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Let π be a finite projective plane of order n. Recall that a *blocking set* S in π is a set of points which does not contain any line but which does intersect every line of π. The first objective is to elaborate on the connection, pointed out by the writers, between blocking sets and complete k-arcs of π. For example, the set of secants of a complete k-arc with $k < n + 2$ dualizes to a blocking set. Using some simple observations, it is shown that a blocking set in a projective plane π of order ten, if π exists, contains at least 16 points. The proof uses a computer result on the nonexistence of complete 6-arcs of π due to R. H. F. Denniston. Using the result, a recent theorem concerning certain codes related to π due to MacWilliams, Sloane, and Thompson is easily established. The result also shows that, in effect, *a set of four mutually orthogonal latin squares of order ten is embeddable in a complete set in at most one way.* **This improves slightly on the bound of R. H. Bruck.**

Later on, new bounds are obtained on the number of points of a blocking set that lie on any line. Examples in finite Desarguesian planes are given to show that these bounds are, in a sense, best possible. In § 4 some miscellaneous remarks on blocking sets are made and an interesting example in $PG(2, 7)$ is discussed.

2. Planes of order 10.

The following theorem is a key to the results of this section.

THEOREM 1. *Let R be a complete k-arc in a projective plane π of order n, and let π' be the plane dual to π. When $k < n + 2$ the mapping that takes the lines of π into the points of π' takes the secants of R into a blocking set S in π'.*

Proof. The assumption that R is complete implies immediately that every point of π lies on a secant of R. Thus every line of π' contains at least one point of S. The maximum number of secants of R that pass through a point of π is $k - 1$: this occurs if and only if the point is on R. By our assumption that $k < n + 2$, every point of π is incident with at most $k - 1 \leqq n$ secants. Thus any line of π' contains at most n points of S, so that S is a blocking set in π.

REMARKS. We shall say that such a blocking set — the dual of the set of secants of a complete k-arc — is *derived from* (or is *obtained from* or *comes from* etc.) a complete k-arc. It is well-known that no k-arc can have more than $n + 2$ points, and only when n is even can we have $k = n + 2$. From the above it is clear that a blocking set cannot be derived from an $n + 2$-arc.

By Theorem 3.9 in [5], a blocking set in π must contain at least $n + \sqrt{n} + 1$ points. Since there are $\binom{k}{2}$ secants to any k-arc we immediately deduce from Theorem 1 that, if the k-arc is complete, $n + \sqrt{n} + 1 \leqq k(k - 1)/2$. It is possible to improve this bound to obtain a result due to M. Sce [13, p. 280].

THEOREM 2 (Sce). *Let R be a complete k-arc in a projective plane π of order n. Then $n \leqq 1/2\,(k - 1)(k - 2)$.*

Proof. We use the notation of Theorem 1. Corresponding to any point M of R is a line of π', call it m, which contains $k - 1$ points of the blocking set S obtained from R. Denoting by $|S|$ the number of points of S we have $|S| = \binom{k}{2}$ = number of secants to R. Let P be any point of m not in S; in symbols $P \in m - S$. Since S is a blocking set, the n lines of π through P that are different from m must each contain at least one point of S. Thus $|S| \geqq k - 1 + n$, so that $k(k - 1)/2 \geqq (k - 1) + n$, as claimed.

DEFINITIONS. A line of π that contains at least 2 points of a subset S of points of π will be called a *line of* S. A line that contains exactly t distinct points of S is called a *t-line* and is said to have *strength* t.

It would be of interest to characterize blocking sets obtained from complete k-arcs. A result in that direction is as follows.

THEOREM 3. *A blocking set S is obtained from a complete k-arc in a projective plane π if and only if the following three conditions are satisfied.*

(i) $|S| \leqq \binom{k}{2}$.

