DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF THE PLANE

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Let $S$ be a 3-convex subset of the plane. If $(\text{cl } S \sim S) \subseteq \text{int } (\text{cl } S)$ or if $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, then $S$ is expressible as a union of four or fewer convex sets. Otherwise, $S$ is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let $S$ be a subset of $\mathbb{R}^d$. Then $S$ is said to be 3-convex iff for every three distinct points in $S$, at least one of the segments determined by these points lies in $S$. Valentine [2] has proved that for $S$ a closed, 3-convex subset of the plane, $S$ is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set $S$ to be closed. The following definitions and results obtained by Valentine will be useful.

For $S \subseteq \mathbb{R}^d$, a point $x$ in $S$ is a point of local convexity of $S$ iff there is some neighborhood $U$ of $x$ such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If $S$ fails to be locally convex at some point $q$ in $S$, then $q$ is called a point of local nonconvexity (lnc point) of $S$.

Let $S$ be a closed, connected, 3-convex subset of the plane, and let $Q$ denote the closure of the set of isolated lnc points of $S$. Valentine has proved that for $S$ not convex, then $\text{card } Q \geq 1$, $Q$ lies in the convex kernel of $S$, and $Q \subseteq \text{bdry } (\text{conv } Q)$. An edge of $\text{bdry } (\text{conv } Q)$ is a closed segment (or ray) in $\text{bdry } (\text{conv } Q)$ whose endpoints are in $Q$. We define a leaf of $S$ in the following manner: In case $\text{card } Q \geq 3$, let $L$ be the line determined by an edge of $\text{bdry } (\text{conv } Q)$, $L_x, L_z$ the corresponding open halfspaces. Then $L$ supports $\text{conv } Q$, and we may assume $\text{conv } Q \subseteq \text{cl } (L_i)$. We define $W = \text{cl } (L_z \cap S)$ to be a leaf of $S$. For $2 \geq \text{card } Q \geq 1$, constructions used by Valentine may be employed to decompose $S$ into two closed convex sets, and we define each of these convex sets to be a leaf of $S$.

By Valentine's results, every point of $S$ is either in $\text{conv } Q$ or in some leaf $W$ of $S$ (or both), and every leaf $W$ is convex. Moreover, Valentine obtains his decomposition of $S$ by showing that for any collection $\{s_i\}$ of disjoint edges of $\text{bdry } (\text{conv } Q)$, with $\{W_i\}$ the corresponding collection of leaves, $\text{conv } Q \cup (\bigcup W_i)$ is closed and convex.

Finally, we will use the following familiar definitions: For $x, y$ in $S$, we say $x$ see $y$ via $S$ iff the corresponding segment $[x, y]$ lies in $S$. A subset $T$ of $S$ is visually independent via $S$ iff for every
Throughout the paper, $S$ will denote a 3-convex subset of the plane, $Q$ the closure of the set of isolated lnc points of $\text{cl} S$.

2. Preliminary lemmas. The following sequence of lemmas will be useful in obtaining the desired representation theorems. We begin with an easy result.

**Lemma 1.** $\text{cl} S$ is 3-convex.

*Proof.* Let $x, y, z$ be distinct points in $\text{cl} S$ and select disjoint sequences $(x_i), (y_i), (z_i)$ in $S$ converging to $x, y, z$ respectively. For each $i$, one of the corresponding segments is in $S$, and for one pair, say $x$ and $y$, infinitely many of the segments $[x_i, y_i]$ lie in $S$. Since these segments converge to $[x, y]$, $[x, y]$ lies in $\text{cl} S$.

The remaining lemmas are technical in nature. Lemmas 2, 3, and 4 reveal various pleasant features of $\text{int} (\text{cl} S) \sim S$, while 5 and 6 are concerned with lnc points of $\text{cl} S$.

**Lemma 2.** If $p \in \text{int} (\text{cl} S) \sim \ker (\text{cl} S) \neq \emptyset$, then $p \in S$.

*Proof.* Since $p \in \ker (\text{cl} S)$, there is some point $x$ in $\text{cl} S$ for which $[x, p] \not\subseteq \text{cl} S$. Moreover, $x$ may be chosen in $S$ (for if $p$ saw every member of $S$ via $\text{cl} S$, then $p$ would see every member of $\text{cl} S$ via $\text{cl} S$ and $p$ would lie in $\ker (\text{cl} S)$).

There is a convex neighborhood $N$ of $p$, no point of which sees $x$ via $\text{cl} S$, with $N \subseteq \text{int} (\text{cl} S)$. For any $s, t$ distinct points in $N \cap S$, necessarily $[s, t] \subseteq S$ by the 3-convexity of $S$, so $N \cap S$ is convex. Since $N \subseteq \text{int} (\text{cl} S)$, $p$ is interior to some triangle $\text{conv} \{w, y, z\}$ with vertices belonging to $N \cap S$. Then since $N \cap S$ is convex, $\text{conv} \{w, y, z\} \subseteq S$, and $p \in S$. In fact, $p \in \text{int} S$.

**Corollary.** If $p \in \text{cl} S \sim S$, then either $p \in \text{bdry} (\text{cl} S)$ or $p \in \ker (\text{cl} S)$ (or both).

**Lemma 3.** Let $T \neq \emptyset$ be the set of points $p$ of $\text{cl} S \sim S$ for which $p \in \text{bdry} (\text{cl} S)$. Then every connected component of $T$ is either an isolated point of $\text{cl} S \sim S$ or an interval. Moreover, there can be at most one isolated point, and all components of $T$ lie on a common line.

*Proof.* If $T$ is a singleton point, the result is immediate, so assume that $T$ contains at least two distinct points $x, y$. Let $L(x, y)$ denote
the line determined by these points. It is clear that not both $x$ and $y$ can be isolated in $\text{cl } S \sim S$, for otherwise, since $x, y \in \text{int } (\text{cl } S)$, it would be easy to select three points of $S$ on $L(x, y)$ visually independent via $S$.

