DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF THE PLANE

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Let $S$ be a 3-convex subset of the plane. If $(\text{cl } S \sim S) \subseteq \text{int } (\text{cl } S)$ or if $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, then $S$ is expressible as a union of four or fewer convex sets. Otherwise, $S$ is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let $S$ be a subset of $R^d$. Then $S$ is said to be 3-convex iff for every three distinct points in $S$, at least one of the segments determined by these points lies in $S$. Valentine [2] has proved that for $S$ a closed, 3-convex subset of the plane, $S$ is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set $S$ to be closed. The following definitions and results obtained by Valentine will be useful.

For $S \subseteq R^d$, a point $x$ in $S$ is a point of local convexity of $S$ iff there is some neighborhood $U$ of $x$ such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If $S$ fails to be locally convex at some point $q$ in $S$, then $q$ is called a point of local nonconvexity (Inc point) of $S$.

Let $S$ be a closed, connected, 3-convex subset of the plane, and let $Q$ denote the closure of the set of isolated Inc points of $S$. Valentine has proved that for $S$ not convex, then $\text{card } Q \geq 1$, $Q$ lies in the convex kernel of $S$, and $Q \subseteq \text{bdry } (\text{conv } Q)$. An edge of $\text{bdry } (\text{conv } Q)$ is a closed segment (or ray) in $\text{bdry } (\text{conv } Q)$ whose endpoints are in $Q$. We define a leaf of $S$ in the following manner: In case $\text{card } Q \geq 3$, let $L$ be the line determined by an edge of $\text{bdry } (\text{conv } Q)$, $L_1, L_2$ the corresponding open halfspaces. Then $L$ supports $\text{conv } Q$, and we may assume $\text{conv } Q \subseteq \text{cl } (L_1)$. We define $W = \text{cl } (L_2 \cap S)$ to be a leaf of $S$. For $2 \geq \text{card } Q \geq 1$, constructions used by Valentine may be employed to decompose $S$ into two closed convex sets, and we define each of these convex sets to be a leaf of $S$.

By Valentine’s results, every point of $S$ is either in $\text{conv } Q$ or in some leaf $W$ of $S$ (or both), and every leaf $W$ is convex. Moreover, Valentine obtains his decomposition of $S$ by showing that for any collection $\{s_i\}$ of disjoint edges of $\text{bdry } (\text{conv } Q)$, with $\{W_i\}$ the corresponding collection of leaves, $\text{conv } Q \cup (\bigcup W_i)$ is closed and convex.

Finally, we will use the following familiar definitions: For $x, y$ in $S$, we say $x$ see $y$ via $S$ iff the corresponding segment $[x, y]$ lies in $S$. A subset $T$ of $S$ is visually independent via $S$ iff for every
Throughout the paper, \( S \) will denote a 3-convex subset of the plane, \( Q \) the closure of the set of isolated lnc points of \( \text{cl} \ S \).

2. Preliminary lemmas. The following sequence of lemmas will be useful in obtaining the desired representation theorems. We begin with an easy result.

**Lemma 1.** \( \text{cl} \ S \) is 3-convex.

*Proof.* Let \( x, y, z \) be distinct points in \( \text{cl} \ S \) and select disjoint sequences \((x_i), (y_i), (z_i)\) in \( S \) converging to \( x, y, z \) respectively. For each \( i \), one of the corresponding segments is in \( S \), and for one pair, say \( x \) and \( y \), infinitely many of the segments \([x_i, y_i]\) lie in \( S \). Since these segments converge to \([x, y]\), \([x, y]\) lies in \( \text{cl} \ S \).

The remaining lemmas are technical in nature. Lemmas 2, 3, and 4 reveal various pleasant features of \( \text{int}(\text{cl} \ S) \sim S \), while 5 and 6 are concerned with lnc points of \( \text{cl} \ S \).

**Lemma 2.** If \( p \in \text{int}(\text{cl} \ S) \sim \text{ker}(\text{cl} \ S) \neq \emptyset \), then \( p \in S \).

*Proof.* Since \( p \in \text{ker}(\text{cl} \ S) \), there is some point \( x \) in \( \text{cl} \ S \) for which \([x, p] \nsubseteq \text{cl} \ S \). Moreover, \( x \) may be chosen in \( S \) (for if \( p \) saw every member of \( S \) via \( \text{cl} \ S \), then \( p \) would see every member of \( \text{cl} \ S \) via \( \text{cl} \ S \) and \( p \) would lie in \( \text{ker}(\text{cl} \ S) \)).

There is a convex neighborhood \( N \) of \( p \), no point of which sees \( x \) via \( \text{cl} \ S \), with \( N \subseteq \text{int}(\text{cl} \ S) \). For any \( s, t \) distinct points in \( N \cap S \), necessarily \([s, t] \subseteq S \) by the 3-convexity of \( S \), so \( N \cap S \) is convex. Since \( N \subseteq \text{int}(\text{cl} \ S) \), \( p \) is interior to some triangle \( \text{conv}\{w, y, z\} \) with vertices belonging to \( N \cap S \). Then since \( N \cap S \) is convex, \( \text{conv}\{w, y, z\} \subseteq S \), and \( p \in S \). In fact, \( p \in \text{int} \ S \).

**Corollary.** If \( p \in \text{cl} \ S \sim S \), then either \( p \in \text{bdry}(\text{cl} \ S) \) or \( p \in \text{ker}(\text{cl} \ S) \) (or both).

**Lemma 3.** Let \( T \neq \emptyset \) be the set of points \( p \) of \( \text{cl} \ S \sim S \) for which \( p \in \text{bdry}(\text{cl} \ S) \). Then every connected component of \( T \) is either an isolated point of \( \text{cl} \ S \sim S \) or an interval. Moreover, there can be at most one isolated point, and all components of \( T \) lie on a common line.

*Proof.* If \( T \) is a singleton point, the result is immediate, so assume that \( T \) contains at least two distinct points \( x, y \). Let \( L(x, y) \) denote
the line determined by these points. It is clear that not both $x$ and $y$ can be isolated in $\text{cl} \, S \sim S$, for otherwise, since $x, y \in \text{int} \,(\text{cl} \, S)$, it would be easy to select three points of $S$ on $L(x, y)$ visually independent via $S$.

Again using the 3-convexity of $S$, $L(x, y) \cap S$ has at most two components, and $L(x, y) \cap T \subseteq \text{ker} \,(\text{cl} \, S)$ has at most three components. By an earlier argument, at most one component of $L(x, y) \cap T$ is an isolated point, and clearly each component is either an isolated point or an interval.

