BLOCKING SETS AND COMPLETE $k$-ARCS

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Let $\pi$ be a finite projective plane of order $n$. Recall that a blocking set $S$ in $\pi$ is a set of points which does not contain any line but which does intersect every line of $\pi$. The first objective is to elaborate on the connection, pointed out by the writers, between blocking sets and complete $k$-arcs of $\pi$. For example, the set of secants of a complete $k$-arc with $k < n + 2$ dualizes to a blocking set. Using some simple observations, it is shown that a blocking set in a projective plane $\pi$ of order ten, if $\pi$ exists, contains at least 16 points. The proof uses a computer result on the nonexistence of complete 6-arcs of $\pi$ due to R. H. F. Denniston. Using the result, a recent theorem concerning certain codes related to $\pi$ due to MacWilliams, Sloane, and Thompson is easily established. The result also shows that, in effect, a set of four mutually orthogonal latin squares of order ten is embeddable in a complete set in at most one way. This improves slightly on the bound of R. H. Bruck.

Later on, new bounds are obtained on the number of points of a blocking set that lie on any line. Examples in finite Desarguesian planes are given to show that these bounds are, in a sense, best possible. In §4 some miscellaneous remarks on blocking sets are made and an interesting example in $PG(2, 7)$ is discussed.

2. Planes of order 10. The following theorem is a key to the results of this section.

THEOREM 1. Let $R$ be a complete $k$-arc in a projective plane $\pi$ of order $n$, and let $\pi'$ be the plane dual to $\pi$. When $k < n + 2$ the mapping that takes the lines of $\pi$ into the points of $\pi'$ takes the secants of $R$ into a blocking set $S$ in $\pi'$.

Proof. The assumption that $R$ is complete implies immediately that every point of $\pi$ lies on a secant of $R$. Thus every line of $\pi'$ contains at least one point of $S$. The maximum number of secants of $R$ that pass through a point of $\pi$ is $k - 1$: this occurs if and only if the point is on $R$. By our assumption that $k < n + 2$, every point of $\pi$ is incident with at most $k - 1 \leq n$ secants. Thus any line of $\pi'$ contains at most $n$ points of $S$, so that $S$ is a blocking set in $\pi$.

REMARKS. We shall say that such a blocking set — the dual of
the set of secants of a complete $k$-arc — is derived from (or is obtained from or comes from etc.) a complete $k$-arc. It is well-known that no $k$-arc can have more than $n + 2$ points, and only when $n$ is even can we have $k = n + 2$. From the above it is clear that a blocking set cannot be derived from an $n + 2$-arc.

By Theorem 3.9 in [5], a blocking set in $\pi$ must contain at least $n + \sqrt{n} + 1$ points. Since there are $\binom{k}{2}$ secants to any $k$-arc we immediately deduce from Theorem 1 that, if the $k$-arc is complete, $n + \sqrt{n} + 1 \leq k(k - 1)/2$. It is possible to improve this bound to obtain a result due to M. See [13, p. 280].

**Theorem 2 (Sce).** Let $R$ be a complete $k$-arc in a projective plane $\pi$ of order $n$. Then $n \leq 1/2 (k - 1)(k - 2)$.

**Proof.** We use the notation of Theorem 1. Corresponding to any point $M$ of $R$ is a line of $\pi'$, call it $m$, which contains $k - 1$ points of the blocking set $S$ obtained from $R$. Denoting by $|S|$ the number of points of $S$ we have $|S| = \binom{k}{2}$ number of secants to $R$. Let $P$ be any point of $m$ not in $S$; in symbols $P \in m - S$. Since $S$ is a blocking set, the $n$ lines of $\pi$ through $P$ that are different from $m$ must each contain at least one point of $S$. Thus $|S| \geq k - 1 + n$, so that $k(k - 1)/2 \geq (k - 1) + n$, as claimed.

**Definitions.** A line of $\pi$ that contains at least 2 points of a subset $S$ of points of $\pi$ will be called a line of $S$. A line that contains exactly $t$ distinct points of $S$ is called a $t$-line and is said to have strength $t$.

It would be of interest to characterize blocking sets obtained from complete $k$-arcs. A result in that direction is as follows.