Again using the 3-convexity of $S$, $L(x, y) \cap S$ has at most two components, and $L(x, y) \cap T \supseteq \text{ker } (\text{cl } S)$ has at most three components. By an earlier argument, at most one component of $L(x, y) \cap T$ is an isolated point, and clearly each component is either an isolated point or an interval.

To complete the proof, it suffices to show that $T \subseteq L(x, y)$. Let $t \in \text{int } (\text{cl } S) \sim L(x, y)$ to show $t \in T$. Since $L(x, y) \cap T$ contains at most one isolated point, $L(x, y) \cap T$ contains at least one interval $(r, s) \subseteq \text{int } (\text{cl } S)$, and we may choose some point $u$ in $S$ for which $(u, t)$ cuts $(r, s)$. Then select a convex neighborhood $N$ of $t$, $N \subseteq \text{int } (\text{cl } S)$, so that for every $q$ in $N$, $(u, q)$ cuts $(r, s)$. By techniques similar to those used in the proof of Lemma 2, $N \cap S$ is convex and $t \in S$. Hence $t \in T$ and $T \subseteq L(x, y)$.

**Lemma 4.** If $\text{cl } S \sim S$ contains an interval $(r, s)$ disjoint from $\text{bdry } (\text{cl } S)$, then every lnc point of $\text{cl } S$ lies on $L(r, s)$.

**Proof.** Assume that for some lnc point $t$ of $\text{cl } S$, $t \in L(r, s)$. As in the proof of Lemma 3, choose a point $u$ and a neighborhood $N$ of $t$ so that $u$ sees no point of $N \cap S$ via $S$. Since $t$ is an lnc point of $\text{cl } S$, $N$ contains points $v, w$ in $S$ which are visually independent via $S$. Hence $u, v, w$ are visually independent via $S$, a contradiction, and $t$ must lie on $L(r, s)$.

**Lemma 5.** If $p$ is in $\text{ker } (\text{cl } S)$ and $q, r$ are in $Q$, then $q \in (p, r)$ (where $p, q, r$ are distinct points).

**Proof.** Assume, on the contrary, that the points are collinear, with $p < q < r$. Let $L$ denote the line containing $p, q, r, L_1, L_2$ the corresponding open halfspaces. Since $p \in \text{ker } (\text{cl } S)$ and $\text{cl } S$ is not convex, there must be some point $x$ of $\text{cl } S$ not on $L$, say in $L_2$. Our hypothesis implies that $\text{cl } S$ is connected, so by [2], Corollary 1, $r \in \text{ker } (\text{cl } S)$, and the triangle $\text{conv } \{p, x, r\}$ has its boundary in $\text{cl } S$. It is easy to see that the closed, 3-convex set $\text{cl } S$ is simply connected, so $\text{conv } \{p, x, r\} \subseteq \text{cl } S$. Thus since $q$ is an lnc point for $\text{cl } S$, there must be some point $y$ of $\text{cl } S$ in $L_1$, $\text{conv } \{p, y, r\} \subseteq \text{cl } S$, and $q$ cannot be an lnc point for $\text{cl } S$, clearly impossible. Our assumption is false, and $q \in (p, r)$.

**Corollary.** No three members of $Q$ are collinear.
Lemma 6. If \( p \in \text{conv } Q, \ q \in Q, \ q \neq p, \) and \( W_1, W_2 \) are leaves of \( \text{cl } S \) containing \( q \), then \( W_1, W_2 \) are in opposite closed halfspaces determined by \( L(p, q) \).

Proof. Clearly the hypothesis implies that \( \text{cl } S \) is connected and that \( \text{card } Q \geq 2 \). If \( \text{card } Q = 2 \), the result is an immediate consequence of an argument used by Valentine (Case 2, Theorem 3 of [2]), so we may assume that \( \text{card } Q \geq 3 \). Let \( r \) lie on the edge of \( \text{bdry } (\text{conv } Q) \) which defines \( W_1, r \neq q \). If \( r \in L(p, q) \equiv L \), then by the definition of \( W_1 \), it is obvious that \( W_1 \) is in one of the closed halfspaces determined by \( L \), say \( \text{cl } L \). Otherwise, without loss of generality, assume that \( r \) is in the open halfspace \( L \). Clearly \( p \) and \( W_1 \) are separated by \( L(r, q) \). Now if any point \( x \) of \( W_1 \) lay in \( L \), then \( q \) would lie interior to the triangle \( \text{conv } \{p, x, r\} \subseteq \text{cl } S \), and \( q \) could not be an lnc point for \( \text{cl } S \), a contradiction. Hence \( W_1 \) lies in \( \text{cl } L \) in either case.

Since \( W_1 \cup \text{conv } Q \) is convex (by Valentine's results) and \( q \) is an lnc point for \( \text{cl } S \), \( W_1 \) necessarily contains points in \( L \), and \( W_1 \subseteq \text{cl } L \), finishing the proof.

3. Decomposition theorems. With the preliminary lemmas behind us, we begin to investigate conditions under which \( S \) may be represented as a union of four or fewer convex sets, dealing primarily with the case for \( \text{(cl } S \sim S) \subseteq \text{int } (\text{cl } S) \).

The first theorem, allowing us to restrict attention to the case for \( \text{cl } S = \text{cl } (\text{int } S) \), will be helpful later.

Theorem 1. If \( \text{cl } S \neq \text{cl } (\text{int } S) \), then \( S \) is a union of two or fewer convex sets.

Proof. Without loss of generality, assume \( S \) is connected, for otherwise the result is trivial. Let \( x \in S \sim \text{cl } (\text{int } S) \neq \emptyset \), and let \( N \) be a convex neighborhood of \( x \) disjoint from \( \text{int } S \). Since \( S \) is connected, \( x \) is not an isolated point of \( S \), and it is clear that \( N \cap S \) contains at least one segment.