To complete the proof, it suffices to show that $T \subseteq L(x, y)$. Let $t \in \text{int} \,(\text{cl} \, S) \sim L(x, y)$ to show $t \notin T$. Since $L(x, y) \cap T$ contains at most one isolated point, $L(x, y) \cap T$ contains at least one interval $(r, s) \subseteq \text{int} \,(\text{cl} \, S)$, and we may choose some point $u$ in $S$ for which $(u, t)$ cuts $(r, s)$. Then select a convex neighborhood $N$ of $t$, $(u, q)$ cuts $(r, s)$. By techniques similar to those used in the proof of Lemma 2, $N \cap S$ is convex and $t \in S$. Hence $t \notin T$ and $T \subseteq L(x, y)$.

**Lemma 4.** If $\text{cl} \, S \sim S$ contains an interval $(r, s)$ disjoint from $\text{bdry} \,(\text{cl} \, S)$, then every lnc point of $\text{cl} \, S$ lies on $L(r, s)$.

**Proof.** Assume that for some lnc point $t$ of $\text{cl} \, S$, $t \notin L(r, s)$. As in the proof of Lemma 3, choose a point $u$ and a neighborhood $N$ of $t$ so that $u$ sees no point of $N \cap S$ via $S$. Since $t$ is an lnc point of $\text{cl} \, S$, $N$ contains points $v, w$ in $S$ which are visually independent via $S$. Hence $u, v, w$ are visually independent via $S$, a contradiction, and $t$ must lie on $L(r, s)$.

**Lemma 5.** If $p$ is in $\text{ker} \,(\text{cl} \, S)$ and $q, r$ are in $Q$, then $q \notin (p, r)$ (where $p, q, r$ are distinct points).

**Proof.** Assume, on the contrary, that the points are collinear, with $p < q < r$. Let $L$ denote the line containing $p, q, r$, $L_1, L_2$ the corresponding open halfspaces. Since $p \in \text{ker} \,(\text{cl} \, S)$ and $\text{cl} \, S$ is not convex, there must be some point $x$ of $\text{cl} \, S$ not on $L$, say in $L_1$. Our hypothesis implies that $\text{cl} \, S$ is connected, so by [2], Corollary 1, $r \in \text{ker} \,(\text{cl} \, S)$, and the triangle $\text{conv} \{p, x, r\}$ has its boundary in $\text{cl} \, S$. It is easy to see that the closed, 3-convex set $\text{cl} \, S$ is simply connected, so $\text{conv} \{p, x, r\} \subseteq \text{cl} \, S$. Thus since $q$ is an lnc point for $\text{cl} \, S$, there must be some point $y$ of $\text{cl} \, S$ in $L_2$, $\text{conv} \{p, y, r\} \subseteq \text{cl} \, S$, and $q$ cannot be an lnc point for $\text{cl} \, S$, clearly impossible. Our assumption is false, and $q \notin (p, r)$.

**Corollary.** No three members of $Q$ are collinear.
LEMMA 6. If \( p \in \text{conv } Q, \ q \in Q, \ q \neq p, \) and \( W_1, \ W_2 \) are leaves of \( \text{cl } S \) containing \( q \), then \( W_1, \ W_2 \) are in opposite closed halfspaces determined by \( L(p, q) \).

Proof. Clearly the hypothesis implies that \( \text{cl } S \) is connected and that \( \text{card } Q \geq 2 \). If \( \text{card } Q = 2 \), the result is an immediate consequence of an argument used by Valentine (Case 2, Theorem 3 of [2]), so we may assume that \( \text{card } Q \geq 3 \). Let \( r \) lie on the edge of \( \text{bdry} (\text{conv } Q) \) which defines \( W_1, \ r \neq q \). If \( r \in L(p, q) = L \), then by the definition of \( W_1 \), it is obvious that \( W_1 \) is in one of the closed halfspaces determined by \( L \), say \( \text{cl } L_1 \). Otherwise, without loss of generality, assume that \( r \) is in the open halfspace \( L_1 \). Clearly \( p \) and \( W_1 \) are separated by \( L(r, q) \). Now if any point \( x \) of \( W_1 \) lay in \( L_2 \), then \( q \) would lie interior to the triangle \( \text{conv } \{p, x, r\} \subseteq \text{cl } S \), and \( q \) could not be an Inc point for \( \text{cl } S \), a contradiction. Hence \( W_1 \) lies in \( \text{cl } L_1 \) in either case.

Since \( W_1 \cup \text{conv } Q \) is convex (by Valentine’s results) and \( q \) is an Inc point for \( \text{cl } S \), \( W_2 \) necessarily contains points in \( L_2 \), and \( W_1 \subseteq \text{cl } L_2 \), finishing the proof.

3. Decomposition theorems. With the preliminary lemmas behind us, we begin to investigate conditions under which \( S \) may be represented as a union of four or fewer convex sets, dealing primarily with the case for \( (\text{cl } S \sim S) \subseteq \text{int } (\text{cl } S) \).

The first theorem, allowing us to restrict attention to the case for \( \text{cl } S = \text{cl } (\text{int } S) \), will be helpful later.

THEOREM 1. If \( \text{cl } S \neq \text{cl } (\text{int } S) \), then \( S \) is a union of two or fewer convex sets.

Proof. Without loss of generality, assume \( S \) is connected, for otherwise the result is trivial. Let \( x \in S \sim \text{cl } (\text{int } S) \neq \emptyset \), and let \( N \) be a convex neighborhood of \( x \) disjoint from \( \text{int } S \). Since \( S \) is connected, \( x \) is not an isolated point of \( S \), and it is clear that \( N \cap S \) contains at least one segment.

We examine the maximal segments of \( N \cap S \) (i.e., the segments which are not proper subsets of segments in \( N \cap S \)). It is easy to show that \( N \cap S \) has at most two maximal segments, for otherwise, the 3-convexity of \( S \) together with the simple connectedness of \( \text{cl } S \) would yield an open region in \( \text{cl } S \cap N \). Since by Lemma 3 the points of \( \text{int } (\text{cl } S) \sim S \) are collinear, this would imply that \( N \cap S \) has interior points, clearly impossible by our choice of \( N \).