**Theorem 3.** A blocking set $S$ is obtained from a complete $k$-arc in a projective plane $\pi$ if and only if the following three conditions are satisfied.

(i) $|S| \leq \binom{k}{2}$.

(ii) The number of $(k - 1)$-lines of $S$ is at least $k$.

(iii) No three of the $(k - 1)$-lines of $S$ are concurrent.

**Proof.** If $S$ is derived from a complete $k$-arc the 3 conditions are clearly satisfied. Conversely, assume that a blocking set $S$ has the three properties. Let $W$ denote the set of $(k - 1)$-lines of $S$ and let $I$ denote the set of incidences of points of $S$ with lines of $W$. By (ii), $|I| \geq k(k - 1)$. The points of intersection of pairs of
lines of $W$ are distinct by (iii). Let $r$ of these points be points of $S$. This leaves $|S| - r \leq \binom{k}{2} - r$ points of $S$ that lie on either one or zero lines of $W$. Thus $|I| \leq r \cdot 2 + \left\lceil \binom{k}{2} - r \right\rceil$. Thus $r + \binom{k}{2} \geq k(k - 1)$, so that $r \geq \binom{k}{2}$. But, by definition, $r \leq |S| \leq \binom{k}{2}$. Thus $r = \binom{k}{2}$, $|I| = k(k - 1)$, $|W| = k$, $|S| = \binom{k}{2}$ and the points of $S$ are precisely the $\binom{k}{2}$ intersections of all pairs of lines of $W$. Let $R$ denote the dual of $W$, that is, the set of points of $\pi'$ corresponding to $W$, where $\pi'$ is the plane dual to $\pi$. Then, from the above, $R$ is a $k$-arc, and the dual of $S$ is the set of secants of $R$. Since $S$ is a blocking set this forces $R$ to be a complete $k$-arc, proving the result.

**Remark.** Every $k$-arc in a finite projective plane gives rise to a system of diophantine equations [13, §179]. In the light of Theorems 1 and 3, these equations can be used to study blocking sets derived from complete $k$-arcs. For example, let $c_i$ be the number of $i$-lines of $S$ and let $j$ denote the greatest integer in $k/2$. We have immediately $c_0 = 0$, $c_{k-1} = k$, $c_i = 0$ for $j < i < k - 1$. Furthermore, it can be shown that

$$\sum_{i=1}^{j} i c_i = 1 + n + n^2 - k$$

$$2 \sum_{i=1}^{j} i^2 c_i = k(k - 1)(n - 1)$$

and

$$4 \sum_{i=1}^{j} i^4 c_i = 2k(k - 1)(n - 1) + k(k - 1)(k - 2)(k - 3).$$

Other relevant identities and inequalities can be found in [13].

We now concentrate on the case $n = 10$. Our objective here is to show that if $S$ is a blocking set in a projective plane $\pi$ of order 10, then $|S| \geq 16$. This was shown in [5] by using an additional assumption, namely that $\pi$ contains no projective subplane of order 2. We proceed to show how this rather awkward assumption can be dropped.

**Theorem 4.** Suppose $S$ is a blocking set in a projective plane $\pi$ of order 10. Then $S$ must contain at least 16 points.

**Proof.** By [5, Theorem 3.9], $|S| \geq 10 + 10^{1/2} + 1$. Assume $|S| = 15 = \binom{6}{2}$. We use the notation of [5] with $(P, 1)$ and $(P, 2)$ being
5-lines. The theorem was proved there under the additional assumption that $\pi$ contained no projective subplane of order 2. The only place that this extra assumption was needed occurred in case B(i).