We examine the maximal segments of \( N \cap S \) (i.e., the segments which are not proper subsets of segments in \( N \cap S \)). It is easy to show that \( N \cap S \) has at most two maximal segments, for otherwise, the 3-convexity of \( S \) together with the simple connectedness of \( \text{cl } S \) would yield an open region in \( \text{cl } S \cap N \). Since by Lemma 3 the points of \( \text{int } (\text{cl } S) \sim S \) are collinear, this would imply that \( N \cap S \) has interior points, clearly impossible by our choice of \( N \).

In case \( N \cap S \) has exactly two maximal segments, an argument similar to the one above may be used to show that any point of \( S \)
lies on one of the corresponding lines, and \( S \) is a union of two segments (possibly infinite). If \( N \cap S \) has just one segment, let \( K \), denote a maximal convex subset of \( S \) containing it, and let \( K_2 = \text{conv}(S \sim K) \). Again using the facts that \( N \) contains no interior points of \( \text{cl} \ S \) and \( \text{cl} \ S \) is simply connected, it is not hard to show that \( K_1 \subseteq S \), and \( S = K_1 \cup K_2 \), completing the proof.

Theorems 2 and 3 show that a decomposition is possible when \( (\text{cl} \ S \sim S) \cap \text{bdry} (\text{cl} \ S) = \emptyset \), and card \( Q = n \) for \( n \) an odd integer, \( n > 1 \), then \( S \) is expressible as a union of four or fewer convex sets.

**Theorem 2.** If \( (\text{cl} \ S \sim S) \cap \text{bdry} (\text{cl} \ S) = \emptyset \), and card \( Q = n \) for \( n \) an odd integer, \( n > 1 \), then \( S \) is expressible as a union of four or fewer convex sets.

**Proof.** Clearly the hypothesis implies that \( \text{cl} \ S = \text{cl} \left( \text{int} \ S \right) \). By the Corollary to Lemma 2, \( \text{cl} \ S \sim S \subseteq \ker (\text{cl} \ S) \), and by Lemma 3, every component of \( \text{cl} \ S \sim S \) is either an isolated point or an interval. Since card \( Q \geq 3 \) and (by the corollary to Lemma 5) no three members of \( Q \) can be collinear, Lemma 4 implies that \( \text{cl} \ S \sim S \) cannot contain an interval. Hence \( \text{cl} \ S \sim S \) consists of exactly one isolated point \( p \) in \( \ker (\text{cl} \ S) \).

Select \( q \in Q \) in the following manner: If \( p \in \text{conv} Q \), choose \( q \in Q \) so that the line \( L(p, q) \) contains no other member of \( Q \). (Clearly this is possible since card \( Q \) is odd and no three members of \( Q \) are collinear.) If \( p \in \text{conv} Q \), let \( \{e_i; 1 \leq i \leq n\} \) denote the edges of \( \text{conv} Q \), \( \{E_i; 1 \leq i \leq n\} \) the corresponding lines, with \( \text{conv} Q \) in the closed halfspace \( \text{cl} (E_{i \alpha}) \) for each \( i \). Then \( p \in E_{i \alpha} \) for exactly one \( i \), for otherwise, if \( p \in E_{i \alpha} \cap E_{i \beta} \), then \( \text{int} \text{conv} ((p) \cup e_i \cup e_\beta) \) would contain an inc point of \( \text{cl} S \), clearly impossible since \( (p) \cup e_i \cup e_\beta \subseteq \ker (\text{cl} S) \) and \( \text{conv} ((p) \cup e_i \cup e_\beta) \subseteq \text{cl} S \). Thus we may choose some \( q \in Q \) so that \( p \in \text{cl} E_{i \alpha} \) for each edge \( e_i \) containing \( q \). Then \( (p, q) \) contains points of \( \text{conv} Q \). Since all points of \( L(p, q) \cap \text{conv} Q \) are on the open ray at \( p \) emanating through \( q \), Lemma 5 implies that \( L(p, q) \) contains no other members of \( Q \) (and in fact \( p \) cannot lie on any line \( E_i \)).

To review, in either case we have chosen \( q \) in \( Q \) so that \( L(p, q) \) contains no other member of \( Q \) and \( (p, q) \) contains points of \( \text{conv} Q \). Letting \( L_1, L_2 \) denote distinct open halfspaces determined by \( L = L(p, q) \), define \( A = \text{cl} (S \cap L_1) \), \( B = \text{cl} (S \cap L_2) \). If \( W_1, W_2 \) are leaves of \( \text{cl} S \) containing \( q \), then by Lemma 6, \( W_1 \) and \( W_2 \) are in opposite closed halfspaces determined by \( L \), say \( W_1 \subseteq \text{cl} L_1 \), \( W_2 \subseteq \text{cl} L_2 \).

Let \( R_1, R_2 \) denote opposite closed rays at \( p \), \( R_1 \cup R_2 = L \), labeled so that \( q \in R_2 \). Each of \( R_1 \cap S, R_2 \cap S \) is an interval by the 3-convexity of \( S \). Points of \( R_1 \cap S \) necessarily lie in \( A \cap B \), for otherwise
would contain an lnc point of \( \text{cl} S \), clearly impossible. If there are any points of \( R_2 \cap S \) not in \( A \cap B \), without loss of generality we may assume such points lie in \( W_1 \) and hence in \( A \sim B \). Then \( R_2 \cap S \subseteq A \).

By Case 4 in Theorems 2 and 3 of [2], \( \text{cl} (S \sim W_2) \) is a union of two closed convex sets \( C_i, C_2 \), selected as in Valentine's proof. Since \( A = \text{cl} [\text{cl} (S \sim W_2) \cap L_i] \), \( A \) is the union of the two closed convex sets \( A_i, A_2 \), where \( A_i = \text{cl} (C_i \cap L_i), \ i = 1, 2 \). Moreover, \( (R_2 \cap S) \cup (p, q] \) lies in one of these sets, say \( A_i \), and \( R_2 \sim (p, q] \) is either in \( A_i \) or in \( A_2 \).

Using an identical argument for \( B \) and \( \text{cl} (S \sim W_1) \), we may write \( B \) as a union of two closed convex sets \( B_i, B_2 \) with \( (R_1 \cap S) \cup (p, q] \) in \( B_i \), and \( R_2 \sim (p, q] \) disjoint from \( B \).