In case \( N \cap S \) has exactly two maximal segments, an argument similar to the one above may be used to show that any point of \( S \)
lies on one of the corresponding lines, and $S$ is a union of two segments (possibly infinite). If $N \cap S$ has just one segment, let $K_i$ denote a maximal convex subset of $S$ containing it, and let $K_i \equiv \text{conv}(S \sim K_i)$. Again using the facts that $N$ contains no interior points of $\text{cl} S$ and $\text{cl} S$ is simply connected, it is not hard to show that $K_i \subseteq S$, and $S = K_1 \cup K_2$, completing the proof.

Theorems 2 and 3 show that a decomposition is possible when $(\text{cl} S \sim S) \subseteq \text{int} (\text{cl} S)$. There are two cases to consider, depending on the cardinality of $Q$.

**Theorem 2.** If $(\text{cl} S \sim S) \cap \text{bdry} (\text{cl} S) = \emptyset$, and $\text{card} Q = n$ for $n$ an odd integer, $n > 1$, then $S$ is expressible as a union of four or fewer convex sets.

**Proof.** Clearly the hypothesis implies that $\text{cl} S = \text{cl} (\text{int} S)$. By the Corollary to Lemma 2, $\text{cl} S \sim S \subseteq \text{ker} (\text{cl} S)$, and by Lemma 3, every component of $\text{cl} S \sim S$ is either an isolated point or an interval. Since $\text{card} Q \geq 3$ and (by the corollary to Lemma 5) no three members of $Q$ can be collinear, Lemma 4 implies that $\text{cl} S \sim S$ cannot contain an interval. Hence $\text{cl} S \sim S$ consists of exactly one isolated point $p$ in $\text{ker} (\text{cl} S)$.

Select $q \in Q$ in the following manner: If $p \in \text{conv} Q$, choose $q \in Q$ so that the line $L(p, q)$ contains no other member of $Q$. (Clearly this is possible since card $Q$ is odd and no three members of $Q$ are collinear.) If $p \notin \text{conv} Q$, let $\{e_i : 1 \leq i \leq n\}$ denote the edges of $\text{conv} Q$, $\{E_i : 1 \leq i \leq n\}$ the corresponding lines, with $\text{conv} Q$ in the closed halfspace $\text{cl} (E_{i_0})$ for each $i$. Then $p \in E_{i_0}$ for exactly one $i$, for otherwise, if $p \in E_{i_1} \cap E_{i_2}$, then $\text{int} \text{conv} \{(p) \cup e_1 \cup e_2\}$ would contain an interior point of $\text{cl} S$, clearly impossible since $\{p\} \cup e_1 \cup e_2 \subseteq \text{ker} (\text{cl} S)$ and $\text{conv} \{(p) \cup e_1 \cup e_2\} \subseteq \text{cl} S$. Thus we may choose some $q \in Q$ so that $p \in \text{cl} E_{i_0}$ for each edge $e_i$ containing $q$. Then $(p, q)$ contains points of $\text{conv} Q$. Since all points of $L(p, q) \cap \text{conv} Q$ are on the open ray at $p$ emanating through $q$, Lemma 5 implies that $L(p, q)$ contains no other members of $Q$ (and in fact $p$ cannot lie on any line $E_i$).

To review, in either case we have chosen $q \in Q$ so that $L(p, q)$ contains no other member of $Q$ and $(p, q)$ contains points of $\text{conv} Q$. Letting $L_1$, $L_2$ denote distinct open halfspaces determined by $L = L(p, q)$, define $A \equiv \text{cl} (S \cap L_1)$, $B \equiv \text{cl} (S \cap L_2)$. If $W_1$, $W_2$ are leaves of $\text{cl} S$ containing $q$, then by Lemma 6, $W_1$ and $W_2$ are in opposite closed halfspaces determined by $L$, say $W_1 \subseteq \text{cl} L_1$, $W_2 \subseteq \text{cl} L_2$.

Let $R_1$, $R_2$ denote opposite closed rays at $p$, $R_1 \cup R_2 = L$, labeled so that $q \in R_2$. Each of $R_1 \cap S$, $R_2 \cap S$ is an interval by the 3-convexity of $S$. Points of $R_1 \cap S$ necessarily lie in $A \cap B$, for otherwise
would contain an lnc point of clS, clearly impossible. If there are any points of \( R_2 \cap S \) not in \( A \cap B \), without loss of generality we may assume such points lie in \( W_i \) and hence in \( A \sim B \). Then \( R_2 \cap S \subseteq A \).

By Case 4 in Theorems 2 and 3 of [2], \( \text{cl} (S \sim W_i) \) is a union of two closed convex sets \( C_i, C_2 \), selected as in Valentine’s proof. Since \( A = \text{cl} [\text{cl} (S \sim W_i) \cap L_i] \), \( A \) is the union of the two closed convex sets \( A_1, A_2 \), where \( A_i = \text{cl} (C_i \cap L_i) \), \( i = 1, 2 \). Moreover, \( (R_i \cap S) \cup (p, q) \) lies in one of these sets, say \( A_i \), and \( R_i \sim (p, q) \) is either in \( A_i \) or in \( A_2 \).

Using an identical argument for \( B \) and \( \text{cl} (S \sim W_i) \), we may write \( B \) as a union of two closed convex sets \( B_i, B_2 \) with \( (R_i \cap S) \cup (p, q) \) in \( B_i \), and \( R_i \sim (p, q) \) disjoint from \( B \).

At last, define sets \( A'_1, A'_2, B'_1, B'_2 \) in the following manner: If \( (R_2 \cap S) \sim (p, q) \subseteq A_2 \), let
\[
A'_1 \equiv A_1 \sim R_2, \quad A'_2 \equiv A_2 \sim R_1, \\
B'_1 \equiv B_1 \sim R_1, \quad B'_2 \equiv B_2 \sim R_2.
\]
And if \( (R_2 \cap S) \sim (p, q) \subseteq A_1 \), let
\[
A'_1 \equiv A_1 \sim R_1, \quad A'_2 \equiv A_2 \sim R_2, \\
B'_1 \equiv B_1 \sim R_2, \quad B'_2 \equiv B_2 \sim R_1.
\]

We assert that these are convex subsets of \( S \) whose union is \( S \): Clearly each is a convex subset of \( S \), and \( S \sim L \) is contained in their union. For \( (R_2 \cap S) \sim (p, q) \subseteq A_2 \), \( R_2 \cap S \subseteq A'_2 \cup B'_2, \ R_1 \cap S \subseteq A'_1 \). For \( (R_2 \cap S) \sim (p, q) \subseteq A_1 \), \( R_2 \cap S \subseteq A'_1 \cup B'_1, \ R_1 \cap S \subseteq B'_1 \). Hence in either case \( S \cap L \) is contained in the union of these sets, and \( S = A'_1 \cup A'_2 \cup B'_1 \cup B'_2 \), completing the proof of the theorem.