Now, as in [5], we have 5, 6, 8 collinear, that is, (5, 6, 8). The following kind of argument is used repeatedly below. The line 5, 6, 8 contains already 3 distinct points of $S$. Suppose this line met the 5-line $(P, 2)$ in a point $X$ not in $S$. Now $S$ is a blocking set, so that, in particular, every line of $\pi$ through $X$ contains at least one point of $S$. Thus $|S| \geq 3 + 5 + 9 = 17$, contradicting $|S| = 15$. Thus the line (5, 6, 8) contains a point of $S$, say 11, different from 2 and from $P$. Similarly, the line (7, 6, 3) must meet $(P, 2)$ in a point of $S$, say 9, with $9 \neq 2$, $P$, 11. Similarly (7, 6, 3) meets $(P, 1)$ in a point 12, with $12 \neq 14$, $P$, 1. Here 14 is where (5, 6, 8) meets $(P, 1)$. In summary, we now have (5, 6, 8, 11, 14) and (3, 6, 7, 9, 12). Join 7 to 8. In [5] it was shown that (7, 8, 4). This last line meets $(P, 2)$ in a point $Q$ of $S$. Because two distinct lines meet in just one point, we know that $Q \neq 9, 11, 2, P$. So $Q = 10$ say. Similarly (7, 8) meets $(P, 1)$ in a point 13 $\neq 14, 12, 1, P$. In short $(4, 7, 8, 10, 13)$.

Finally consider the six 5-lines $(1, 2, 3, 4, 5)$, $(1, 12, 13, 14, P)$, $(2, 9, 10, 11, P)$, $(5, 6, 8, 11, 14)$, $(3, 6, 7, 9, 10)$, $(4, 7, 8, 10, 13)$, no 3 of which are concurrent. It is clear that, in $\pi'$, the plane dual to $\pi$, these six lines correspond to a 6-arc, which is also complete since $S$ is a blocking set. However, the main result of [7], which was obtained by R. H. F. Denniston on a computer, is that no plane of order 10 can contain a complete 6-arc. Thus Denniston's result eliminates case B(i) and the proof of Theorem 4 is complete.

**Corollary 5.** A net of order 10 which contains 6 or more parallel classes can be completed to a plane of order 10 in at most one way.

**Proof.** Let $\pi$, be an affine plane of order 10 containing a net $N$ which has exactly $t$ parallel classes. Let $\pi$ be a different affine plane of order 10 defined on the same points as $\pi$, $N$ and containing $N$. It is shown in [4, Theorem 2.1] that each line of $\pi$, which is not a line of $\pi$, gives rise to a blocking set $S$ in the projective completion of $\pi$, with $|S| = 10 + (11 - t)$. An application of Theorem 4 completes the proof.

3. Blocking sets. In this section we are concerned with blocking sets $S$ in a finite projective plane $\pi$ of order $n$. The main result in [5] was that $|S| \geq n + \sqrt{n} + 1$. The proof depended on the fact that if $S$ is as small as possible (that is, if $|S| = n + \sqrt{n} + 1$) then some line of $\pi$ contains at least $\sqrt{n} + 1$ points of $S$. One is
naturally tempted to try to show that for any blocking set there is some line containing at least $\sqrt{n} + 1$ points of $S$. Surprisingly, this turn out to be quite false. We can show that, in general, some line of $\pi$ must contain at least four points of $S$. This last result does not sound too impressive until it is shown (in Theorem 12) that this is the best one can hope to do.

**Theorem 6.** Let $S$ be any blocking set in a finite projective plane $\pi$ of order $n$. Then

(i) $n > 2$.

(ii) when $n = 3$ some line of $\pi$ contains exactly 3 points of $S$.

(iii) when $n = 4$ some line of $\pi$ contains exactly 4 points of $S$ unless $S$ has 3 points on each of its lines and is such that its points are the points of a projective subplane of order 2 or the points of an affine subplane of order 3.

(iv) when $n \geq 5$, some line of $\pi$ must contain at least 4 points of $S$.

**Proof.** If no 3 points of $S$ were collinear then $S$ would be a $k$-arc, so that $|S| = k \leq n + 2$. However by [5, Theorem 3.9], $|S| \geq n + \sqrt{n} + 1$. This proves (ii). It is pointed out in [5] that no blocking sets exist in the plane of order 2, and (i) follows.

We now assume that $n \geq 4$ and that no line of $\pi$ contains more than 3 points of $S$. We must show that this forces the points of $S$ to be the points of a subplane of the projective plane of order 4. The proof is broken into four parts. The following notation is adhered to: $m$ is a line containing exactly 3 (distinct) points of $S$, $B$ is the set of lines of $S$ passing through points of $m - S$, $I$ denotes the set of incidences of points of $S$ with lines of $B$, and $|S| = n + t$. Each of the four parts utilizes Lemma 3.3 in [5, p.381]. (Incidentally, the condition in Lemma 3.3 of [5] should read $b \leq a \leq 2b$.)