At last, define sets \( A_i', A_2', B_i', B_2' \) in the following manner: If \( (R_2 \cap S) \sim (p, q] \subseteq A_2 \), let

\[
A_i' \equiv A_i \sim R_2, \quad A_2' \equiv A_2 \sim R_1, \\
B_i' \equiv B_i \sim R_1, \quad B_2' \equiv B_2 \sim R_2.
\]

And if \( (R_2 \cap S) \sim (p, q] \subseteq A_i \), let

\[
A_i' \equiv A_i \sim R_1, \quad A_2' \equiv A_2 \sim R_2, \\
B_i' \equiv B_i \sim R_2, \quad B_2' \equiv B_2 \sim R_1.
\]

We assert that these are convex subsets of \( S \) whose union is \( S \): Clearly each is a convex subset of \( S \), and \( S \sim L \) is contained in their union. For \( (R_2 \cap S) \sim (p, q] \subseteq A_2 \), \( R_2 \cap S \subseteq A_2' \cup B_2' \), \( R_1 \cap S \subseteq A_1' \). For \( (R_1 \cap S) \sim (p, q] \subseteq A_i \), \( R_1 \cap S \subseteq A_i' \), \( R_1 \cap S \subseteq B_i \). Hence in either case \( S \cap L \) is contained in the union of these sets, and \( S = A_1' \cup A_2' \cup B_1' \cup B_2' \), completing the proof of the theorem.

**Theorem 3.** If \( (\text{cl} S \sim S) \cap \text{bdry} (\text{cl} S) = \emptyset \) and \( \text{card} \ Q = n \geq 0 \), where \( n \) (possibly infinite) is not an odd integer greater than one, then \( S \) is expressible as a union of four or fewer convex sets.

**Proof.** If \( S \) is not connected, the result is trivial. Otherwise, by Theorem 3 of Valentine [2], \( \text{cl} S \) may be expressed as a union of two or fewer closed convex sets \( A, B \). Using Lemma 3, let \( L \) be a line containing \( \text{cl} S \sim S \), \( L_1, L_2 \) the corresponding open halfspaces. Since \( S \) is 3-convex and \( A \) is convex, \( S \cap A \) is 3-convex, and hence \( (S \cap A) \cap L \) has at most two components, say \( C_1, C_2 \). Let \( R_1, R_2 \) denote opposite rays on \( L \) with \( C_1 \subseteq R_1, \ C_2 \subseteq R_2 \).

Define

\[
A_1 \equiv (A \cap S \cap \text{cl} L_i) \sim R_1, \\
A_2 \equiv (A \cap S \cap \text{cl} L_i) \sim R_2.
\]
Then $A_1$, $A_2$ are convex subsets of $S$ whose union is $A \cap S$.

Similarly define convex sets $B_1$, $B_2$ whose union is $B \cap S$. Clearly $S = A_1 \cup A_2 \cup B_1 \cup B_2$, the desired result.

**Corollary.** If $(\text{cl } S \sim S) \cap \text{bdry (cl } S) = \emptyset$, then $S$ is expressible as a union of four or fewer convex sets. The number four is best possible.

That the number four in the corollary is best possible is evident from Example 1.

**Example 1.** Let $S$ be the set in Figure 1, with $p \in S$. Then $S$ is not expressible as a union of fewer than four convex sets.

\[
\begin{align*}
|| &  \\
| & |
\end{align*}
\]

**Figure 1**

The preceding theorems allow us to obtain the following decomposition for open sets.

**Theorem 4.** If $S$ is open, then $S$ is expressible as a union of four or fewer convex sets. The result is best possible.

**Proof.** Let $T = S \cup \text{bdry (cl } S)$. Applying arguments identical to those used in the proofs of Theorems 2 and 3, $T$ is expressible as a union of four or fewer convex sets $A_i$, $1 \leq i \leq 4$. Define $B_i = A_i \cap S$, $1 \leq i \leq 4$. We assert that each $B_i$ is convex. The proof follows:

By Valentine’s results, cl $S$ is expressible as a union of three or fewer closed convex sets $C_j$, $1 \leq j \leq 3$, each consisting of an appropriate selection of leaves of cl $S$, together with conv $Q$. Examining the proofs of Theorems 2 and 3, it is clear that each $A_i$ may be considered as a subset of some $C_j$ set. Thus we may assume $B_i \subseteq A_i \subseteq C_i$ for an appropriate $C_i$.

Let $x, y \in B_i$, and let $p \in (x, y)$ to show $p \in B_i$. If $x$ (or $y$) is interior to some leaf $W$, then $W \subseteq C_i$, $y$ sees a neighborhood of $x$ via
$C_1$, and $p$ is interior to $\text{cl} S$. Since $p \in A_i$ and $p \in \text{bdry} (\text{cl} S)$, $p$ is in $A_i \cap S = B_i$. A similar argument holds if $x$ (or $y$) is interior to $\text{conv} Q$. Since neither $x$ nor $y$ is in $\text{bdry} (\text{cl} S)$, the only other possibility to consider is the case in which $x, y \in \text{bdry} (\text{conv} Q) \sim Q \subseteq \ker (\text{cl} S)$. Then $x \in \text{int} (\text{cl} S), y \in \ker (\text{cl} S), y$ sees some neighborhood of $x$ via $\text{cl} S$, and $p \in \text{int} (\text{cl} S)$. Again $p \in A_i \cap S = B_i$ and $B_i$ is indeed convex. Thus $S$ is the union of the convex sets $B_i, 1 \leq i \leq 4$, and the theorem is proved.

To see that the number four is best possible, let $S$ denote the set in Example 1 with its boundary deleted. Then $S$ is an open 3-convex set not expressible as a union of fewer than four convex sets.

4. The general case. It remains to investigate the case for $S$ an arbitrary 3-convex subset of the plane. A decomposition of $S$ into six convex sets may be obtained from our previous results, together with Theorems 5 and 6, which deal with the case for $(\text{cl} S \sim S) \subseteq \text{bdry} (\text{cl} S)$.

