of the two closed convex sets A_1, A_2, where $A_i = \mathrm{cl}\,(C_i \cap L_1)$, $i = 1, 2$. Moreover, $(R_1 \cap S) \cup (p, q]$ lies in one of these sets, say A_1, and $R_2 \sim (p, q]$ is either in A_1 or in A_2.

Using an identical argument for B and $\mathrm{cl}\,(S \sim W_1)$, we may write B as a union of two closed convex sets B_1, B_2 with $(R_1 \cap S) \cup (p, q]$ in B_1, and $R_2 \sim (p, q]$ disjoint from B.

At last, define sets A_1', A_2', B_1', B_2' in the following manner: If $(R_2 \cap S) \sim (p, q] \subseteqq A_2$, let

$$A_1' \equiv A_1 \sim R_2\,, \quad A_2' \equiv A_2 \sim R_1\,,$$
$$B_1' \equiv B_1 \sim R_1\,, \quad B_2' \equiv B_2 \sim R_2\,.$$

And if $(R_2 \cap S) \sim (p, q] \subseteqq A_1$, let

$$A_1' \equiv A_1 \sim R_1\,, \quad A_2' \equiv A_2 \sim R_2\,,$$
$$B_1' \equiv B_1 \sim R_2\,, \quad B_2' \equiv B_2 \sim R_1\,.$$

We assert that these are convex subsets of S whose union is S: Clearly each is a convex subset of S, and $S \sim L$ is contained in their union. For $(R_2 \cap S) \sim (p, q] \subseteqq A_2$, $R_2 \cap S \subseteqq A_2' \cup B_1'$, $R_1 \cap S \subseteqq A_1'$. For $(R_2 \cap S) \sim (p, q] \subseteqq A_1$, $R_2 \cap S \subseteqq A_1'$, $R_1 \cap S \subseteqq B_1'$. Hence in either case $S \cap L$ is contained in the union of these sets, and $S = A_1' \cup A_2' \cup B_1' \cup B_2'$, completing the proof of the theorem.

THEOREM 3. *If* $(\mathrm{cl}\, S \sim S) \cap \mathrm{bdry}\,(\mathrm{cl}\, S) = \emptyset$ *and* $\mathrm{card}\, Q = n \geqq 0$, *where* n (possibly infinite) *is not an odd integer greater than one, then* S *is expressible as a union of four or fewer convex sets.*

Proof. If S is not connected, the result is trivial. Otherwise, by Theorem 3 of Valentine [2], $\mathrm{cl}\, S$ may be expressed as a union of two or fewer closed convex sets A, B. Using Lemma 3, let L be a line containing $\mathrm{cl}\, S \sim S$, L_1, L_2 the corresponding open halfspaces. Since S is 3-convex and A is convex, $S \cap A$ is 3-convex, and hence $(S \cap A) \cap L$ has at most two components, say C_1, C_2. Let R_1, R_2 denote opposite rays on L with $C_1 \subseteqq R_1$, $C_2 \subseteqq R_2$.

Define

$$A_1 \equiv (A \cap S \cap \mathrm{cl}\, L_1) \sim R_1\,,$$
$$A_2 \equiv (A \cap S \cap \mathrm{cl}\, L_2) \sim R_2\,.$$

Then A_1, A_2 are convex subsets of S whose union is $A \cap S$.

Similarly define convex sets B_1, B_2 whose union is $B \cap S$. Clearly $S = A_1 \cup A_2 \cup B_1 \cup B_2$, the desired result.

COROLLARY. *If* $(\mathrm{cl}\, S \sim S) \cap \mathrm{bdry}\,(\mathrm{cl}\, S) = \varnothing$, *then S is expressible as a union of four or fewer convex sets. The number four is best possible.*

That the number four in the corollary is best possible is evident from Example 1.

EXAMPLE 1. Let S be the set in Figure 1, with $p \in S$. Then S is not expressible as a union of fewer than four convex sets.