**Part 1.** If no line of $\pi$ contains more than 3 points of $S$, and $n \geq 4$, then $2n - 1 \leq |S| \leq 2n + 3$.

**Proof of Part 1.** The $n + t - 3$ points of $S - m$ are distributed among the $n$ lines through any point $P$ of $m - S$. Since there are at most 3 points of $S$ on any of the $n + 1$ lines through a point of $S$, we have $|S| \leq 1 + 2(n + 1) = 2n + 3$. Thus, $n < |S - m| \leq 2n$ and we can apply Lemma 3.3 in [5] which says that the maximum number of incidences in $I$ that can come from lines of $B$ through $P$ occurs when every line of $B$ through $P$ is a 2-line. Thus $|I| \leq 2(|S - m| - n) = 2(t - 3)$. We note also here that each 3-line through $P$ will lower this total by 1. Summing over all the $n + 1 - 3 = n - 2$
points of $m - S$, we obtain

$$|I| \leq 2(n - 2)(t - 3).$$

We now obtain a lower bound on $|I|$. There can be at most 7 points of $S$ on the three lines joining a point $P$ of $S - m$ to a point of $S$ on $m$. So there are at least $n + t - 7$ points of $S$ different from $P$ that are incident with lines of $B$ through $P$. If there be $I(P)$ such lines then

$$|I(P)| \geq \frac{n + t - 7}{2}.$$

Summing over all points of $S - m$ we obtain

$$|I| \geq \frac{1}{2}(n + t - 7)(n + t - 3).$$

This, together with (A) yields

$$4(n - 2)(t - 3) \geq (n + t - 7)(n + t - 3).$$

This yields that $(n + 1 - t)^2 \leq 4$. Thus $n - 1 \leq t \leq n + 3$, so that $2n - 1 \leq |S| \leq 2n + 3$.

**Part 2.** Under the hypotheses of Part 1, $|S|$ cannot be $2n$, and $|S|$ cannot be $2n + 2$.

**Proof of Part 2.** We note in formula (B) that $|I(P)|$ must be an integer. Since $n + t - 7$ is odd in the two cases under consideration, we get $2|I(P)| \geq n + t - 6$, so that (D) is improved to $4(n - 2)(t - 3) \geq (n + t - 3)(n + t - 6)$. Putting $t = n$ or $t = n + 2$ in this inequality leads to a contradiction.

**Part 3.** Under the hypotheses of Part 1 there can exist no 2-line of $B$.

**Proof of Part 3.** Suppose to the contrary that there exist points $Q, R \in S - m$ such that $QR$ is a 2-line of $S$. As in the argument for formula (B) in Part 1, $|I(Q)| \geq 1/2(n + t - 8) + 1$ and $|I(R)| \geq 1/2(n + t - 8) + 1$. Since $|S| = n + t$ is odd (by Parts 1,2), we can improve this to get $|I(Q)| \geq 1/2(n + t - 7) + 1$, and $|I(R)| \geq 1/2(n + t - 7) + 1$. Thus the lower bound in (C) is increased by at least two. The right side of (D) is thus increased by two, so the inequality becomes $2 \geq (n + 1 - t)^2$. The only possibility allowed by Part 2 is $t = n + 1$ and $|S| = 2n + 1$. This also is not possible. For example, suppose $n = 4$, so that $|S - m| = 6$. Thus if $P = QR \cdot m$
the other 3 lines through $P$ must include a second 2-line. The points of $S$ in that 2-line would each increase the right hand side of (D) by one. Furthermore, the other point of $m - S$ would necessarily lie on either a 3-line (and the left hand side of (D) would be lowered by one as in the calculation for (A)) or a pair of 2-lines (and the right hand side of (D) would be raised). Either possibility yields a contradiction. Finally, if $n > 4$ and $|S| = 2n + 1$ a similar argument would show the existence of enough 2-lines to increase the lower bound, or sufficiently many 3-lines to lower the upper bound (of $|I|$) to obtain the desired result.