(ii) *The number of $(k - 1)$-lines of S is at least k.*

(iii) *No three of the $(k - 1)$-lines of S are concurrent.*

Proof. If S is derived from a complete k-arc the 3 conditions are clearly satisfied. Conversely, assume that a blocking set S has the three properties. Let W denote the set of $(k - 1)$-lines of S and let I denote the set of incidences of points of S with lines of W. By (ii), $|I| \geqq k(k - 1)$. The points of intersection of pairs of lines of W are distinct by (iii). Let r of these points be points of S. This leaves $|S| - r \leqq \binom{k}{2} - r$ points of S that lie on either one or zero lines of W. Thus $|I| \leqq r \cdot 2 + \left[\binom{k}{2} - r\right]$. Thus $r + \binom{k}{2} \geqq k(k-1)$, so that $r \geqq \binom{k}{2}$. But, by definition, $r \leqq |S| \leqq \binom{k}{2}$. Thus $r = \binom{k}{2}$, $|I| = k(k-1)$, $|W| = k$, $|S| = \binom{k}{2}$ and the points of S are precisely the $\binom{k}{2}$ intersections of all pairs of lines of W. Let R denote the dual of W, that is, the set of points of π' corresponding to W, where π' is the plane dual to π. Then, from the above, R is a k-arc, and the dual of S is the set of secants of R. Since S is a blocking set this forces R to be a complete k-arc, proving the result.

REMARK. Every k-arc in a finite projective plane gives rise to a system of diophantine equations [13, § 179]. In the light of Theorems 1 and 3, these equations can be used to study blocking sets derived from complete k-arcs. For example, let c_i be the number of i-lines of S and let j denote the greatest integer in $k/2$. We have immediately $c_0 = 0$, $c_{k-1} = k$, $c_i = 0$ for $j < i < k-1$. Furthermore, it can be shown that

$$\sum_{i=1}^{j} c_i = 1 + n + n^2 - k$$

$$2 \sum_{i=1}^{j} i c_i = k(k-1)(n-1)$$

and

$$4 \sum_{i=1}^{j} i^2 c_i = 2k(k-1)(n-1) + k(k-1)(k-2)(k-3)\ .$$

Other relevant identities and inequalities can be found in [13].

We now concentrate on the case $n = 10$. Our objective here is to show that if S is a blocking set in a projective plane π of order 10, then $|S| \geqq 16$. This was shown in [5] by using an additional assumption, namely that π contains no projective subplane of order 2. We proceed to show how this rather awkward assumption can be dropped.

THEOREM 4. *Suppose S is a blocking set in a projective plane π of order* 10. *Then S must contain at least* 16 *points.*

Proof. By [5, Theorem 3.9], $|S| \geqq 10 + 10^{1/2} + 1$. Assume $|S| = 15 = \binom{6}{2}$. We use the notation of [5] with $(P, 1)$ and $(P, 2)$ being

5-lines. The theorem was proved there under the additional assumption that π contained no projective subplane of order 2. The only place that this extra assumption was needed occurred in case B(i). Now, as in [5], we have 5, 6, 8 collinear, that is, (5, 6, 8). The following kind of argument is used repeatedly below. The line 5, 6, 8 contains already 3 distinct points of S. Suppose this line met the 5-line $(P, 2)$ in a point X not in S. Now S is a blocking set, so that, in particular, every line of π through X contains at least one point of S. Thus $|S| \geqq 3 + 5 + 9 = 17$, contradicting $|S| = 15$. Thus the line (5, 6, 8) contains a point of S, say 11, different from 2 and from P. Similarly, the line (7, 6, 3) must meet $(P, 2)$ in a point of S, say 9, with $9 \neq 2$, P, 11. Similarly (7, 6, 3) meets $(P, 1)$ in a point 12, with $12 \neq 14$, P, 1. Here 14 is where (5, 6, 8) meets $(P, 1)$. In summary, we now have (5, 6, 8, 11, 14) and (3, 6, 7, 9, 12). Join 7 to 8. In [5] it was shown that (7, 8, 4). This last line meets $(P, 2)$ in a point Q of S. Because two distinct lines meet in just one point, we know that $Q \neq 9, 11, 2, P$. So $Q = 10$ say. Similarly (7, 8) meets $(P, 1)$ in a point $13 \neq 14, 12, 1, P$. In short (4, 7, 8, 10, 13).

Finally consider the six 5-lines (1, 2, 3, 4, 5), (1, 12, 13, 14, P), (2, 9, 10, 11, P), (5, 6, 8, 11, 14), (3, 6, 7, 9, 10), (4, 7, 8, 10, 13), no 3 of which are concurrent. It is clear that, in π', the plane dual to π, these six lines correspond to a 6-arc, which is also complete since S is a blocking set. However, the main result of [7], which was obtained by R. H. F. Denniston on a computer, is that no plane of order 10 can contain a complete 6-arc. Thus Denniston's result eliminates case B(i) and the proof of Theorem 4 is complete.