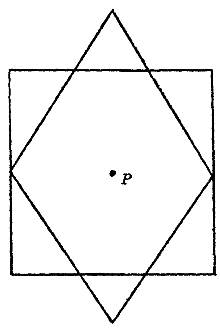

FIGURE 1

The preceding theorems allow us to obtain the following decomposition for open sets.

THEOREM 4. *If S is open, then S is expressible as a union of four or fewer convex sets. The result is best possible.*

Proof. Let $T \equiv S \cup \mathrm{bdry}\,(\mathrm{cl}\, S)$. Applying arguments identical to those used in the proofs of Theorems 2 and 3, T is expressible as a union of four or fewer convex sets A_i, $1 \leqq i \leqq 4$. Define $B_i \equiv A_i \cap S$, $1 \leqq i \leqq 4$. We assert that each B_i is convex. The proof follows:

By Valentine's results, $\mathrm{cl}\, S$ is expressible as a union of three or fewer closed convex sets C_j, $1 \leqq j \leqq 3$, each consisting of an appropriate selection of leaves of $\mathrm{cl}\, S$, together with $\mathrm{conv}\, Q$. Examining the proofs of Theorems 2 and 3, it is clear that each A_i may be considered as a subset of some C_j set. Thus we may assume $B_1 \subseteqq A_1 \subseteqq C_1$ for an appropriate C_1.

Let $x, y \in B_1$, and let $p \in (x, y)$ to show $p \in B_1$. If x (or y) is interior to some leaf W, then $W \subseteqq C_1$, y sees a neighborhood of x via

C_1, and p is interior to cl S. Since $p \in A_1$ and $p \notin$ bdry (cl S), p is in $A_1 \cap S = B_1$. A similar argument holds if x (or y) is interior to conv Q. Since neither x nor y is in bdry (cl S), the only other possibility to consider is the case in which $x, y \in$ bdry (conv Q) $\sim Q \subseteq$ ker (cl S). Then $x \in$ int (cl S), $y \in$ ker (cl S), y sees some neighborhood of x via cl S, and $p \in$ int (cl S). Again $p \in A_1 \cap S = B_1$ and B_1 is indeed convex. Thus S is the union of the convex sets B_i, $1 \leqq i \leqq 4$, and the theorem is proved.

To see that the number four is best possible, let S denote the set in Example 1 with its boundary deleted. Then S is an open 3-convex set not expressible as a union of fewer than four convex sets.

4. The general case.

It remains to investigate the case for S an arbitrary 3-convex subset of the plane. A decomposition of S into six convex sets may be obtained from our previous results, together with Theorems 5 and 6, which deal with the case for (cl $S \sim S) \subseteq$ bdry (cl S).

The following result by Lawrence, Hare, and Kenelly [1, Theorem 2] will be useful:

Lawrence, Hare, Kenelly Theorem. Let T be a subset of a linear space such that each finite subset $F \subseteq T$ has a k-partition, $\{F_1, \cdots, F_k\}$, where conv $F_i \subseteq T$, $1 \leqq i \leqq k$. Then T is a union of k convex sets.

THEOREM 5. *If* cl S *is convex and* (cl $S \sim S) \subseteq$ bdry (cl S), *then* S *is a union of three or fewer convex sets. The bound of three is best possible.*

Proof. Consider the collection of all intervals in bdry (cl S) having endpoints in S and some relatively interior point not in S. Each interval determines a line L, and by the 3-convexity of S, $L \cap S$ has exactly two components. Let $\mathscr{L}$ denote the collection of all such lines. By the Lawrence, Hare, Kenelly Theorem, without loss of generality we may assume that $\mathscr{L}$ is finite. Hence the set $\bigcup \{L \cap S: L \text{ in } \mathscr{L}\}$ has finitely many components, and we may order these components in a clockwise direction along bdry (cl S). If c_i denotes the ith component in our ordering, let

$$\begin{aligned} A' &\equiv \{c_i: i \text{ odd}, \ i < n\} , \\ B' &\equiv \{c_i: i \text{ even}, \ i < n\} , \\ C' &\equiv \{c_n\} . \end{aligned}$$

