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A number of questions involving T-nilpotence are studied. §1 contains characterizations of left and two-sided T-nilpotent rings in terms of (transfinite) annihilator series and a list of ring constructions which preserve T-nilpotence. In §2 the radical theory of T-nilpotence is investigated. It is shown that a left T-nilpotent ring belongs to a radical (resp. semi-simple) class precisely when the zeroring on its additive group does so, and that there are no interesting radical classes which consist entirely of left T-nilpotent rings. §3 is devoted to an examination of the effect which chain conditions on the type set of a suitably restricted torsion-free abelian group G have on the kinds of ring multiplication which G admits. Some conditions are given which are sufficient to ensure that every multiplication on G is (two-sided) T-nilpotent. A result from §2 is used to show that certain homogeneous groups do not admit nontrivial nilpotent multiplications. In the final brief section an example is used to show that whereas two-sided T-nilpotent rings satisfy the idealizer condition, the same need not be true of a left T-nilpotent ring.

1. Generalities. A ring (all rings considered are associative) is *left T-nilpotent* if for every sequence $x_1, x_2, \cdots$ of its elements, there exists an index n for which $x_1 \cdots x_n = 0$. *Right T-nilpotence* is similarly defined. The terms are due to Bass [2] though the concepts were introduced by Levitzki [13]. The class of left T-nilpotent rings lies strictly between the class of nilpotent rings and the Baer lower (= prime) radical class $\mathscr{B}$. Our first result lists some closure properties of the class of left T-nilpotent rings. Here, and elsewhere throughout the paper, the symbol $\lhd$ is used to denote ideals.

THEOREM 1.1. *The class of left T-nilpotent rings is closed under formation of subrings, homomorphic images, extensions and direct sums. A ring is left T-nilpotent if and only if every countable subring is.*

Proof. Subrings and homomorphic images present no difficulties.

Suppose $I \lhd R$ where I and R/I are left T-nilpotent. For any sequence $x_1, x_2, \cdots$ of elements of R, we have $y_1 = x_1 \cdots x_{n_1} \in I$ for some n_1. Similarly there exists $n_2 > n_1$ for which $y_2 = x_{n_1+1} \cdots x_{n_2} \in I$. Proceeding thus, we construct a sequence $y_1, y_2, \cdots$ of elements of I

and consequently for some n_k we have $x_1 \cdots x_{n_k} = y_1 \cdots y_k = 0$, so that R is left T-nilpotent.

Let $\{R_\lambda \mid \lambda \in \Lambda\}$ be a set of left T-nilpotent rings and let (x_λ) denote a typical element of $\bigoplus_\Lambda R_\lambda$ (ring direct sum). Suppose there exists a sequence $(x_{\lambda,1}), (x_{\lambda,2}), \cdots$ with $(x_{\lambda,1}) \cdots (x_{\lambda,n}) = (\prod_{i=1}^n x_{\lambda,i}) \neq 0$ for each n. There is no loss of generality in assuming that the same set of indices λ is nontrivially involved with each term. But then $\prod_{i=1}^n x_{\lambda,i} \neq 0$ for each n, contradicting the assumed left T-nilpotence of R_λ for each relevant value of λ. Hence there is no such sequence and $\bigoplus_\Lambda R_\lambda$ is left T-nilpotent.

The last assertion of the theorem is clear.

We are going to obtain another characterization of T-nilpotence, but before proceeding to this we need to introduce some notation and definitions.