COROLLARY 5. *A net of order* 10 *which contains* 6 *or more parallel classes can be completed to a plane of order* 10 *in at most one way.*

Proof. Let π_1 be an affine plane of order 10 containing a net N which has exactly t parallel classes. Let π_2 be a different affine plane of order 10 defined on the same points as π_1, N and containing N. It is shown in [4, Theorem 2.1] that each line of π_2 which is not a line of π_1 gives rise to a blocking set S in the projective completion of π_1 with $|S| = 10 + (11 - t)$. An application of Theorem 4 completes the proof.

3. **Blocking sets.** In this scetion we are concerned with blocking sets S in a finite projective plane π of order n. The main result in [5] was that $|S| \geqq n + \sqrt{n} + 1$. The proof depended on the fact that if S is as small as possible (that is, if $|S| = n + \sqrt{n} + 1$) then some line of π contains at least $\sqrt{n} + 1$ points of S. One is

naturally tempted to try to show that for any blocking set there is some line containing at least $\sqrt{n}+1$ points of S. Surprisingly, this turn out to be quite false. We can show that, in general, some line of π must contain at least four points of S. This last result does not sound too impressive until it is shown (in Theorem 12) that this is the best one can hope to do.

THEOREM 6. *Let S be any blocking set in a finite projective plane π of order n. Then*

(i) $n > 2$.

(ii) *when $n = 3$ some line of π contains exactly 3 points of S.*

(iii) *when $n = 4$ some line of π contains exactly 4 points of S unless S has 3 points on each of its lines and is such that its points are the points of a projective subplane of order 2 or the points of an affine subplane of order 3.*

(iv) *when $n \geqq 5$, some line of π must contain at least 4 points of S.*

Proof. If no 3 points of S were collinear then S would be a k-arc, so that $|S| = k \leqq n + 2$. However by [5, Theorem 3.9], $|S| \geqq n + \sqrt{n} + 1$. This proves (ii). It is pointed out in [5] that no blocking sets exist in the plane of order 2, and (i) follows.

We now assume that $n \geqq 4$ and that *no* line of π contains more than 3 points of S. We must show that this forces the points of S to be the points of a subplane of the projective plane of order 4. The proof is broken into four parts. The following notation is adhered to: m is a line containing exactly 3 (distinct) points of S, B is the set of lines of S passing through points of $m - S$, I denotes the set of incidences of points of S with lines of B, and $|S| = n + t$. Each of the four parts utilizes Lemma 3.3 in [5, p. 381]. (Incidentally, the condition in Lemma 3.3 of [5] should read $b \leqq a \leqq 2b$.)

Part 1. *If no line of π contains more than 3 points of S, and $n \geqq 4$, then $2n - 1 \leqq |S| \leqq 2n + 3$.*

Proof of Part 1. The $n + t - 3$ points of $S - m$ are distributed among the n lines through any point P of $m - S$. Since there are at most 3 points of S on any of the $n + 1$ lines through a point of S, we have $|S| \leqq 1 + 2(n + 1) = 2n + 3$. Thus, $n < |S - m| \leqq 2n$ and we can apply Lemma 3.3 in [5] which says that the maximum number of incidences in I that can come from lines of B through P occurs when every line of B through P is a 2-line. Thus $|I| \leqq 2(|S - m| - n) = 2(t - 3)$. *We note also here that each 3-line through P will lower this total by* 1. Summing over all the $n + 1 - 3 = n - 2$

points of $m - S$, we obtain

$$\text{(A)} \qquad |I| \leqq 2(n-2)(t-3) .$$

We now obtain a lower bound on $|I|$. There can be at most 7 points of S on the three lines joining a point P of $S - m$ to a point of S on m. So there are at least $n + t - 7$ points of S different from P that are incident with lines of B through P. If there be $I(P)$ such lines then

$$\text{(B)} \qquad |I(P)| \geqq \frac{n+t-7}{2} .$$

Summing over all points of $S - m$ we obtain

$$\text{(C)} \qquad |I| \geqq \frac{1}{2}(n+t-7)(n+t-3) .$$

This, together with (A) yields

$$\text{(D)} \qquad 4(n-2)(t-3) \geqq (n+t-7)(n+t-3) .$$

This yields that $(n + 1 - t)^2 \leqq 4$. Thus $n - 1 \leqq t \leqq n + 3$, so that $2n - 1 \leqq |S| \leqq 2n + 3$.