**Theorem 3.** If \( (\text{cl} S \sim S) \cap \text{bdry} (\text{cl} S) = \emptyset \) and \( \text{card} \ Q = n \geq 0 \), where \( n \) (possibly infinite) is not an odd integer greater than one, then \( S \) is expressible as a union of four or fewer convex sets.

**Proof.** If \( S \) is not connected, the result is trivial. Otherwise, by Theorem 3 of Valentine [2], \( \text{cl} S \) may be expressed as a union of two or fewer closed convex sets \( A, B \). Using Lemma 3, let \( L \) be a line containing \( \text{cl} S \sim S, \ L_1, \ L_2 \) the corresponding open halfspaces. Since \( S \) is 3-convex and \( A \) is convex, \( S \cap A \) is 3-convex, and hence \( (S \cap A) \cap L \) has at most two components, say \( C_1, C_2 \). Let \( R_1, R_2 \) denote opposite rays on \( L \) with \( C_1 \subseteq R_1, \ C_2 \subseteq R_2 \).

Define
\[
A_1 \equiv (A \cap S \cap \text{cl} L_1) \sim R_1, \quad A_2 \equiv (A \cap S \cap \text{cl} L_2) \sim R_2.
\]
Then $A_1, A_2$ are convex subsets of $S$ whose union is $A \cap S$.

Similarly define convex sets $B_1, B_2$ whose union is $B \cap S$. Clearly $S = A_1 \cup A_2 \cup B_1 \cup B_2$, the desired result.

**Corollary.** If $(\text{cl } S \sim S) \cap \text{bdry (cl } S) = \emptyset$, then $S$ is expressible as a union of four or fewer convex sets. The number four is best possible.

That the number four in the corollary is best possible is evident from Example 1.

**Example 1.** Let $S$ be the set in Figure 1, with $p \in S$. Then $S$ is not expressible as a union of fewer than four convex sets.

![Figure 1](image)

The preceding theorems allow us to obtain the following decomposition for open sets.

**Theorem 4.** If $S$ is open, then $S$ is expressible as a union of four or fewer convex sets. The result is best possible.

*Proof.* Let $T = S \cup \text{bdry (cl } S)$. Applying arguments identical to those used in the proofs of Theorems 2 and 3, $T$ is expressible as a union of four or fewer convex sets $A_i$, $1 \leq i \leq 4$. Define $B_i = A_i \cap S, 1 \leq i \leq 4$. We assert that each $B_i$ is convex. The proof follows:

By Valentine's results, cl $S$ is expressible as a union of three or fewer closed convex sets $C_j, 1 \leq j \leq 3$, each consisting of an appropriate selection of leaves of cl $S$, together with conv $Q$. Examining the proofs of Theorems 2 and 3, it is clear that each $A_i$ may be considered as a subset of some $C_j$ set. Thus we may assume $B_1 \subseteq A_i \subseteq C_i$ for an appropriate $C_i$.

Let $x, y \in B_1$, and let $p \in (x, y)$ to show $p \in B_1$. If $x$ (or $y$) is interior to some leaf $W$, then $W \subseteq C_i, y$ sees a neighborhood of $x$ via
Clf and \( p \) is interior to \( \text{cl} \ S \). Since \( p \in A_1 \) and \( p \not\in \text{bdry} (\text{cl} \ S) \), \( p \) is in \( A_1 \cap S = B_1 \). A similar argument holds if \( x \) (or \( y \)) is interior to \( \text{conv} \ Q \). Since neither \( x \) nor \( y \) is in \( \text{bdry} (\text{cl} \ S) \), the only other possibility to consider is the case in which \( x, y \in \text{bdry} (\text{conv} \ Q) \sim Q \subseteq \text{ker} (\text{cl} \ S) \). Then \( x \in \text{int} (\text{cl} \ S), y \in \text{ker} (\text{cl} \ S), y \) sees some neighborhood of \( x \) via \( \text{cl} \ S \), and \( p \in \text{int} (\text{cl} \ S) \). Again \( p \in A_i \cap S = B_i \) and \( B_i \) is indeed convex. Thus \( S \) is the union of the convex sets \( B_i, 1 \leq i \leq 4 \), and the theorem is proved.

To see that the number four is best possible, let \( S \) denote the set in Example 1 with its boundary deleted. Then \( S \) is an open 3-convex set not expressible as a union of fewer than four convex sets.

4. The general case. It remains to investigate the case for \( S \) an arbitrary 3-convex subset of the plane. A decomposition of \( S \) into six convex sets may be obtained from our previous results, together with Theorems 5 and 6, which deal with the case for \( (\text{cl} \ S \sim S) \subseteq \text{bdry} (\text{cl} \ S) \).

The following result by Lawrence, Hare, and Kenelly [1, Theorem 2] will be useful:

**Lawrence, Hare, Kenelly Theorem.** Let \( T \) be a subset of a linear space such that each finite subset \( F \subseteq T \) has a \( k \)-partition, \( \{F_1, \ldots, F_k\} \), where \( \text{conv} \ F_i \subseteq T, 1 \leq i \leq k \). Then \( T \) is a union of \( k \) convex sets.

**THEOREM 5.** If \( \text{cl} \ S \) is convex and \( (\text{cl} \ S \sim S) \subseteq \text{bdry} (\text{cl} \ S) \), then \( S \) is a union of three or fewer convex sets. The bound of three is best possible.