Define

$$A \equiv S \sim (B' \cup C') ,$$
$$B \equiv S \sim (A' \cup C') ,$$
$$C \equiv S \sim (A' \cup B') .$$

We assert that A, B, C are convex sets whose union is S. The proof follows:

For x, y in A, if $[x, y]$ contains any point of $\operatorname{int}(\operatorname{cl} S)$, then $(x, y) \subseteqq \operatorname{int}(\operatorname{cl} S) \subseteqq A$, and $[x, y] \subseteqq A$. Otherwise, $[x, y]$ lies in the boundary of the convex set $\operatorname{cl} S$. If the corresponding line $L(x, y)$ is not in $\mathscr{L}$, the result is clear, so suppose $L(x, y) \in \mathscr{L}$. Then x, y must lie in the same c_i set for some i odd, $i < n$, again giving the desired result. Hence A is convex. Similarly, B, C are convex. It is easy to see that $A \cup B \cup C = S$ and the proof is complete.

The surprising fact that three is best possible is illustrated by Example 2.

EXAMPLE 2. Let S denote the set in Figure 2, where dotted lines represent segments not in S. Then S is not expressible as a union of fewer than three convex sets.

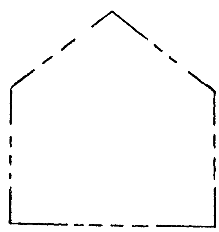

FIGURE 2

THEOREM 6. *If* $(\operatorname{cl} S \sim S) \subseteqq \operatorname{bdry}(\operatorname{cl} S)$, *then S is a union of four or fewer convex sets. The number four is best possible.*

Proof. We assume that S is connected and $\operatorname{cl} S = \operatorname{cl}(\operatorname{int} S)$, for otherwise S is a union of two convex sets. Furthermore, by the Lawrence, Hare, Kenelly Theorem, we may assume that $\operatorname{cl} S$ has finitely many leaves, and hence $\operatorname{card} Q = n$ is finite. Notice also that since $\operatorname{cl} S$ is simply connnected and $(\operatorname{cl} S \sim S) \subseteqq \operatorname{bdry}(\operatorname{cl} S)$, S is simply connected.

For the moment, suppose $3 \leqq n$. Order the points of Q in a clockwise direction along $\operatorname{bdry}(\operatorname{conv} Q)$, letting W_i denote the leaf of $\operatorname{cl} S$ determined by lnc points q_i, q_{i+1} (where $n + 1 \equiv 1$). By Valentine's results in [2], for any pair of disjoint leaves W_i, W_j of $\operatorname{cl} S$, the set $R \equiv \operatorname{conv} Q \cup W_i \cup W_j$ is a closed convex set. (In case there are no disjoint leaves, $n = 3$, $W_j = \emptyset$, and $R \equiv \operatorname{conv} Q \cup W_i$ is closed and convex.) Consider the collection of intervals in $\operatorname{bdry} R$ having endpoints x, y in S and some relatively interior point p not in S. Either such an interval is contained in one leaf, or $x \in W_i \cup \operatorname{conv} Q$, $y \in W_j \cup \operatorname{conv} Q$. We examine the latter case. It is clear that for an appropriate labeling, $j = i + 2$, so to simplify notation, say $i = 1$, $j = 3$, and $L(x, y)$ supports W_2. Clearly not both x, y can lie in $\operatorname{conv} Q$, for then $p \in \operatorname{int} S \subseteq S$. However, we assert that either x or y must lie in $\operatorname{conv} Q$ and that $W_2 \cap S$ is convex. The proof follows:

Assume that x is not an lnc point and that $x < p \leqq q_2 < q_3$, where q_2, q_3 are the lnc points in $W_1 \cap W_2$, $W_2 \cap W_3$ respectively. Then $q_2 \leqq y$. For w in $W_2 \cap S$, w cannot see x via S, so necessarily w sees y via S, by the 3-convexity of S. This implies that $y \leqq q_3$ (for otherwise q_3 could not be an lnc point for $\operatorname{cl} S$). Moreover, since no two points of $W_2 \cap S$ see x via S, the 3-convexity of S together with the convexity of W_2 imply that $W_2 \cap S$ is convex.