The following result by Lawrence, Hare, and Kenelly [1, Theorem 2] will be useful:

**Lawrence, Hare, Kenelly Theorem.** Let $T$ be a subset of a linear space such that each finite subset $F \subseteq T$ has a $\lambda$-partition, $\{F_1, \ldots, F_k\}$, where $\text{conv} F_i \subseteq T, 1 \leq i \leq k$. Then $T$ is a union of $k$ convex sets.

**Theorem 5.** If $\text{cl} S$ is convex and $(\text{cl} S \sim S) \subseteq \text{bdry} (\text{cl} S)$, then $S$ is a union of three or fewer convex sets. The bound of three is best possible.

**Proof.** Consider the collection of all intervals in $\text{bdry} (\text{cl} S)$ having endpoints in $S$ and some relatively interior point not in $S$. Each interval determines a line $L$, and by the 3-convexity of $S$, $L \cap S$ has exactly two components. Let $\mathcal{L}$ denote the collection of all such lines. By the Lawrence, Hare, Kenelly Theorem, without loss of generality we may assume that $\mathcal{L}$ is finite. Hence the set $\bigcup \{L \cap S: L \in \mathcal{L}\}$ has finitely many components, and we may order these components in a clockwise direction along $\text{bdry} (\text{cl} S)$. If $c_i$ denotes the $i$th component in our ordering, let

$$A' = \{c_i: i \text{ odd}, i < n\},$$

$$B' = \{c_i: i \text{ even}, i < n\},$$

$$C' = \{c_n\}.$$

Define
We assert that $A, B, C$ are convex sets whose union is $S$. The proof follows:

For $x, y$ in $A$, if $[x, y]$ contains any point of $\text{int}(\text{cl } S)$, then $(x, y) \subseteq \text{int}(\text{cl } S) \subseteq A$, and $[x, y] \subseteq A$. Otherwise, $[x, y]$ lies in the boundary of the convex set $\text{cl } S$. If the corresponding line $L(x, y)$ is not in $\mathcal{L}$, the result is clear, so suppose $L(x, y) \in \mathcal{L}$. Then $x, y$ must lie in the same $C_i$ set for some $i$ odd, $i < n$, again giving the desired result. Hence $A$ is convex. Similarly, $B, C$ are convex. It is easy to see that $A \cup B \cup C = S$ and the proof is complete.

The surprising fact that three is best possible is illustrated by Example 2.

**Example 2.** Let $S$ denote the set in Figure 2, where dotted lines represent segments not in $S$. Then $S$ is not expressible as a union of fewer than three convex sets.

![Figure 2](image)

**Theorem 6.** If $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, then $S$ is a union of four or fewer convex sets. The number four is best possible.

**Proof.** We assume that $S$ is connected and $\text{cl } S = \text{cl } (\text{int } S)$, for otherwise $S$ is a union of two convex sets. Furthermore, by the Lawrence, Hare, Kenelly Theorem, we may assume that $\text{cl } S$ has finitely many leaves, and hence card $Q = n$ is finite. Notice also that since $\text{cl } S$ is simply connected and $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, $S$ is simply connected.

For the moment, suppose $3 \leq n$. Order the points of $Q$ in a clockwise direction along $\text{bdry } (\text{conv } Q)$, letting $W_i$ denote the leaf of $\text{cl } S$ determined by $n$ points $q_i, q_{i+1}$ (where $n + 1 = 1$). By Valentine's results in [2], for any pair of disjoint leaves $W_i, W_j$ of $\text{cl } S$, the set $R = \text{conv } Q \cup W_i \cup W_j$ is a closed convex set. (In case there are no disjoint leaves, $n = 3, W_j = \emptyset$, and $R = \text{conv } Q \cup W_i$ is closed and convex.) Consider the collection of intervals in $\text{bdry } R$ having end-
points $x, y$ in $S$ and some relatively interior point $p$ not in $S$. Either such an interval is contained in one leaf, or $x \in W_i \cup \text{conv } Q$, $y \in W_j \cup \text{conv } Q$. We examine the latter case. It is clear that for an appropriate labeling, $j = i + 2$, so to simplify notation, say $i = 1$, $j = 3$, and $L(x, y)$ supports $W_2$. Clearly not both $x, y$ can lie in conv $Q$, for then $p \in \text{int } S \subseteq S$. However, we assert that either $x$ or $y$ must lie in conv $Q$ and that $W_2 \cap S$ is convex. The proof follows:

Assume that $x$ is not an lnc point and that $x < p$ or $y < q$, where $q_2, q_3$ are the lnc points in $W_1 \cap W_2$, $W_1 \cap W_3$ respectively. Then $q_2 \leq y$. For $w$ in $W_2 \cap S$, $w$ cannot see $x$ via $S$, so necessarily $w$ sees $y$ via $S$, by the 3-convexity of $S$. This implies that $y \leq q_3$ (for otherwise $q_3$ could not be an lnc point for $cl S$). Moreover, since no two points of $W_2 \cap S$ see $x$ via $S$, the 3-convexity of $S$ together with the convexity of $W_2$ imply that $W_2 \cap S$ is convex.

Here we digress briefly for future reference. The set $L(x, y) \cap S$ has two components, and by the above argument, one must lie in the interval $[q_2, q_3]$, the other in $W_1 \sim Q$ (by our labeling). For general $W_{i-1}, W_{i+1}$ (disjoint if and only if $n > 3$), we let $T_i$ denote the connected set of all the somewhat troublesome points $y$ in $[q_i, q_{i+1}] \cap S$ having the above property. That is, there exist points $x$ in exactly one of $(W_{i-1} \cap S) \sim Q$, $(W_{i+1} \cap S) \sim Q$ for which $[x, y] \not\subseteq S (n + 1 \equiv 1)$.