**Proof.** Consider the collection of all intervals in \( \text{bdry} (\text{cl} \ S) \) having endpoints in \( S \) and some relatively interior point not in \( S \). Each interval determines a line \( L \), and by the 3-convexity of \( S \), \( L \cap S \) has exactly two components. Let \( \mathcal{L} \) denote the collection of all such lines. By the Lawrence, Hare, Kenelly Theorem, without loss of generality we may assume that \( \mathcal{L} \) is finite. Hence the set \( \mathcal{U} \{L \cap S \mid L \in \mathcal{L} \} \) has finitely many components, and we may order these components in a clockwise direction along \( \text{bdry} (\text{cl} \ S) \). If \( c_i \) denotes the \( i \)th component in our ordering, let

\[
A' = \{c_i \mid i \text{ odd}, i < n\}, \\
B' = \{c_i \mid i \text{ even}, i < n\}, \\
C' = \{c_n\}.
\]

Define
We assert that $A$, $B$, $C$ are convex sets whose union is $S$. The proof follows:

For $x, y$ in $A$, if $[x, y]$ contains any point of $\text{int}(\text{cl } S)$, then $(x, y) \subseteq \text{int}(\text{cl } S) \subseteq A$, and $[x, y] \subseteq A$. Otherwise, $[x, y]$ lies in the boundary of the convex set $\text{cl } S$. If the corresponding line $L(x, y)$ is not in $\mathcal{L}$, the result is clear, so suppose $L(x, y) \in \mathcal{L}$. Then $x, y$ must lie in the same $c_i$ set for some $i$ odd, $i < n$, again giving the desired result. Hence $A$ is convex. Similarly, $B, C$ are convex. It is easy to see that $A \cup B \cup C = S$ and the proof is complete.

The surprising fact that three is best possible is illustrated by Example 2.

**Example 2.** Let $S$ denote the set in Figure 2, where dotted lines represent segments not in $S$. Then $S$ is not expressible as a union of fewer than three convex sets.

**Figure 2**

**Theorem 6.** If $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, then $S$ is a union of four or fewer convex sets. The number four is best possible.

**Proof.** We assume that $S$ is connected and $\text{cl } S = \text{cl } (\text{int } S)$, for otherwise $S$ is a union of two convex sets. Furthermore, by the Lawrence, Hare, Kenelly Theorem, we may assume that $\text{cl } S$ has finitely many leaves, and hence card $Q = n$ is finite. Notice also that since $\text{cl } S$ is simply connected and $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$, $S$ is simply connected.

For the moment, suppose $3 \leq n$. Order the points of $Q$ in a clockwise direction along $\text{bdry } (\text{conv } Q)$, letting $W_i$ denote the leaf of $\text{cl } S$ determined by $\text{Int } Q$ points $q_i, q_{i+1}$ (where $n + 1 \equiv 1$). By Valentine's results in [2], for any pair of disjoint leaves $W_i, W_j$ of $\text{cl } S$, the set $R = \text{conv } Q \cup W_i \cup W_j$ is a closed convex set. (In case there are no disjoint leaves, $n = 3$, $W_j = \emptyset$, and $R = \text{conv } Q \cup W_i$ is closed and convex.) Consider the collection of intervals in $\text{bdry } R$ having end-
points \(x, y\) in \(S\) and some relatively interior point \(p\) not in \(S\). Either such an interval is contained in one leaf, or \(x \in W_i \cup \text{conv } Q, y \in W_j \cup \text{conv } Q\). We examine the latter case. It is clear that for an appropriate labeling, \(j = i + 2\), so to simplify notation, say \(i = 1, j = 3,\) and \(L(x, y)\) supports \(W_i\). Clearly not both \(x, y\) can lie in \(\text{conv } Q\), for then \(p \in \text{int } S \subseteq S\). However, we assert that either \(x\) or \(y\) must lie in \(\text{conv } Q\) and that \(W_i \cap S\) is convex. The proof follows: Assume that \(x\) is not an lnc point and that \(x < p \leq q_2 < q_3\), where \(q_2, q_3\) are the lnc points in \(W_i \cap W_z, W_z \cap W_i\) respectively. Then \(q_2 \leq y\). For \(w\) in \(W_z \cap S, w\) cannot see \(x\) via \(S\), so necessarily \(w\) sees \(y\) via \(S\), by the 3-convexity of \(S\). This implies that \(y \leq q_3\) (for otherwise \(q_3\) could not be an lnc point for \(\text{cl } S\)). Moreover, since no two points of \(W_z \cap S\) see \(x\) via \(S\), the 3-convexity of \(S\) together with the convexity of \(W_i\) imply that \(W_z \cap S\) is convex.

Here we digress briefly for future reference. The set \(L(x, y) \cap S\) has two components, and by the above argument, one must lie in the interval \([q_2, q_3]\), the other in \(W_1 \sim Q\) (by our labeling). For general \(W_{i-1}, W_{i+1}\) (disjoint if and only if \(n > 3\)), we let \(T_i\) denote the connected set of all the somewhat troublesome points \(y\) in \([q_i, q_{i+1}] \cap S\) having the above property. That is, there exist points \(x\) in exactly one of \((W_{i-1} \cap S) \sim Q, (W_{i+1} \cap S) \sim Q\) for which \([x, y] \subseteq S\) \((n + 1 = 1)\).

Continuing the argument, delete \(W_z\) and consider the 3-convex set \((S \sim W_z) \cup (S \cap L(x, y))\). Renumber the lnc points and leaves for this set so that the old \(W_i\) and \(W_z\) are contained in the new leaf \(U_i\). Since we are assuming \(\text{card } Q\) is finite, repeating the procedure finitely many times yields a 3-convex set \(S_o\) having the following property: For \(V_i, V_j\) disjoint leaves of \(\text{cl } S_o, x\) in \(V_i \cap S_o, y\) in \(V_j \cap S_o\), then \([x, y] \subseteq S_o\). In addition, without loss of generality we may assume that for each leaf \(V_i\) of \(\text{cl } S_o, V_i \cap S_o\) is not convex, for otherwise, \(V_i\) may be deleted by the above procedure.

To avoid confusion, let \(Q_0\) denote the set of lnc points of \(\text{cl } S_o, Q_0 \subseteq Q, \text{card } Q_0 = m \leq n\). For \(3 \leq m\), let \(V_i\) denote the leaf determined by lnc points \(p_i, p_{i+1}\) in \(Q_0\) (where \(p_{m+1} = p_i\)). For \(m = 2\), let \(V_1, V_2\) denote the leaves of \(\text{cl } S_o\) as defined in the introduction to this paper. If \(0 \leq m \leq 1\), let \(V_1 = V_2 = \text{cl } S_o\).