Here we digress briefly for future reference. The set $L(x, y) \cap S$ has two components, and by the above argument, one must lie in the interval $[q_2, q_3]$, the other in $W_1 \sim Q$ (by our labeling). For general W_{i-1}, W_{i+1} (disjoint if and only if $n > 3$), we let T_i denote the connected set of all the somewhat troublesome points y in $[q_i, q_{i+1}] \cap S$ having the above property. That is, there exist points x in exactly one of $(W_{i-1} \cap S) \sim Q$, $(W_{i+1} \cap S) \sim Q$ for which $[x, y] \nsubseteq S$ $(n + 1 \equiv 1)$.

Continuing the argument, delete W_2 and consider the 3-convex set $(S \sim W_2) \cup (S \cap L(x, y))$. Renumber the lnc points and leaves for this set so that the old W_1 and W_3 are contained in the new leaf U_1. Since we are assuming card Q is finite, repeating the procedure finitely many times yields a 3-convex set S_0 having the following property: For V_i, V_j disjoint leaves of $\operatorname{cl} S_0$, x in $V_i \cap S_0$, y in $V_j \cap S_0$, then $[x, y] \subseteq S_0$. In addition, without loss of generality we may assume that for each leaf V_i of $\operatorname{cl} S_0$, $V_i \cap S_0$ is not convex, for otherwise, V_i may be deleted by the above procedure.

To avoid confusion, let Q_0 denote the set of lnc points of $\operatorname{cl} S_0$, $Q_0 \subseteq Q$, $\operatorname{card} Q_0 = m \leqq n$. For $3 \leqq m$, let V_i denote the leaf determined by lnc points p_i, p_{i+1} in Q_0 (where $p_{m+1} = p_1$). For $m = 2$, let V_1, V_2 denote the leaves of $\operatorname{cl} S_0$ as defined in the introduction to this paper. If $0 \leqq m \leqq 1$, let $V_1 = V_2 = \operatorname{cl} S_0$.

For each i, consider the collection of intervals in bdry V_i having endpoints in $V_i \cap S_0$ and some relatively interior point not in S_0. Each interval determines a line L, and for $m \neq 1$, $L \cap V_i \cap S_0$ has exactly two components, each in bdry V_i. In case $m = 1$, an obvious adjustment may be made (by deleting any ray of L which contains interior points of $\operatorname{cl} S_0$) to yield the same result. For each i, let $\mathscr{L}_i$ denote the collection of all such lines. Again using the Lawrence, Hare, Kenelly Theorem, we may assume that each $\mathscr{L}_i$ is finite. The set $\bigcup \{L \cap V_i \cap S_0 \colon L \text{ in } \mathscr{L}_i\}$ has finitely many components, and we may order them in a clockwise direction along bdry V_i. Let c_{ij} denote the jth such component for V_i, and let $\mathscr{C}_i$ denote the collection of all the c_{ij} sets corresponding to V_i. Clearly each c_{ij} is either a point, an interval, or the union of two noncollinear intervals. Moreover, for $m \geqq 2$, no components for V_i, V_{i+1} may have common points. (Such a point would necessarily be p_{i+1}, and if $s_i \in V_i \cap S_0$, $s_{i+1} \in V_{i+1} \cap S_0$ with some interior point of each of $[s_i, p_{i+1}]$, $[p_{i+1}, s_{i+1}]$ not in S_0, then s_i, p_{i+1}, s_{i+1} would be visually independent via S_0, clearly impossible.)