If S is a subset of a ring R, we denote the left (resp. right, resp. two-sided) annihilator of S in R by $(0:S)$ (resp. $(S:0)$, resp. $(0:S)_*$). A *left annihilator* series of a ring R is a (transfinite) series

$$0 = R_0 \subseteqq R_1 \subseteqq \cdots \subseteqq R_\alpha \subseteqq \cdots \subseteqq R_\mu = R$$

of subrings of R such that $R_\alpha \lhd R_{\alpha+1}$ and $R_{\alpha+1}R \subseteqq R_\alpha$ for each α, and $R_\beta = \bigcup_{\alpha<\beta} R_\alpha$ if β is a limit ordinal. For an arbitrary ring R, we define the ideals $R^{(\alpha)}$ inductively as follows: $R^{(0)} = 0$, $R^{(\alpha+1)}/R^{(\alpha)} = (0: R/R^{(\alpha)})$ and $R^{(\beta)} = \bigcup_{\alpha<\beta} R^{(\alpha)}$ if β is a limit ordinal. If $R = R^{(\mu)}$ for some ordinal μ, the series

$$0 = R^{(0)} \subseteqq R^{(1)} \subseteqq \cdots \subseteqq R^{(\alpha)} \subseteqq \cdots \subseteqq R^{(\mu)} = R$$

is called the *upper left annihilator series* of R.

THEOREM 1.2. *The following conditions are equivalent for a ring R.*

(i) *The upper left annihilator series of R exists.*

(ii) *R has a left annihilator series.*

(iii) *Every nonzero homomorphic image of R has nonzero left annihilator.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let $I(\neq R)$ be an ideal,

$$0 = R_0 \subseteqq R_1 \subseteqq \cdots \subseteqq R_\alpha \subseteqq \cdots \subseteqq R_\mu = R$$

a left annihilator series of R and let $\beta = \mathrm{Min.}\{\alpha \mid R_\alpha \nsubseteq I\}$. Then β is not a limit ordinal, $(R_\beta + I)/I \neq 0$ and

$$[(R_\beta + I)/I][R/I] = (R_\beta R + I)/I \subseteqq (R_{\beta-1} + I)/I = 0 .$$

(iii) $\Rightarrow$ (i): The chain

$$0 = R^{(0)} \subseteqq R^{(1)} \cdots$$

must terminate, at $R^{(\mu)}$, say. But then $(0: R/R^{(\mu)}) = R^{(\mu+1)}/R^{(\mu)} = 0$, whence $R = R^{(\mu)}$.

The following characterization of left T-nilpotence resembles a group-theoretic result of Chernikov ([3], Theorem 1; see also [11], p. 219) which characterizes ZA groups. The ring-theoretic result was obtained by Levitzki [13] but we include a proof for the sake of completeness.

THEOREM 1.3. *A ring R is left T-nilpotent if and only if it satisfies the conditions of Theorem* 1.2.

Proof. Suppose R has a left annihilator series

$$0 = R_0 \subseteqq R_1 \subseteqq \cdots \subseteqq R_\alpha \subseteqq \cdots \subseteqq R_\mu = R .$$

If R is not left T-nilpotent, it has elements $x_1, x_2, \cdots$ such that $a_k = x_1 \cdots x_k \neq 0$ for all values of k. Let β be the least ordinal α for which R_α contains some a_k, and let a_m be in R_β. Clearly β is not a limit ordinal and $\beta > 0$. But then $a_{m+1} = a_m x_{m+1} \in R_\beta R \subseteqq R_{\beta-1}$, contrary to our choice of β.

For the converse, it suffices, by Theorem 1.1, to prove that every nonzero left T-nilpotent ring has nonzero left annihilator. Let R be such a ring, x_1 any nonzero element of R. Either $x_1 \in (0: R)$ or $x_1x_2 \neq 0$ for some $x_2 \in R$. Either $x_1x_2 \in (0: R)$ or $x_1x_2x_3 \neq 0$ for some x_3. This cannot go on, so there are elements $x_1, \cdots, x_k$ with $0 \neq x_1 \cdots x_k \in (0: R)$.

Theorem 1.3 enables us to demonstrate some further closure properties of the class of left T-nilpotent rings. In the sequel, $R(n \times n)$ denotes the ring of $n \times n$ matrices, $R[x]$ the polynomial ring, $R[G]$ the group ring corresponding to a group G, over a ring R.