For each \(i\), consider the collection of intervals in \(\text{bdry } V_i\) having endpoints in \(V_i \cap S_o\) and some relatively interior point not in \(S_o\). Each interval determines a line \(L\), and for \(m \neq 1\), \(L \cap V_i \cap S_o\) has exactly two components, each in \(\text{bdry } V_i\). In case \(m = 1\), an obvious adjustment may be made (by deleting any ray of \(L\) which contains interior points of \(\text{cl } S_o\)) to yield the same result. For each \(i\), let \(\mathcal{L}_i\) denote the collection of all such lines. Again using the Lawrence, Hare, Kenelly Theorem, we may assume that each \(\mathcal{L}_i\) is finite. The set \(\bigcup \{L \cap V_i \cap S_o : L \text{ in } \mathcal{L}_i\}\) has finitely many components, and we
may order them in a clockwise direction along bdry \( V_t \). Let \( c_{ij} \) denote the \( j \)th such component for \( V_t \), and let \( \mathcal{C}_i \) denote the collection of all the \( c_{ij} \) sets corresponding to \( V_t \). Clearly each \( c_{ij} \) is either a point, an interval, or the union of two noncollinear intervals. Moreover, for \( m \geq 2 \), no components for \( V_t \) for \( V_t \) may have common points. (Such a point would necessarily be \( p_{i+1} \), and if \( s_i \in V_t \cap S_o \), \( s_{i+1} \in V_{i+1} \cap S_o \) with some interior point of each of \( [s_i, p_{i+1}] \) \( [p_{i+1}, s_{i+1}] \) not in \( S_o \), then \( s_i, p_{i+1}, s_{i+1} \) would be visually independent via \( S_o \), clearly impossible.)

For each \( V_i \), select every \( c_{i,ij} \). That is, select the members of \( \mathcal{C}_i \) having second subscript even. No two components selected correspond to the same line, and for \( m \neq 0 \), we have chosen one component corresponding to each line in \( L_i \). If \( m = 0 \), without loss of generality we may assume \( \mathcal{C}_i \) is ordered in a clockwise direction from some point in \( \mathcal{Q} \cap cl S_o = \emptyset \). In case no component has been chosen for some line \( L \) in \( L_i \), then \( L \) must contain points of both the first and last members of \( \mathcal{C}_i \), and by a previous argument, one of these components must lie in \( conv \mathcal{Q} \).

For \( m \neq 1 \), since \( V_i \) is convex, it is easy to show that \( conv \{ c_{i,ij} : 1 \leq j \} \) is a subset of \( S_o \) (and this is certainly true even if \( cl S_o \) is convex). We will prove that \( B_o \equiv conv \{ c_{i,ij} : 1 \leq i \leq m, 1 \leq j \} \) is in \( S_o \) and hence in \( S \). If \( cl S_o \) is convex (or empty) the result is immediate, so assume \( cl S_o \) has at least one lnc point. For convenience, in case \( cl S_o \) has only one lnc point, call it \( p_2 \), and let \( V_1 = V_2 \) follow \( p_2 \) in our clockwise ordering.

Recall that \( V_i \cap S_o \) is not convex for any \( i \), so no \( \mathcal{C}_i \) is empty. Let \( c_o \) denote the last member of \( \mathcal{C}_i \) selected, \( x \) the last point of \( cl c_o \) (relative to our ordering). If \( x \neq p_2 \), let \( L = L(x, p_2) \). Otherwise, by the 3-convexity of \( S_o \), \( c_o = \{ p_2 \} \), and in this case let \( L \) denote the corresponding member of \( L_i \). Let \( L_1, L_2 \) be the open halfspaces determined by \( L \), with \( Q_o \subseteq cl L_1 \). Since \( p_2 \) is an lnc point of \( S \) and \( S_o \) is 3-convex, it is clear that at most one member of \( \mathcal{C}_2 \), namely \( c_{21} \), may contain points in \( L_2 \). We assert that \( c_o \) sees \( c_{22} \) via \( S_o \). The proof follows:

In case \( L \in L_i \), \( L \cap V_1 \cap S_o \) has two components, each in bdry \( V_i \), and one of these must be \( \{ p_2 \} \). Then by the 3-convexity of \( S_o \), \( c_{22} \subseteq L_1 \) and \( c_o \) sees \( c_{22} \) via \( S_o \). Otherwise, \( c_o \sim \{ x \} \subseteq L_1 \). If \( x \in S_o \), then since \( c_{22} \subseteq cl L_1 \), it is clear that \( c_o \) sees \( c_{22} \) via \( S_o \). If \( x \in S_o \) and \( p_2 \in S_o \), then again the result is clear. If \( x \in S_o \) and \( p_2 \in S_o \), then \( c_{22} \subseteq L_1 \) and \( c_o \) sees \( c_{22} \) via \( S_o \), finishing the argument.

In case \( V_1, V_2 \) are the only leaves for \( cl S_o \), \( V_1 \neq V_2 \), then repeating the argument for the last member of \( \mathcal{C}_2 \) and \( c_{12} \) and using the fact that \( S_o \) is simply connected, we have \( B_o \subseteq S_o \subseteq S \). (If \( V_1 = V_2 \), the result is immediate.) Otherwise, \( 3 \leq m \) and an inductive argument may be used to show that \( B_o \) is in \( S \).
Using Valentine's results, write cl $S$ as a union of three or fewer convex sets $A_j$, $j = 1, 2, 3$, where for $n$ odd

$$A_1 \equiv \bigcup \{W_i; i \text{ odd, } i < n\} \cup \text{conv } Q,$$
$$A_2 \equiv \bigcup \{W_i; i \text{ even, } i < n\} \cup \text{conv } Q,$$
$$A_3 \equiv W_n \cup \text{conv } Q,$$

and for $n$ even

$$A_1 \equiv \bigcup \{W_i; i \text{ odd, } i \leq n\} \cup \text{conv } Q,$$
$$A_2 \equiv \bigcup \{W_i; i \text{ even, } i \leq n\} \cup \text{conv } Q,$$
$$A_3 = \emptyset.$$

Define $B_j \equiv S \cap \{A_j \sim ((\text{bdry } S) \cap B_0)\}$, $j = 1, 2, 3$.

Recall the $T_i$ sets defined previously, $T_i \subseteq [q_i, q_{i+1}] \subseteq W_i$, $1 \leq i \leq n$. To simplify notation, let $L_i = L(q_i, q_{i+1})$, and define sets $F_i, G_i$ in the following manner: For $i$ even, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in $B_i$, $F_i = \emptyset$ otherwise. Similarly for $i$ odd, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in $B_i$, $F_i = \emptyset$ otherwise. For $i = 1$, $i = n - 1$, let $G_i = T_i$ if points from both components of $L_i \cap S$ are in $B_i$, $G_i = \emptyset$ otherwise. By previous remarks, at least one of $G_1, F_1$ is empty, and at least one of $G_{n-1}, F_{n-1}$ is empty.