For each V_i, select every c_{i2j}. That is, select the members of $\mathscr{C}_i$ having second subscript even. No two components selected correspond to the same line, and for $m \neq 0$, we have chosen one component corresponding to each line in $\mathscr{L}_i$. If $m = 0$, without loss of generality we may assume $\mathscr{C}_1$ is ordered in a clockwise direction from some point in $Q \cap \mathrm{cl}\, S_0 \neq \emptyset$. In case no component has been chosen for some line L in $\mathscr{L}_1$, then L must contain points of both the first and last members of $\mathscr{C}_1$, and by a previous argument, one of these components must lie in conv Q.

For $m \neq 1$, since V_i is convex, it is easy to show that conv $\{c_{i2j}: 1 \leqq j\}$ is a subset of S_0 (and this is certainly true even if cl S_0 is convex). We will prove that $B_0 \equiv \mathrm{conv}\, \{c_{i2j}: 1 \leqq i \leqq m,\ 1 \leqq j\}$ is in S_0 and hence in S. If cl S_0 is convex (or empty) the result is immediate, so assume cl S_0 has at least one lnc point. For convenience, in case cl S_0 has only one lnc point, call it p_2, and let $V_1 = V_2$ follow p_2 in our clockwise ordering.

Recall that $V_i \cap S_0$ is not convex for any i, so no $\mathscr{C}_i$ is empty. Let c_0 denote the last member of $\mathscr{C}_1$ selected, x the last point of cl c_0 (relative to our ordering). If $x \neq p_2$, let $L = L(x, p_2)$. Otherwise, by the 3-convexity of S_0, $c_0 = \{p_2\}$, and in this case let L denote the corresponding member of $\mathscr{L}_1$. Let L_1, L_2 be the open halfspaces determined by L, with $Q_0 \subseteqq \mathrm{cl}\, L_1$. Since p_2 is an lnc point of S_0 and S_0 is 3-convex, it is clear that at most one member of $\mathscr{C}_2$, namely c_{21}, may contain points in L_2. We assert that c_0 sees c_{22} via S_0. The proof follows:

In case $L \in \mathscr{L}_1$, $L \cap V_1 \cap S_0$ has two components, each in bdry V_1, and one of these must be $\{p_2\}$. Then by the 3-convexity of S_0, $c_{22} \subseteqq L_1$ and c_0 sees c_{22} via S_0. Otherwise, $c_0 \sim \{x\} \subseteqq L_1$. If $x \notin S_0$, then since $c_{22} \subseteqq \mathrm{cl}\, L_1$, it is clear that c_0 sees c_{22} via S_0. If $x \in S_0$ and $p_2 \in S_0$, then again the result is clear. If $x \in S_0$ and $p_2 \notin S_0$, then $c_{22} \subseteqq L_1$ and c_0 sees c_{22} via S_0, finishing the argument.

In case V_1, V_2 are the only leaves for cl S_0, $V_1 \neq V_2$, then repeating the argument for the last member of $\mathscr{C}_2$ and c_{12} and using the fact that S_0 is simply connected, we have $B_0 \subseteqq S_0 \subseteqq S$. (If $V_1 = V_2$, the result is immediate.) Otherwise, $3 \leqq m$ and an inductive argument may be used to show that B_0 is in S.

Using Valentine's results, write $\mathrm{cl}\, S$ as a union of three or fewer convex sets A_j, $j = 1, 2, 3$, where for n odd

$$A_1 \equiv \bigcup \{W_i \colon i \text{ odd}, \; i < n\} \cup \mathrm{conv}\, Q ,$$
$$A_2 \equiv \bigcup \{W_i \colon i \text{ even}, \; i < n\} \cup \mathrm{conv}\, Q ,$$
$$A_3 \equiv W_n \cup \mathrm{conv}\, Q ,$$

and for n even

$$A_1 \equiv \bigcup \{W_i \colon i \text{ odd}, \; i \leqq n\} \cup \mathrm{conv}\, Q ,$$
$$A_2 = \bigcup \{W_i \colon i \text{ even}, \; i \leqq n\} \cup \mathrm{conv}\, Q ,$$
$$A_3 = \varnothing .$$

Define $B_j \equiv S \cap [A_j \sim ((\mathrm{bdry}\, S) \cap B_0)]$, $j = 1, 2, 3$.