Part 2. *Under the hypotheses of Part* 1, $|S|$ *cannot be* $2n$, *and* $|S|$ *cannot be* $2n + 2$.

Proof of Part 2. We note in formula (B) that $|I(P)|$ must be an integer. Since $n + t - 7$ is odd in the two cases under consideration, we get $2|I(P)| \geqq n + t - 6$, so that (D) is improved to $4(n - 2)(t - 3) \geqq (n + t - 3)(n + t - 6)$. Putting $t = n$ or $t = n + 2$ in this inequality leads to a contradiction.

Part 3. *Under the hypotheses of Part* 1 *there can exist no* 2-*line of* B.

Proof of Part 3. Suppose to the contrary that there exist points $Q, R \in S - m$ such that QR is a 2-line of S. As in the argument for formula (B) in Part 1, $|I(Q)| \geqq 1/2(n + t - 8) + 1$ and $|I(R)| \geqq 1/2(n + t - 8) + 1$. Since $|S| = n + t$ is odd (by Parts 1,2), we can improve this to get $|I(Q)| \geqq 1/2(n + t - 7) + 1$, and $|I(R)| \geqq 1/2(n + t - 7) + 1$. Thus the lower bound in (C) is increased by at least two. The right side of (D) is thus increased by two, so the inequality becomes $2 \geqq (n + 1 - t)^2$. The only possibility allowed by Part 2 is $t = n + 1$ and $|S| = 2n + 1$. This also is not possible. For example, suppose $n = 4$, so that $|S - m| = 6$. Thus if $P = QR \cdot m$

the other 3 lines through P must include a second 2-line. The points of S in that 2-line would each increase the right hand side of (D) by one. Furthermore, the other point of $m - S$ would necessarily lie on either a 3-line (and the left hand side of (D) would be lowered by one as in the calculation for (A)) or a pair of 2-lines (and the right hand side of (D) would be raised). Either possibility yields a contradiction. Finally, if $n > 4$ and $|S| = 2n + 1$ a similar argument would show the existence of enough 2-lines to increase the lower bound, or sufficiently many 3-lines to lower the upper bound (of $|I|$) to obtain the desired result.

Part 4. *Under the hypotheses of Part* 1 *every line of B is a* 3-*line, $n = 4$ and the points of S are the points of an affine subplane of order* 3 *or a projective subplane having order* 2.

Proof. By Part 3 every line of B is a 3-line. We can now get a better lower bound for $|I|$. Let P be any point of $m - S$. Every line of π through P apart from m contains either 1 or 3 points of S. Thus, there are exactly $(t - 3)/2$ 3-lines, that is, lines of B through P, each yielding 3 incidences in I. There are $n - 2$ points of $m - S$. Thus (D) becomes

$$3(t - 3)(n - 2) \geqq (n + t - 3)(n + t - 7) .$$

Because of Parts 1 and 2, we need only test the values $t = n + 3$, $t = n - 1$, $t = n + 1$. Now $t = n + 3$ yields a contradiction as does $t = n - 1$, $t = n + 1$ unless $n = 4$. The case $n = 4$, $t = 4 - 1 = 3$, $|S| = n + t = 7$ is possible. In fact, by [3, Theorem 1], the points of S are the points of a Baer subplane of $PG(2, 4)$. Finally the case $n = 4$, $t = 4 + 1 = 5$ $|S| = n + t = 9$ is also possible. By studying how the inequality was obtained and using the fact that each line of B is a 3-line, we see that, in order for this case to occur, *every* line of S is a 3-line and there are 4 3-lines of S through each of the 9 points of S. This implies that S is the set of points of an affine subplane of order 3. It can be checked that such a subplane π_0 does exist in $\pi = PG(2, 4)$ and that, in fact, the points of π_0 do form a blocking set in π (see [11, 12]). This completes the proof of Theorem 6.

REMARK. In *Math. Reviews*, 42 no. 8389, it is stated that a more complete description of the projective plane π of order 4 can be found in [9]. It turns out that the 12 lines of the subplane π_0 are partitioned into 4 "parallel" classes. One can easily show (since the diagonal points of any quadrangle are collinear) that the 3 lines in each "parallel" class are actually the sides of a triangle in π. Any pair of the 4 resulting triangles is perspective in 6 ways. The 6 centres of perspectivity are the vertices of the other 2 triangles, the axes being the corresponding opposite sides. Corresponding sides of each perspective pair meet in points of S. Finally, a line from any centre of perspectivity (containing one vertex from each of the triangles in perspective from that centre) meets the axis of the perspectivity in a point of S.