Continuing the argument, delete $W_2$ and consider the 3-convex set $(S \sim W_2) \cup (S \cap L(x, y))$. Renumbe the lnc points and leaves for this set so that the old $W_1$ and $W_2$ are contained in the new leaf $U_i$. Since we are assuming card $Q$ is finite, repeating the procedure finitely many times yields a 3-convex set $S_0$ having the following property: For $V_i, V_j$ disjoint leaves of $cl S_0$, $x$ in $V_i \cap S_0$, $y$ in $V_j \cap S_0$, then $[x, y] \not\subseteq S_0$. In addition, without loss of generality we may assume that for each leaf $V_i$ of $cl S_0$, $V_i \cap S_0$ is not convex, for otherwise, $V_i$ may be deleted by the above procedure.

To avoid confusion, let $Q_0$ denote the set of lnc points of $cl S_0$, $Q_0 \subseteq Q$, card $Q_0 = m = n$. For $3 \leq m$, let $V_i$ denote the leaf determined by lnc points $p_i, p_{i+1}$ in $Q_0$ (where $p_{m+1} = p_0$). For $m = 2$, let $V_1, V_2$ denote the leaves of $cl S_0$ as defined in the introduction to this paper. If $0 \leq m \leq 1$, let $V_1 = V_2 = cl S_0$.

For each $i$, consider the collection of intervals in bdry $V_i$ having endpoints in $V_i \cap S_0$ and some relatively interior point not in $S_0$. Each interval determines a line $L$, and for $m \neq 1$, $L \cap V_i \cap S_0$ has exactly two components, each in bdry $V_i$. In case $m = 1$, an obvious adjustment may be made (by deleting any ray of $L$ which contains interior points of $cl S_0$) to yield the same result. For each $i$, let $\mathcal{L}_i$ denote the collection of all such lines. Again using the Lawrence, Hare, Kenelly Theorem, we may assume that each $\mathcal{L}_i$ is finite. The set $\bigcup \{L \cap V_i \cap S_0 : L \in \mathcal{L}_i\}$ has finitely many components, and we
may order them in a clockwise direction along bdry $V_i$. Let $c_{ij}$ denote the $j$th such component for $V_i$, and let $\mathcal{C}_i$ denote the collection of all the $c_{ij}$ sets corresponding to $V_i$. Clearly each $c_{ij}$ is either a point, an interval, or the union of two noncollinear intervals. Moreover, for $m \geq 2$, no components for $V_i$, $V_{i+1}$ may have common points. (Such a point would necessarily be $p_{i+1}$, and if $s_i \in V_i \cap S_0$, $s_{i+1} \in V_{i+1} \cap S_0$ with some interior point of each of $[s_i, p_{i+1}], [p_{i+1}, s_{i+1}]$ not in $S_0$, then $s_i, p_{i+1}, s_{i+1}$ would be visually independent via $S_0$, clearly impossible.) Otherwise, $3 \leq m$ and an inductive argument may be used to show that $B_0$ is in $S$.

For each $V_i$, select every $c_{ij}$. That is, select the members of $\mathcal{C}_i$ having second subscript even. No two components selected correspond to the same line, and for $m \neq 0$, we have chosen one component corresponding to each line in $\mathcal{L}_i$. If $m = 0$, without loss of generality we may assume $\mathcal{C}_i$ is ordered in a clockwise direction from some point in $Q \cap cl S_0 \neq \emptyset$. In case no component has been chosen for some line $L$ in $\mathcal{L}_i$, then $L$ must contain points of both the first and last members of $\mathcal{C}_i$, and by a previous argument, one of these components must lie in $conv Q$.

For $m \neq 1$, since $V_i$ is convex, it is easy to show that $conv \{c_{ij}: 1 \leq j\}$ is a subset of $S_i$ (and this is certainly true even if $cl S_0$ is convex). We will prove that $B_0 = conv \{c_{ij}: 1 \leq i \leq m, 1 \leq j\}$ is in $S_0$ and hence in $S$. If $cl S_0$ is convex (or empty) the result is immediate, so assume $cl S_0$ has at least one lnc point. For convenience, in case $cl S_0$ has only one lnc point, call it $p_2$, and let $V_1 = V_2$ follow $p_2$ in our clockwise ordering.

Recall that $V_i \cap S_0$ is not convex for any $i$, so no $\mathcal{C}_i$ is empty. Let $c_v$ denote the last member of $\mathcal{C}_i$, selected, $x$ the last point of $cl c_v$ (relative to our ordering). If $x \neq p_2$, let $L = L(x, p_2)$. Otherwise, by the 3-convexity of $S_0$, $c_v = \{p_2\}$, and in this case let $L$ denote the corresponding member of $\mathcal{L}_i$. Let $L_{i_1}, L_{i_2}$ be the open halfspaces determined by $L$, with $Q \subseteq cl L_{i_1}$. Since $p_2$ is an lnc point of $S_0$ and $S_i$ is 3-convex, it is clear that at most one member of $\mathcal{C}_i$, namely $c_{21}$, may contain points in $L_{i_2}$. We assert that $c_v$ sees $c_{22}$ via $S_v$. The proof follows:

In case $L \in \mathcal{L}_i$, $L \cap V_1 \cap S_0$ has two components, each in bdry $V_i$, and one of these must be $\{p_2\}$. Then by the 3-convexity of $S_0$, $c_{22} \subseteq L_{i_1}$ and $c_v$ sees $c_{22}$ via $S_v$. Otherwise, $c_v \sim \{x\} \subseteq L_{i_1}$. If $x \notin S_0$, then since $c_{22} \subseteq cl L_{i_1}$, it is clear that $c_v$ sees $c_{22}$ via $S_v$. If $x \in S_0$ and $p_2 \in S_0$, then again the result is clear. If $x \in S_0$ and $p_2 \notin S_0$, then $c_{22} \subseteq L_{i_1}$ and $c_v$ sees $c_{22}$ via $S_v$, finishing the argument.

In case $V_i, V_{i+1}$ are the only leaves for $cl S_0$, $V_i \neq V_{i+1}$, then repeating the argument for the last member of $\mathcal{C}_i$ and $c_{12}$ and using the fact that $S_0$ is simply connected, we have $B_0 \subseteq S_0 \subseteq S$. (If $V_1 = V_2$, the result is immediate.) Otherwise, $3 \leq m$ and an inductive argument may be used to show that $B_0$ is in $S$. 