Recall the T_i sets defined previously, $T_i \subseteqq [q_i, q_{i+1}] \subseteqq W_i$, $1 \leqq i \leqq n$. To simplify notation, let $L_i = L(q_i, q_{i+1})$, and define sets F_i, G_i in the following manner: For i even, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in B_1, $F_i = \varnothing$ otherwise. Similarly for i odd, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in B_2, $F_i = \varnothing$ otherwise. For $i = 1$, $i = n - 1$, let $G_i = T_i$ if points from both components of $L_i \cap S$ are in B_3, $G_i = \varnothing$ otherwise. By previous remarks, at least one of G_1, F_1 is empty, and at least one of G_{n-1}, F_{n-1} is empty.

Define

$$D_1 \equiv B_1 \sim \bigcup \{F_i \colon i \text{ even}\} ,$$
$$D_2 \equiv B_2 \sim \bigcup \{F_i \colon i \text{ odd}\} ,$$
$$D_3 \equiv B_3 \sim \bigcup \{G_i, G_{n-1}\} .$$

Finally, letting $P = \{F_i \cap F_j \colon 1 \leqq i < j \leqq n\} \cup \{G_i \cap F_j \colon i = 1, n - 1, 1 \leqq j \leqq n\}$, define $D_0 \equiv \mathrm{conv}\,(B_0 \cup P)$. We assert that the sets D_j, $0 \leqq j \leqq 3$, are convex sets whose union is S. The proof follows:

Suppose that one of the sets D_1, D_2, D_3, say D_1, is not convex to obtain a contradiction. Choose x, y in D_1 for which $[x, y] \nsubseteqq D_1$. It is clear that $[x, y] \subseteqq \mathrm{bdry}\,(\mathrm{cl}\, D_1) = \mathrm{bdry}\, A_1$. Furthermore, x, y cannot both belong to $W \sim Q$ for any leaf W of $\mathrm{cl}\, S$, for otherwise they would belong to the same leaf of $\mathrm{cl}\, S_0$, and one of x, y would lie in $(\mathrm{bdry}\, S) \cap B_0$ and hence not in D_1, a contradiction. Employing a previous argument, the set $L(x, y) \cap S$ has two components, each having points in B_1, and one of these components is the set $[q_i, q_{i+1}] \cap S = T_i$ for some i even $(n + 1 \equiv 1)$. Let R_i denote the other component of $L(x, y) \cap S$. If $R_i \cap B_0 \neq \varnothing$, then R_i, T_i would lie on the boundary of a leaf of $\mathrm{cl}\, S_0$, $R_i \subseteqq B_0$, $T_i \subseteqq B_1$, and $[x, y] \subseteqq T_i \subseteqq D_1$, a contradiction. Thus $R_i \cap B_0 = \varnothing$ and $R_i \subseteqq D_1$. However, this implies that one of x, y must lie in F_i and not in D_1, again a contradiction. Our assumption is false and D_1 is convex. Similarly D_2, D_3 are convex, and clearly each is a subset of S.