Define

$$D_1 \equiv B_1 \sim \bigcup \{F_i; i \text{ even}\},$$
$$D_2 \equiv B_2 \sim \bigcup \{F_i; i \text{ odd}\},$$
$$D_3 \equiv B_3 \sim \bigcup \{G_i, G_{n-1}\}.$$

Finally, letting $P = \{F_i \cap F_j; 1 \leq i < j \leq n\} \cup \{G_i \cap F_j; i = 1, n - 1, 1 \leq j \leq n\}$, define $D_0 \equiv \text{conv } (B_0 \cup P)$. We assert that the sets $D_j$, $0 \leq j \leq 3$, are convex sets whose union is $S$. The proof follows:

Suppose that one of the sets $D_1, D_2, D_3$, say $D_1$, is not convex to obtain a contradiction. Choose $x, y$ in $D_1$ for which $[x, y] \not\subseteq D_1$. It is clear that $[x, y] \subseteq \text{bdry } (\text{cl } D_1) = \text{bdry } A_1$. Furthermore, $x, y$ cannot both belong to $W \sim Q$ for any leaf $W$ of cl $S$, for otherwise they would belong to the same leaf of cl $S_{0}$, and one of $x, y$ would lie in $(\text{bdry } S) \cap B_0$ and hence not in $D_1$, a contradiction. Employing a previous argument, the set $L(x, y) \cap S$ has two components, each having points in $B_i$, and one of these components is the set $[q_0, q_{i+1}] \cap S = T_i$, for some $i$ even ($n + 1 = 1$). Let $R_i$ denote the other component of $L(x, y) \cap S$. If $R_i \cap B_0 \neq \emptyset$, then $R_i, T_i$ would lie on the boundary of a leaf of cl $S_0$, $R_i \subseteq B_i$, $T_i \subseteq B_i$, and $[x, y] \subseteq T_i \subseteq D_1$, a contradiction. Thus $R_i \cap B_0 = \emptyset$ and $R_i \subseteq D_i$. However, this implies that one of $x, y$ must lie in $F_i$ and not in $D_i$, again a contradiction. Our assumption is false and $D_i$ is convex. Similarly $D_2, D_3$ are convex,
and clearly each is a subset of $S$.

It remains to show that the convex set $D_0$ lies in $S$. Examining the set $P$, if $F_i \cap F_j \neq \emptyset$ for some $i \neq j$ (or if $G_i \cap F_j \neq \emptyset$), then $F_i = T_i$, $F_j = T_j$, for an appropriate labeling $j = i + 1$, and $F_i \cap F_{i+1} = \{q_{i+1}\} \subseteq S$. We will show that for each $z$ in $B_0$, $[q_{i+1}, z] \subseteq S$. The proof follows:

We have seen that $W_i \cap S$, $W_{i+1} \cap S$ are both convex, so for every $z$ in one of these sets, $[q_{i+1}, z] \subseteq S$. Moreover, we assert that the components of $L(q_i, q_{i+1}) \cap S$, $L(q_{i+1}, q_{i+2}) \cap S$ not in conv $Q$, call them $R_i$, $R_{i+1}$, are disjoint from $B_0$. If $R_i \cap B_0 \neq \emptyset$, then by an earlier argument, $R_i \subseteq B_0$, $T_i \cap B_0 = \emptyset$, $T_i \subseteq D_1 \cap D_2 \cap D_3$, and $F_i = \emptyset$, a contradiction. Hence for $z$ in $B_0 \sim (W_i \cup W_{i+1})$, $(q_{i+1}, z) \subseteq \text{int } S$, and $[q_{i+1}, z] \subseteq S$ whenever $z \in B_0$, the desired result.

Certainly for $q_i, q_j, q_k$ in $P \subseteq S$, $\text{conv } \{q_i, q_j, q_k\} \subseteq S$.

By Carathéodory's theorem in the plane, to prove that $D_0 \equiv \text{conv } (B_0 \cup P)$ is in $S$, it is sufficient to show that the convex hull of any three points of $B_0 \cup P$ is in $S$, and from the remarks above, clearly we need only show $\text{conv } \{q_i, q_j, z\} \subseteq S$ for $q_i, q_j$ in $P$, $z$ in $B_0$. However, since $S$ is simply connected and $\text{bdry } (\text{conv } \{q_i, q_j, z\}) \subseteq S$, $\text{conv } \{q_i, q_j, z\} \subseteq S$ and $D_0 \subseteq S$, the desired result.

Finally, by inspection, each $F_i \neq \emptyset$ fails to belong to at most one of the sets $D_1, D_2, D_3$. Points in intersecting $F_i$ sets are in $D_0$, so $\bigcup \{D_j; 0 \leq j \leq 3\} = S$ and the argument for $3 \leq \text{card } Q$ is complete.

To finish the proof, we must examine the cases for $0 \leq \text{card } Q \leq 2$. If card $Q = 2$ or if card $Q = 1$ and $S \sim Q$ is connected, then let $W_1, W_2$ denote the corresponding leaves of $\text{cl } S$, and use a simplified version of the previous proof to define $B_0, B_1, B_2$. If one of $B_1, B_2$, say $B_1$, is not convex, then letting $T = W_1 \cap W_2 \cap S$, $W_2 \cap S = B_2$ is convex, $T \subseteq B_2$, and $B_0, B_1 \sim T$, $B_2$ are the desired convex sets.

In case card $Q = 1$ and $S \sim Q$ is not connected, then for $W_1, W_2$ the corresponding leaves of $\text{cl } S$, each of $W_1 \cap S$, $W_2 \cap S$ is convex. For card $Q = 0$, the result follows from Theorem 5, and the proof of Theorem 6 is complete.

The number four in Theorem 6 is best possible, as the following example illustrates.

![Figure 3](image-url)
Example 3. Let $S$ denote the set in Figure 3, where dotted segments are in $\text{bdry} \ (\text{cl} \ S) \sim S$. Then $S$ is a union of no fewer than four convex sets.

At last, using Theorem 6, we have a decomposition theorem for $S$ an arbitrary 3-convex subset of the plane.

Theorem 7. The set $S$ is a union of six or fewer convex sets. The result is best possible.