DECOMPOSITION FOR 3-CONVEX SETS 53
Using Valentine's results, write \( \text{cl } S \) as a union of three or fewer convex sets \( A_j, j = 1, 2, 3 \), where for \( n \) odd
\[
A_1 = \bigcup \{ W_i : i \text{ odd}, \ i < n \} \cup \text{conv } Q ,
A_2 = \bigcup \{ W_i : i \text{ even}, \ i < n \} \cup \text{conv } Q ,
A_3 = W_n \cup \text{conv } Q ,
\]
and for \( n \) even
\[
A_1 = \bigcup \{ W_i : i \text{ odd}, \ i \leq n \} \cup \text{conv } Q ,
A_2 = \bigcup \{ W_i : i \text{ even}, \ i \leq n \} \cup \text{conv } Q ,
A_3 = \emptyset .
\]
Define \( B_j = S \cap [A_j \sim ((\text{bdry } S) \cap B_0)], \ j = 1, 2, 3 \).

Recall the \( T_i \) sets defined previously, \( T_i \subseteq [q_i, q_{i+1}] \equiv W_i, \ 1 \leq i \leq n \).

To simplify notation, let \( L_i = L(q_i, q_{i+1}) \), and define sets \( F_i, G_i \) in the following manner: For \( i \) even, let \( F_i = T_i \) if points from both components of \( L_i \cap S \) are in \( B_i \), \( F_i = \emptyset \) otherwise. Similarly for \( i \) odd, let \( F_i = T_i \) if points from both components of \( L_i \cap S \) are in \( B_i \), \( F_i = \emptyset \) otherwise. For \( i = 1, i = n - 1 \), let \( G_i = T_i \) if points from both components of \( L_i \cap S \) are in \( B_i \), \( G_i = \emptyset \) otherwise. By previous remarks, at least one of \( G_i, F_i \) is empty, and at least one of \( G_{n-1}, F_{n-1} \) is empty.

Define
\[
D_1 = B_1 \sim \bigcup \{ F_j : i \text{ even} \} ,
D_2 = B_1 \sim \bigcup \{ F_j : i \text{ odd} \} ,
D_3 = B_3 \sim \bigcup \{ G_i, G_{n-1} \} .
\]
Finally, letting \( P = \{ F_i \cap F_j : 1 \leq i < j \leq n \} \cup \{ G_i \cap F_j : i = 1, n - 1, 1 \leq j \leq n \} \), define \( D_0 = \text{conv } (B_0 \cup P) \). We assert that the sets \( D_j, 0 \leq j \leq 3 \), are convex sets whose union is \( S \). The proof follows:

Suppose that one of the sets \( D_1, D_2, D_3 \), say \( D_1 \), is not convex to obtain a contradiction. Choose \( x, y \) in \( D_1 \) for which \( [x, y] \not\subseteq D_1 \). It is clear that \( [x, y] \subseteq \text{bdry } (\text{cl } D_1) = \text{bdry } A_1 \). Furthermore, \( x, y \) cannot both belong to \( W \sim Q \) for any leaf \( W \) of \( \text{cl } S \), for otherwise they would belong to the same leaf of \( \text{cl } S_0 \) and one of \( x, y \) would lie in \( (\text{bdry } S) \cap B_0 \) and hence not in \( D_1 \), a contradiction. Employing a previous argument, the set \( L(x, y) \cap S \) has two components, each having points in \( B_1 \), and one of these components is the set \( [q_0, q_{i+1}] \cap S = T_i \) for some \( i \) even \((n + 1 = 1) \). Let \( R_i \) denote the other component of \( L(x, y) \cap S \). If \( R_i \cap B_0 = \emptyset \), then \( R_i, T_i \) would lie on the boundary of a leaf of \( \text{cl } S_0 \), \( R_i \subseteq B_0, \ T_i \subseteq B_1 \), and \( [x, y] \subseteq T_i \subseteq D_1 \), a contradiction. Thus \( R_i \cap B_0 = \emptyset \) and \( R_i \subseteq D_1 \). However, this implies that one of \( x, y \) must lie in \( F_i \) and not in \( D_1 \), again a contradiction. Our assumption is false and \( D_1 \) is convex. Similarly \( D_2, D_3 \) are convex,
and clearly each is a subset of $S$.

It remains to show that the convex set $D_0$ lies in $S$. Examining the set $P$, if $F_i \cap F_j \neq \emptyset$ for some $i \neq j$ (or if $G_i \cap F_j \neq \emptyset$), then $F_i = T_i$, $F_j = T_j$, for an appropriate labeling $j = i + 1$, and $F_i \cap F_{i+1} = \{q_{i+1}\} \subseteq S$. We will show that for each $z$ in $B_o$, $[q_{i+1}, z] \subseteq S$. The proof follows:

We have seen that $W_i \cap S, W_i \cap S$ are both convex, so for every $z$ in one of these sets, $[q_{i+1}, z] \subseteq S$. Moreover, we assert that the components of $L(q_i, q_{i+1}) \cap S, L(q_{i+1}, q_{i+2}) \cap S$ not in $\text{conv} Q$, call them $R_i, R_i^j$, are disjoint from $B_o$: If $R_i \cap B_o \neq \emptyset$, then by an earlier argument, $R_i \subseteq B_o$, $T_i \cap B_o = \emptyset$, $T_i \subseteq D_i \cap D_{i+1} \cap D_o$ and $F_i = \emptyset$, a contradiction. Hence for $z$ in $B_o \sim (W_i \cup W_i \cup 1)$, $(q_{i+1}, z) \subseteq \text{int} S$, and $[q_{i+1}, z] \subseteq S$ whenever $z \in B_o$, the desired result.

Certainly for $q_i, q_j, q_k$ in $P \subseteq S$, $\text{conv} \{q_i, q_j, q_k\} \subseteq S$.