It remains to show that the convex set D_0 lies in S. Examining the set P, if $F_i \cap F_j \neq \emptyset$ for some $i \neq j$ (or if $G_i \cap F_j \neq \emptyset$), then $F_i = T_i$, $F_j = T_j$, for an appropriate labeling $j = i+1$, and $F_i \cap F_{i+1} = \{q_{i+1}\} \subseteqq S$. We will show that for each z in B_0, $[q_{i+1}, z] \subseteqq S$. The proof follows:

We have seen that $W_i \cap S$, $W_{i+1} \cap S$ are both convex, so for every z in one of these sets, $[q_{i+1}, z] \subseteqq S$. Moreover, we assert that the components of $L(q_i, q_{i+1}) \cap S$, $L(q_{i+1}, q_{i+2}) \cap S$ not in $\operatorname{conv} Q$, call them R_i, R_{i+1}, are disjoint from B_0: If $R_i \cap B_0 \neq \emptyset$, then by an earlier argument, $R_i \subseteqq B_0$, $T_i \cap B_0 = \emptyset$, $T_i \subseteqq D_1 \cap D_2 \cap D_3$, and $F_i = \emptyset$, a contradiction. Hence for z in $B_0 \sim (W_i \cup W_{i+1})$, $(q_{i+1}, z) \subseteqq \operatorname{int} S$, and $[q_{i+1}, z] \subseteqq S$ whenever $z \in B_0$, the desired result.

Certainly for q_i, q_j, q_k in $P \subseteqq S$, $\operatorname{conv}\{q_i, q_j, q_k\} \subseteqq S$.

By Carathéodory's theorem in the plane, to prove that $D_0 \equiv \operatorname{conv}(B_0 \cup P)$ is in S, it is sufficient to show that the convex hull of any three points of $B_0 \cup P$ is in S, and from the remarks above, clearly we need only show $\operatorname{conv}\{q_i, q_j, z\} \subseteqq S$ for q_i, q_j in P, z in B_0. However, since S is simply connected and $\operatorname{bdry}(\operatorname{conv}\{q_i, q_j, z\}) \subseteqq S$, $\operatorname{conv}\{q_i, q_j, z\} \subseteqq S$ and $D_0 \subseteqq S$, the desired result.

Finally, by inspection, each $F_i \neq \emptyset$ fails to belong to at most one of the sets D_1, D_2, D_3. Points in intersecting F_i sets are in D_0, so $\bigcup \{D_j : 0 \leqq j \leqq 3\} = S$ and the argument for $3 \leqq \operatorname{card} Q$ is complete.

To finish the proof, we must examine the cases for $0 \leqq \operatorname{card} Q \leqq 2$. If $\operatorname{card} Q = 2$ or if $\operatorname{card} Q = 1$ and $S \sim Q$ is connected, then let W_1, W_2 denote the corresponding leaves of $\operatorname{cl} S$, and use a simplified version of the previous proof to define B_0, B_1, B_2. If one of B_1, B_2, say B_1, is not convex, then letting $T = W_1 \cap W_2 \cap S$, $W_2 \cap S = B_2$ is convex, $T \subseteqq B_2$, and B_0, $B_1 \sim T$, B_2 are the desired convex sets.

In case $\operatorname{card} Q = 1$ and $S \sim Q$ is not connected, then for W_1, W_2 the corresponding leaves of $\operatorname{cl} S$, each of $W_1 \cap S$, $W_2 \cap S$ is convex. For $\operatorname{card} Q = 0$, the result follows from Theorem 5, and the proof of Theorem 6 is complete.

The number four in Theorem 6 is best possible, as the following example illustrates.

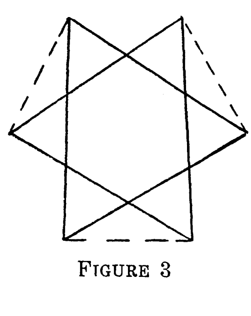

FIGURE 3

EXAMPLE 3. Let S denote the set in Figure 3, where dotted segments are in bdry (cl S) $\sim S$. Then S is a union of no fewer than four convex sets.

At last, using Theorem 6, we have a decomposition theorem for S an arbitrary 3-convex subset of the plane.