4. Desarguesian planes. We now want to find blocking sets S in planes π of "large" order such that no line of π contains more than 4 points of S. A natural thing to try is two conics, but this works only in the infinite case. We then try a pair of suitable cubics and, fortunately, this works in some finite planes. The proof is preceded by the prerequisite results on the solution of cubic equations.

NOTATION. $y^3 + ay^2 + by + c$ is a cubic over the field F, with roots y_1, y_2, y_3 in some extension field. The discriminant D is given by $D = [(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)]^2$. If $x_1, x_2, \cdots, x_n$ are algebraic over F, then $F(x_1, x_2, \cdots, x_n)$ will denote the extension of F got by adjoining them.

LEMMA 7. *If the characteristic of F is not 2, then $F(y_1, y_2, y_3) = F(\sqrt{D}, y_1)$.*

Proof. Simply note that the proof in [1, p. 449] is valid for any field not of characteristic 2.

LEMMA 8. *Any cubic that is irreducible over the finite field $F = GF(q)$ has all its roots in $F(y_1)$. Thus $F(y_1) = F(y_1, y_2, y_3)$.*

Proof. The field $K = F(y_1)$ is an extension of F of degree 3 so that $K = GF(q^3)$ a field of order q^3. Every element of K is a root of the polynomial $x^{q^3} - x$ over F. In fact, K is the splitting field for this polynomial over F. Thus K is a normal extension of F by [1, p. 439].

LEMMA 9. *If the cubic is irreducible over $F = GF(q)$ then $\sqrt{D} \in F$.*

Proof. When q is even every element of F has a square root in F. If q is odd, $\sqrt{D} \in F(y_1, \sqrt{D}) = F(y_1, y_2, y_3) = F(y_1)$ by Lemmas 7, 8. Since $F(y_1)$ is an extension of F of degree 3 all of its elements that are not in F must have degree 3 [1, Corollary 2, p. 407]. Thus $\sqrt{D} \in F$, since otherwise $\sqrt{D}$ would be of degree 2 over F, a contradiction.

LEMMA 10. *Let $F = GF(q)$, q odd and $D \neq 0$. Then a cubic has exactly one root in F if and only if $\sqrt{D} \notin F$. A cubic has either no roots or 3 roots in F if and only if $\sqrt{D} \in F$. Finally, a cubic can have 2 equal roots if and only if $D = 0$.*

Proof. The claim concerning $D = 0$ follows immediately from the definition of D. Assume now that D is a nonzero square in F. By Lemma 7, $F(y_1, y_2, y_3) = F(\sqrt{D}, y_1)$. Thus either no root is in F or all 3 roots are in F. Conversely, if no root is in F, then $\sqrt{D} \in F$ by Lemma 9 and if all 3 roots are in F then $\sqrt{D} \in F$ by definition. We cannot have exactly 2 roots in F since the product of the 3 roots is $-c \in F$. The rest of the lemma is immediate. The following is a straightforward verification ([1, p. 448]).

LEMMA 11. *The discriminant D of $y^3 + by + c$ is $-4b^3 - 27c^2$.*

THEOREM 12. *Let π be the Desarguesian projective plane over the field $F = GF(3^s)$ where $s \geqq 2$. Suppose $t \neq 0$ is a nonsquare in F. Write*

$$S = A \cup B \cup C$$

where

$$\begin{aligned} A &= \{(x, x^3) \mid x \in F\} \\ B &= \{(x, x^3 t) \mid x \in F\} \\ C &= \{(\infty)\} . \end{aligned}$$

Then S is a blocking set in π with $|S| = 2n = 2.3^s$. Moreover, no line of π contains more than 4 points of S.