By Carathéodory’s theorem in the plane, to prove that $D_0 = \text{conv} (B_o \cup P)$ is in $S$, it is sufficient to show that the convex hull of any three points of $B_o \cup P$ is in $S$, and from the remarks above, clearly we need only show $\text{conv} \{q_i, q_j, z\} \subseteq S$ for $q_i, q_j$ in $P$, $z$ in $B_o$. However, since $S$ is simply connected and $\text{bdry} (\text{conv} \{q_i, q_j, z\}) \subseteq S$, $\text{conv} \{q_i, q_j, z\} \subseteq S$ and $D_o \subseteq S$, the desired result.

Finally, by inspection, each $F_i \neq \emptyset$ fails to belong to at most one of the sets $D_i, D_{i+1}$, $D_o$. Points in intersecting $F_i$ sets are in $D_o$, so $\bigcup \{D_j: 0 \leq j \leq 3\} = S$ and the argument for $3 \leq \text{card} Q$ is complete.

To finish the proof, we must examine the cases for $0 \leq \text{card} Q \leq 2$. If $\text{card} Q = 2$ or if $\text{card} Q = 1$ and $S \sim Q$ is connected, then let $W_i, W_2$ denote the corresponding leaves of $cl S$, and use a simplified version of the previous proof to define $B_o, B_1, B_2$. If one of $B_1, B_2$, say $B_1$, is not convex, then letting $T = W_1 \cap W_2 \cap S, W_2 \cap S = B_2$ is convex, $T \subseteq B_2$, and $B_o, B_1 \sim T, B_2$ are the desired convex sets.

In case $\text{card} Q = 1$ and $S \sim Q$ is not connected, then for $W_1, W_2$ the corresponding leaves of $cl S$, each of $W_1 \cap S, W_2 \cap S$ is convex. For $\text{card} Q = 0$, the result follows from Theorem 5, and the proof of Theorem 6 is complete.

The number four in Theorem 6 is best possible, as the following example illustrates.

Figure 3
EXAMPLE 3. Let $S$ denote the set in Figure 3, where dotted segments are in $\text{bdry} \ (\text{cl } S) \sim S$. Then $S$ is a union of no fewer than four convex sets.

At last, using Theorem 6, we have a decomposition theorem for $S$ an arbitrary 3-convex subset of the plane.

THEOREM 7. The set $S$ is a union of six or fewer convex sets. The result is best possible.

Proof. By earlier comments, we may assume that $S$ is connected, $\text{cl } S = \text{cl} \ (\text{int } S)$, and $Q$ is finite. Furthermore, we assume $\text{int} \ (\text{cl } S) \sim S \neq \emptyset$, for otherwise the result is an immediate consequence of Theorem 6. Let $T = S \cup \text{bdry} \ (\text{cl } S)$, and let $L$ be the line containing $\text{cl } T \sim T$ described in Theorem 2 or Theorem 3 (whichever is appropriate). Clearly $L$ may be chosen to contain an lnc point $q$ of $\text{cl } S$. If $L_1, L_2$ are the corresponding open halfspaces, then each of $T_1 = \text{cl} \ (T \cap L_1), T_2 = \text{cl} \ (T \cap L_2) = \text{cl} \ (S \cap L_2)$ is 3-convex.

Define $S_i = T_i \cap S, i = 1, 2$. We assert that each $S_i$ is 3-convex: For $x, y, z$ in $S_i = T_i \cap S$, assume $[x, y] \subseteq S_i$. If $x$ or $y$ is in $L_i$, then certainly $[x, y] \subseteq L_i \cap S \subseteq T_i$, and $[x, y] \subseteq S_i$. If $x, y$ are on $L$, then since no lnc points of the closed set $T_i$ are on $L$, $x, y$ lie in the same leaf of $T_i$, and $[x, y] \subseteq T_i \cap S = S_i$. Thus $S_i$ is 3-convex. Similarly $S_2$ is 3-convex. Moreover, $(\text{cl } S_i \sim S_i) \subseteq \text{bdry} \ (\text{cl } S_i), i = 1, 2$.

Using Theorem 6, we will show that each $S_i$ is a union of three convex sets: By the proofs of Theorems 2 and 3, $\text{cl } S_i = T_i$ is a union of two convex sets $A_i, A_{i+1}$, and each $A_i$ may be considered a subset of an appropriate $C_i$ set, $1 \leq j \leq 3$, where the $C_i$ sets are those described in Valentine's paper with $\text{cl } S = C_1 \cup C_2 \cup C_3$. In case $T_i$ has one leaf or an even number of leaves, then clearly the proof of Theorem 6 may be used to write $S_i$ as a union of three convex sets. If $T_i$ has $n$ leaves for $n$ odd, $n > 1$, let $V$ be the leaf of $T_i$ bounded by $L_q$, $q \in Q \cap L \subseteq A_i \cap A_{i+1}$. Order the lnc points of $T_i$ in a clockwise direction so that $V$ is determined by $q_n, q_1$, and let $U_i, U_{i+1}$ denote the closed subsets of $V$ bounded by $L(q_n, q), L(q, q_1)$ respectively. Treating $U_1, \ldots, U_n, U_{n+1}$ as leaves of $T_i$, $U_i$ determined by lnc points $q_i, q_{i+1}$, $1 \leq i < n$, the proof of Theorem 6 may be applied to write $S_i$ as a union of three convex sets. (Of course, in defining $B_i$, points of $V$ in $S'_i$ belong to the same leaf of $S'_i$.)

By a parallel argument $S_2$ is a union of three convex sets, and $S = S_1 \cup S_2$ is a union of six or fewer convex sets, finishing the proof of the theorem.

Our final example shows that the bound of six in Theorem 7 is
best possible.

**Example 4.** Let $S$ be the set in Figure 4, with dotted segments in $\text{bdry} \ (\text{cl} \ S) \sim S$ and $p \in \text{int} \ (\text{cl} \ S) \sim S$. Then $S$ cannot be expressed as a union of fewer than six convex sets.

![Figure 4](image)

**References**


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