**Part 4.** Under the hypotheses of Part 1 every line of $B$ is a 3-line, $n = 4$ and the points of $S$ are the points of an affine subplane of order 3 or a projective subplane having order 2.

**Proof.** By Part 3 every line of $B$ is a 3-line. We can now get a better lower bound for $|I|$. Let $P$ be any point of $m - S$. Every line of $\pi$ through $P$ apart from $m$ contains either 1 or 3 points of $S$. Thus, there are exactly $(t - 3)/2$ 3-lines, that is, lines of $B$ through $P$, each yielding 3 incidences in $I$. There are $n - 2$ points of $m - S$. Thus (D) becomes

$$3(t - 3)(n - 2) \geq (n + t - 3)(n + t - 7).$$

Because of Parts 1 and 2, we need only test the values $t = n + 3$, $t = n - 1$, $t = n + 1$. Now $t = n + 3$ yields a contradiction as does $t = n - 1$, $t = n + 1$ unless $n = 4$. The case $n = 4$, $t = 4 - 1 = 3$, $|S| = n + t = 7$ is possible. In fact, by [3, Theorem 1], the points of $S$ are the points of a Baer subplane of $PG(2, 4)$. Finally the case $n = 4$, $t = 4 + 1 = 5$ $|S| = n + t = 9$ is also possible. By studying how the inequality was obtained and using the fact that each line of $B$ is a 3-line, we see that, in order for this case to occur, every line of $S$ is a 3-line and there are 4 3-lines of $S$ through each of the 9 points of $S$. This implies that $S$ is the set of points of an affine subplane of order 3. It can be checked that such a subplane $\pi_0$ does exist in $\pi = PG(2, 4)$ and that, in fact, the points of $\pi_0$ do form a blocking set in $\pi$ (see [11, 12]). This completes the proof of Theorem 6.

**Remark.** In Math. Reviews, 42 no. 8389, it is stated that a more complete description of the projective plane $\pi$ of order 4 can be found in [9]. It turns out that the 12 lines of the subplane $\pi_0$ are partitioned into 4 “parallel” classes. One can easily show (since the diagonal points of any quadrangle are collinear) that the 3 lines...
in each “parallel” class are actually the sides of a triangle in $\pi$. Any pair of the 4 resulting triangles is perspective in 6 ways. The 6 centres of perspectivity are the vertices of the other 2 triangles, the axes being the corresponding opposite sides. Corresponding sides of each perspective pair meet in points of $S$. Finally, a line from any centre of perspectivity (containing one vertex from each of the triangles in perspective from that centre) meets the axis of the perspectivity in a point of $S$.

4. Desarguesian planes. We now want to find blocking sets $S$ in planes $\pi$ of “large” order such that no line of $\pi$ contains more than 4 points of $S$. A natural thing to try is two conics, but this works only in the infinite case. We then try a pair of suitable cubics and, fortunately, this works in some finite planes. The proof is preceded by the prerequisite results on the solution of cubic equations.

**Notation.** $y^3 + ay^2 + by + c$ is a cubic over the field $F$, with roots $y_1, y_2, y_3$ in some extension field. The discriminant $D$ is given by $D = [(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)]^2$. If $x_1, x_2, \ldots, x_n$ are algebraic over $F$, then $F(x_1, x_2, \ldots, x_n)$ will denote the extension of $F$ got by adjoining them.

**Lemma 7.** If the characteristic of $F$ is not 2, then $F(y_1, y_2, y_3) = F(\sqrt{D}, y_1)$.

**Proof.** Simply note that the proof in [1, p. 449] is valid for any field not of characteristic 2.

**Lemma 8.** Any cubic that is irreducible over the finite field $F = GF(q)$ has all its roots in $F(y_1)$. Thus $F(y_1) = F(y_1, y_2, y_3)$.

**Proof.** The field $K = F(y_1)$ is an extension of $F$ of degree 3 so that $K = GF(q^3)$ a field of order $q^3$. Every element of $K$ is a root of the polynomial $x^{q^3} - x$ over $F$. In fact, $K$ is the splitting field for this polynomial over $F$. Thus $K$ is a normal extension of $F$ by [1, p. 439].