Proof. By earlier comments, we may assume that $S$ is connected, $\text{cl} \ S = \text{cl} \ (\text{int} \ S)$, and $Q$ is finite. Furthermore, we assume $\text{int} \ (\text{cl} \ S) \sim S \neq \emptyset$, for otherwise the result is an immediate consequence of Theorem 6. Let $T = S \cup \text{bdry} \ (\text{cl} \ S)$, and let $L$ be the line containing $\text{cl} \ T \sim T$ described in Theorem 2 or Theorem 3 (whichever is appropriate). Clearly $L$ may be chosen to contain an lnc point $q$ of $\text{cl} \ S$.

If $L_1, L_2$ are the corresponding open halfspaces, then each of $T_i = \text{cl} \ (T \cap L_i) = \text{cl} \ (S \cap L_i)$, $T_2 = \text{cl} \ (T \cap L_2) = \text{cl} \ (S \cap L_2)$ is 3-convex.

Define $S_i = T_i \cap S$, $i = 1, 2$. We assert that each $S_i$ is 3-convex: For $x, y, z$ in $S_i = T_i \cap S$, assume $[x, y] \subseteq S_i$. If $x$ or $y$ is in $T_i$, then certainly $(x, y) \subseteq L_1 \cap S \subseteq T_1$, and $[x, y] \subseteq S_i$. If $x, y$ are on $L$, then since no lnc points of the closed set $T_i$ are on $L$, $x, y$ lie in the same leaf of $T_i$, and $[x, y] \subseteq T_1 \cap S = S_i$. Thus $S_i$ is 3-convex. Similarly $S_2$ is 3-convex. Moreover, $(\text{cl} \ S_i \sim S_i) \subseteq \text{bdry} \ (\text{cl} S_i)$, $i = 1, 2$.

Using Theorem 6, we will show that each $S_i$ is a union of three convex sets: By the proofs of Theorems 2 and 3, $\text{cl} \ S_i = T_i$ is a union of two convex sets $A_1, A_2$, and each $A_i$ may be considered a subset of an appropriate $C_j$ set, $1 \leq j \leq 3$, where the $C_j$ sets are those described in Valentine's paper with $\text{cl} \ S = C_1 \cup C_2 \cup C_3$. In case $T_i$ has one leaf or an even number of leaves, then clearly the proof of Theorem 6 may be used to write $S_i$ as a union of three convex sets. If $T_i$ has $n$ leaves for $n$ odd, $n > 1$, let $V$ be the leaf of $T_i$ bounded by $L$, $q \in Q \cap L \subseteq A_1 \cap A_2$. Order the lnc points of $T_i$ in a clockwise direction so that $V$ is determined by $q_n, q_1$, and let $U_n, U_{n+1}$ denote the closed subsets of $V$ bounded by $L(q_n, q)$, $L(q, q_1)$ respectively. Treating $U_1, \ldots, U_n, U_{n+1}$ as leaves of $T_i$, $U_i$ determined by lnc points $q_i, q_{i+1}$, $1 \leq i < n$, the proof of Theorem 6 may be applied to write $S_i$ as a union of three convex sets. (Of course, in defining $B_n$, points of $V$ in $S_6$ belong to the same leaf of $S_6$.)

By a parallel argument $S_2$ is a union of three convex sets, and $S = S_1 \cup S_2$ is a union of six or fewer convex sets, finishing the proof of the theorem.

Our final example shows that the bound of six in Theorem 7 is
best possible.

**EXAMPLE 4.** Let $S$ be the set in Figure 4, with dotted segments in $\text{bdry}(\text{cl } S) \sim S$ and $p \in \text{int}(\text{cl } S) \sim S$. Then $S$ cannot be expressed as a union of fewer than six convex sets.

![Figure 4](image)

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The University of Oklahoma
Martin Bartelt, *Strongly unique best approximates to a function on a set, and a finite subset thereof* .......................................................... 1
S. J. Bernau, *Theorems of Korovkin type for $L_p$-spaces* .................................. 11
S. J. Bernau and Howard E. Lacey, *The range of a contractive projection on an $L_p$-space* .......................................................... 21
Marilyn Breen, *Decomposition theorems for 3-convex subsets of the plane* .......... 43
Ronald Elroy Bruck, Jr., *A common fixed point theorem for a commuting family of nonexpansive mappings* .................................. 59
Aiden A. Bruen and J. C. Fisher, *Blocking sets and complete k-arcs* ................. 73
R. Creighton Buck, *Approximation properties of vector valued functions* .......... 85
Mary Rodriguez Embry and Marvin Rosenblum, *Spectra, tensor products, and linear operator equations* .................................. 95
Edward William Formanek, *Maximal quotient rings of group rings* .................. 109
Barry J. Gardner, *Some aspects of T-nilpotence* ........................................... 117
Juan A. Gatica and William A. Kirk, *A fixed point theorem for k-set-contractions defined in a cone* ................................ 131
Kenneth R. Goodearl, *Localization and splitting in hereditary noetherian prime rings* .......................................................... 137
James Victor Herod, *Generators for evolution systems with quasi continuous trajectories* .......................................................... 153
C. V. Hinkle, *The extended centralizer of an S-set* ......................................... 163
I. Martin (Irving) Isaacs, *Lifting Brauer characters of p-solvable groups* .......... 171
Bruce R. Johnson, *Generalized Lerch zeta function* ......................................... 189
Erwin Kleinfeld, *A generalization of $(-1, 1)$ rings* ........................................ 195
Horst Leptin, *On symmetry of some Banach algebras* ....................................... 203
Paul Weldon Lewis, *Strongly bounded operators* ............................................. 207
Arthur Larry Lieberman, *Spectral distribution of the sum of self-adjoint operators* .......................................................... 211
I. J. Maddox and Michael A. L. Willey, *Continuous operators on paranormed spaces and matrix transformations* .......................................... 217
James Dolan Reid, *On rings on groups* ........................................................... 229
Richard Miles Schori and James Edward West, *Hyperspaces of graphs are Hilbert cubes* .......................................................... 239
William H. Specht, *A factorization theorem for p-constrained groups* ............... 253
Robert L. Thele, *Iterative techniques for approximation of fixed points of certain nonlinear mappings in Banach spaces* .......................................... 259
Tim Eden Traynor, *An elementary proof of the lifting theorem* ......................... 267
Charles Irvin Vinsonhaler and William Jennings Wickless, *Completely decomposable groups which admit only nilpotent multiplications* .......... 273
Raymond O’Neil Wells, Jr, *Comparison of de Rham and Dolbeault cohomology for proper surjective mappings* .......................................... 281
David Lee Wright, *The non-minimality of induced central representations* .......... 301
Bertram Yood, *Commutativity properties in Banach $^*$-algebras* ...................... 307