THEOREM 7. *The set S is a union of six or fewer convex sets. The result is best possible.*

Proof. By earlier comments, we may assume that S is connected, $\mathrm{cl}\, S = \mathrm{cl}\,(\mathrm{int}\, S)$, and Q is finite. Furthermore, we assume $\mathrm{int}\,(\mathrm{cl}\, S) \sim S \neq \emptyset$, for otherwise the result is an immediate consequence of Theorem 6. Let $T \equiv S \cup \mathrm{bdry}\,(\mathrm{cl}\, S)$, and let L be the line containing $\mathrm{cl}\, T \sim T$ described in Theorem 2 or Theorem 3 (whichever is appropriate). Clearly L may be chosen to contain an lnc point q of cl S. If L_1, L_2 are the corresponding open halfspaces, then each of $T_1 \equiv \mathrm{cl}\,(T \cap L_1) = \mathrm{cl}\,(S \cap L_1)$, $T_2 \equiv \mathrm{cl}\,(T \cap L_2) = \mathrm{cl}\,(S \cap L_2)$ is 3-convex.

Define $S_i \equiv T_i \cap S$, $i = 1, 2,$. We assert that each S_i is 3-convex: For x, y, z in $S_1 = T_1 \cap S$, assume $[x, y]$ lies in the 3-convex set S to show $[x, y] \subseteq S_1$. If x or y is in L_1, then certainly $(x, y) \subseteq L_1 \cap S \subseteq T_1$, and $[x, y] \subseteq S_1$. If x, y are on L, then since no lnc points of the closed set T_1 are on L, x, y lie in the same leaf of T_1, and $[x, y] \subseteq T_1 \cap S = S_1$. Thus S_1 is 3-convex. Similarly S_2 is 3-convex. Moreover, $(\mathrm{cl}\, S_i \sim S_i) \subseteq \mathrm{bdry}\,(\mathrm{cl}\, S_i)$, $i = 1, 2$.

Using Theorem 6, we will show that each S_i is a union of three convex sets: By the proofs of Theorems 2 and 3, $\mathrm{cl}\, S_1 = T_1$ is a union of two convex sets A_1, A_2, and each A_i may be considered a subset of an appropriate C_j set, $1 \leqq j \leqq 3$, where the C_j sets are those described in Valentine's paper with $\mathrm{cl}\, S = C_1 \cup C_2 \cup C_3$. In case T_1 has one leaf or an even number of leaves, then clearly the proof of Theorem 6 may be used to write S_1 as a union of three convex sets. If T_1 has n leaves for n odd, $n > 1$, let V be the leaf of T_1 bounded by L, $q \in Q \cap L \subseteq A_1 \cap A_2$. Order the lnc points of T_1 in a clockwise direction so that V is determined by q_n, q_1, and let U_n, U_{n+1} denote the closed subsets of V bounded by $L(q_n, q)$, $L(q, q_1)$ respectively. Treating $U_1, \cdots, U_n, U_{n+1}$ as leaves of T_1, U_i determined by lnc points q_i, q_{i+1}, $1 \leqq i < n$, the proof of Theorem 6 may be applied to write S_1 as a union of three convex sets. (Of course, in defining B_0, points of V in S_0 belong to the same leaf of S_0.)

By a parallel argument S_2 is a union of three convex sets, and $S = S_1 \cup S_2$ is a union of six or fewer convex sets, finishing the proof of the theorem.

Our final example shows that the bound of six in Theorem 7 is best possible.

EXAMPLE 4. Let S be the set in Figure 4, with dotted segments in $\mathrm{bdry}\,(\mathrm{cl}\,S) \sim S$ and $p \in \mathrm{int}\,(\mathrm{cl}\,S) \sim S$. Then S cannot be expressed as a union of fewer than six convex sets.

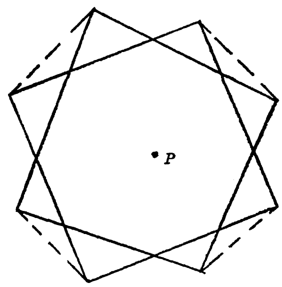

FIGURE 4

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