**Lemma 9.** If the cubic is irreducible over $F = GF(q)$ then $\sqrt{D} \in F$.

**Proof.** When $q$ is even every element of $F$ has a square root in $F$. If $q$ is odd, $\sqrt{D} \in F(y_1, \sqrt{D}) = F(y_1, y_2, y_3) = F(y_1)$ by Lemmas 7, 8. Since $F(y_1)$ is an extension of $F$ of degree 3 all of its elements
that are not in $F$ must have degree 3 [1, Corollary 2, p. 407]. Thus $\sqrt[3]{D} \in F$, since otherwise $\sqrt[3]{D}$ would be of degree 2 over $F$, a contradiction.

**Lemma 10.** Let $F = GF(q)$, $q$ odd and $D \neq 0$. Then a cubic has exactly one root in $F$ if and only if $\sqrt[3]{D} \in F$. A cubic has either no roots or 3 roots in $F$ if and only if $\sqrt[3]{D} \in F$. Finally, a cubic can have 2 equal roots if and only if $D = 0$.

**Proof.** The claim concerning $D = 0$ follows immediately from the definition of $D$. Assume now that $D$ is a nonzero square in $F$. By Lemma 7, $F(y_0, y_1, y_2) = F(\sqrt[3]{D}, y_0)$. Thus either no root is in $F$ or all 3 roots are in $F$. Conversely, if no root is in $F$, then $\sqrt[3]{D} \in F$ by Lemma 9 and if all 3 roots are in $F$ then $\sqrt[3]{D} \in F$ by definition. We cannot have exactly 2 roots in $F$ since the product of the 3 roots is $-c \in F$. The rest of the lemma is immediate. The following is a straightforward verification ([1, p. 448]).

**Lemma 11.** The discriminant $D$ of $y^3 + by + c$ is $-4b^3 - 27c^2$.

**Theorem 12.** Let $\pi$ be the Desarguesian projective plane over the field $F = GF(3^s)$ where $s \geq 2$. Suppose $t \neq 0$ is a nonsquare in $F$. Write

$$S = A \cup B \cup C$$

where

$$A = \{(x, x^3) \mid x \in F\}$$

$$B = \{(x, x^3t) \mid x \in F\}$$

$$C = \{\infty\}.$$

Then $S$ is a blocking set in $\pi$ with $|S| = 2n = 2 \cdot 3^s$. Moreover, no line of $\pi$ contains more than 4 points of $S$.

**Proof.** The line at infinity contains exactly one point of $S$. The line $x = 0$ contains exactly 2 points of $S$, while each line $x = k$, $k$ a nonzero constant, contains 3 points of $S$, one in each of $A, B, C$. The line $y = 0$ contains one point of $S$. Since each element of $F$ has a unique cube root in $F$, each line $y = k$ with $k \neq 0$ contains one point of $A$ and one of $B$. We look at the line $y = mx + b$, $m \neq 0$. This line meets $A$ at all those points $(x, x^3)$ for which $x^3 - mx - b = 0$: it meets $B$ at all points $(x, x^3t)$ such that $x^3 - (m/t)x - b/t = 0$, and it does not meet $C$. By Lemma 11, recalling that $F$ has characteristic 3, the discriminants of the two cubics are $m^3$ and $(m/t)^3$ respectively. If $m$ is a nonzero square in $F$ then so is $m^3$ while $(m/t)^3$ is a nonsquare. By Lemma 10, the line has either 0 or exactly 3 points in
common with $A$ and 1 point in common with $B$. If $m$ is a nonsquare
a similar argument shows that the line contains 1 point of $A$ and
either 0 or 3 points of $B$. Thus any line $y = mx + b$ with $m \neq 0$
has either 1 or 4 points in common with $S$. Thus every line of $\pi$
contains at least one point of $S$. When $3^2 \geq 9$ all lines contain more
than 4 points so that no line can have all its points in $S$. This
shows that $S$ is a blocking set such that no line of $\pi$ contains more
than 4 points of $S$, completing the proof.