Proof. The line at infinity contains exactly one point of S. The line $x = 0$ contains exactly 2 points of S, while each line $x = k$, k a nonzero constant, contains 3 points of S, one in each of A, B, C. The line $y = 0$ contains one point of S. Since each element of F has a unique cube root in F, each line $y = k$ with $k \neq 0$ contains one point of A and one of B. We look at the line $y = mx + b$, $m \neq 0$. This line meets A at all those points (x, x^3) for which $x^3 - mx - b = 0$: it meets B at all points $(x, x^3 t)$ such that $x^3 - (m/t)x - b/t = 0$, and it does not meet C. By Lemma 11, recalling that F has characteristic 3, the discriminants of the two cubics are m^3 and $(m/t)^3$ respectively. If m is a nonzero square in F then so is m^3 while $(m/t)^3$ is a nonsquare. By Lemma 10, the line has either 0 or exactly 3 points in common with A and 1 point in common with B. If m is a nonsquare a similar argument shows that the line contains 1 point of A and either 0 or 3 points of B. Thus any line $y = mx + b$ with $m \neq 0$ has either 1 or 4 points in common with S. Thus every line of π contains at least one point of S. When $3^s \geqq 9$ all lines contain more than 4 points so that no line can have all its points in S. This shows that S is a blocking set such that no line of π contains more than 4 points of S, completing the proof.

5. Concluding comments. In [3] it was shown that when S is a blocking set in a projective plane π of order n with $|S| = n + \sqrt{n} + 1$, then the points of S are the points of a Baer subplane. As soon as $|S| > n + \sqrt{n} + 1$ however, things seem to get more complicated. For example, we have

THEOREM 13. *There exists a projective plane π or order n and a blocking set S in π with $|S| = n + \sqrt{n} + 2$ such that S is not obtained by adjoining a point to a Baer subplane of π.*

Proof. Let $\pi = PG(2, 4)$. Let S be the 8 points that remain on the sides of triangle ABC when B, C and 2 other points of BC are removed. S is then a blocking set having $2^2 + 2 + 2$ points. It contains no subplane, and so is not obtained by adjoining a point to a subplane. This follows from the fact that such a configuration would have exactly one 4-line, whereas there exist two 4-lines of S.

REMARK. It still seems reasonable to conjecture that, in planes π of "large" order, if S is a blocking set with $|S| = n + \sqrt{n} + 2$ then S contains the points of a Baer subplane of π. Up to this stage, we have been unable to decide whether the case $n = 4$ is truly exceptional in this context.

It is possible to imitate the proof of Theorem 6 and get a general lower bound on the number of points of a blocking set which must lie on some line. In particular one can show

THEOREM 14. *Let S be a blocking set in a finite projective plane π of order n and let k be the maximum number of points of S which lie on any line of π. Then $k(1 + n) \geqq |S| + n$.*

REMARK. From the above inequality we get $k \geqq 1 + (|S| - 1)(1 + n)^{-1}$. Thus, for a fixed n, k increases with $|S|$. Note that if $|S|$ is as small as possible, namely $|S| = n + \sqrt{n} + 1$, we get only that $k \geqq 3$. Thus this general result is not as strong as Theorem 6. Furthermore, the true relationship between k and S is not linear since when $|S|$ is as small as possible, with n being a square, then, in fact, $k = \sqrt{n} + 1$ [3, Theorem 1]. Note too that when $|S|$ is as large as possible, namely when $|S| = n^2 - \sqrt{n}$ (see [5, Theorem 3.9]), Theorem 14 does tell us that $k = n$.

Finally we want to comment on blocking sets S in $\pi = PG(2, 7)$. It is known (see [5], [8]) that $|S| \geqq 12$. The example given in [5] is a *projective triangle* (see [5, p. 390]). However, we want to point out that this example is not unique, as follows. Let π' be the plane dual to π (in fact, π' is isomorphic to π). It is known (see [11, 12]) that π' contains an affine subplane π_0 having order 3. Now one can easily see that through each point of π' passes one of the 12 lines of π_0 and also a line of π other that a line of π_0. The 12 lines of π_0 in π' thus dualize to yield a blocking set S in π with $|S| = 12$. No line of π contains more than 4 points of S, so that S is not a projective triangle since, in the case of a projective triangle, some lines contain 5 points of the blocking set. The above example also suggests the possibility of further connections between blocking sets and the work in [11], [12].

Added in proof. In this paper and in [6] we have seen that the existence of a blocking set with 15 points is (by duality) equivalent to the existence of a complete 6-arc in a plane π of order 10. In [10, Results 2.4, 3.1] it is shown that a codeword of weight 15 yields a blocking set in π. Thus the nonexistence of a complete 6-arc (which is the main result in [7]) implies the main result in [10] (namely, that a codeword of weight 15 does not exist). In fact, the referee has informed us that the main results in [7], [10] are actually equivalent, one being the dual of the other.

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