5. Concluding comments. In [3] it was shown that when $S$
is a blocking set in a projective plane $\pi$ of order $n$ with $|S| = n + \sqrt{n} + 1$, then the points of $S$ are the points of a Baer subplane.
As soon as $|S| > n + \sqrt{n} + 1$ however, things seem to get more
complicated. For example, we have

**Theorem 13.** There exists a projective plane $\pi$ of order $n$ and
a blocking set $S$ in $\pi$ with $|S| = n + \sqrt{n} + 2$ such that $S$ is not
obtained by adjoining a point to a Baer subplane of $\pi$.

**Proof.** Let $\pi = PG(2, 4)$. Let $S$ be the 8 points that remain
on the sides of triangle $ABC$ when $B, C$ and 2 other points of $BC$
are removed. $S$ is then a blocking set having $2^2 + 2 + 2$ points. It
contains no subplane, and so is not obtained by adjoining a point to
a subplane. This follows from the fact that such a configuration
would have exactly one 4-line, whereas there exist two 4-lines of $S$.

**Remark.** It still seems reasonable to conjecture that, in planes
$\pi$ of “large” order, if $S$ is a blocking set with $|S| = n + \sqrt{n} + 2$
then $S$ contains the points of a Baer subplane of $\pi$. Up to this
stage, we have been unable to decide whether the case $n = 4$ is
truly exceptional in this context.

It is possible to imitate the proof of Theorem 6 and get a
general lower bound on the number of points of a blocking set
which must lie on some line. In particular one can show

**Theorem 14.** Let $S$ be a blocking set in a finite projective
plane $\pi$ of order $n$ and let $k$ be the maximum number of points of
$S$ which lie on any line of $\pi$. Then $k(1 + n) \geq |S| + n$.

**Remark.** From the above inequality we get $k \geq 1 + (|S| - 1)
(1 + n)^{-1}$. Thus, for a fixed $n$, $k$ increases with $|S|$. Note that if
$|S|$ is as small as possible, namely $|S| = n + \sqrt{n} + 1$, we get only
that $k \geq 3$. Thus this general result is not as strong as Theorem 6.
Furthermore, the true relationship between $k$ and $S$ is not linear
since when $|S|$ is as small as possible, with $n$ being a square, then, in fact, $k = \sqrt{n} + 1$ [3, Theorem 1]. Note too that when $|S|$ is as large as possible, namely when $|S| = n^2 - \sqrt{n}$ (see [5, Theorem 3.9]), Theorem 14 does tell us that $k = n$.

Finally we want to comment on blocking sets $S$ in $\pi = PG(2, 7)$. It is known (see [5], [8]) that $|S| \geq 12$. The example given in [5] is a projective triangle (see [5, p. 390]). However, we want to point out that this example is not unique, as follows. Let $\pi'$ be the plane dual to $\pi$ (in fact, $\pi'$ is isomorphic to $\pi$). It is known (see [11, 12]) that $\pi'$ contains an affine subplane $\pi_0$ having order 3. Now one can easily see that through each point of $\pi'$ passes one of the 12 lines of $\pi_0$ and also a line of $\pi$ other that a line of $\pi_0$. The 12 lines of $\pi_0$ in $\pi'$ thus dualize to yield a blocking set $S$ in $\pi$ with $|S| = 12$. No line of $\pi$ contains more than 4 points of $S$, so that $S$ is not a projective triangle since, in the case of a projective triangle, some lines contain 5 points of the blocking set. The above example also suggests the possibility of further connections between blocking sets and the work in [11], [12].

\textit{Added in proof.} In this paper and in [6] we have seen that the existence of a blocking set with 15 points is (by duality) equivalent to the existence of a complete 6-arc in a plane $\pi$ of order 10. In [10, Results 2.4, 3.1] it is shown that a codeword of weight 15 yields a blocking set in $\pi$. Thus the nonexistence of a complete 6-arc (which is the main result in [7]) implies the main result in [10] (namely, that a codeword of weight 15 does not exist). In fact, the referee has informed us that the main results in [7], [10] are actually equivalent, one being the dual of the other.

\textbf{References}


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