PROPOSITION 1.4. *If R is left T-nilpotent, then so are $R(n \times n)$, $R[x]$ and $R[G]$ for any group G.*

Proof. We shall prove that $R[x]$ is left T-nilpotent; the other statements can be proved similarly. Let

$$0 = R^{(0)} \subseteqq R^{(1)} \subseteqq \cdots \subseteqq R^{(\alpha)} \subseteqq \cdots \subseteqq R^{(\mu)} = R$$

be the upper left annihilator series of R. For any polynomial $p(x) = a_0 + \cdots + a_n x^n \in R[x]$ such that $p(x)R[x] \subseteqq R^{(\alpha)}[x]$, we have, in particular, $a_0a + \cdots + a_nax^n \in R^{(\alpha)}[x]$, and so $a_0a, \cdots, a_na \in R^{(\alpha)}$ for every $a \in R$, i.e., $p(x)$ belongs to $R^{(\alpha+1)}[x]$. Conversely, if $p(x) \in R^{(\alpha+1)}[x]$, then for any $g(x) = b_0 + \cdots + b_mx^m \in R[x]$, we have $p(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots$ with each coefficient belonging to $R^{(\alpha)}$, i.e., $p(x)g(x) \in R^{(\alpha)}[x]$. If β is a limit ordinal, then $R^{(\beta)}[x] = (\bigcup_{\alpha<\beta} R^{(\alpha)})[x] = \bigcup_{\alpha<\beta} R^{(\alpha)}[x]$. It follows that

$$0 = R^{(0)}[x] \subseteqq R^{(1)}[x] \subseteqq \cdots \subseteqq R^{(\alpha)}[x] \subseteqq \cdots \subseteqq R^{(\mu)}[x] = R[x]$$

is the upper left annihilator series of $R[x]$.

One obtains similar results by considering two-sided annihilators, a *two-sided annihilator series*

$$0 = R_{0*} \subseteqq R_{1*} \subseteqq \cdots \subseteqq R_{\alpha*} \subseteqq \cdots \subseteqq R_{\mu*} = R$$

and an *upper two-sided annihilator series*

$$0 = R_*^{(0)} \subseteqq R_*^{(1)} \subseteqq \cdots \subseteqq R_*^{(\alpha)} \subseteqq \cdots \subseteqq R_*^{(\mu)} = R$$

being defined in the obvious way.

We need give no proof of

THEOREM 1.5. *The following conditions are equivalent for a ring R.*

(i) *The upper two-sided annihilator series of R exists.*

(ii) *R has a two-sided annihilator series.*

(iii) *Every nonzero homomorphic image of R has a nonzero two-sided annihilator.*

THEOREM 1.6. *A ring R is left and right T-nilpotent if and only if it satisfies the conditions of Theorem* 1.5.

Proof. If R has a two-sided annihilator series, the latter is both a left and a right annihilator series, so R is left and right T-nilpotent.

Conversely, if R is left and right T-nilpotent, then $(0: R) \neq 0$. Let x_1 be a nonzero element of $(0: R)$. Either $x_1 \in (R: 0)$ or $x_2x_1 \neq 0$ for some x_2. The right T-nilpotence of R requires that repetitions of this process eventually produce a nonzero element $x_k \cdots x_1 \in (R: 0)$. But $(0: R) \lhd R$, so $x_k \cdots x_1 \in (0: R)_*$. Since the class of rings which are left and right T-nilpotent is homomorphically closed, the result follows.

We conclude this section by noting a connection between T-nilpotence and another generalization of nilpotence first discussed by Levitzki [13]. A topological variant has subsequently been studied by Leptin [12] and Wiegandt [19]. The *left transfinite powers* $\bar{R}^{\alpha}$ of a ring R are defined as follows: $\bar{R}^1 = R$, $\bar{R}^{\alpha+1} = R\bar{R}^{\alpha}$ and $\bar{R}^{\beta} = \bigcap_{\alpha<\beta} \bar{R}^{\alpha}$ if β is a limit ordinal. The result which follows is due to Levitzki [13].

PROPOSITION 1.7. *Let R be a left T-nilpotent ring with $R^{(\mu)} = R$. Then $\bar{R}^{\mu+1} = 0$.*

Proof. It may be assumed that $R \neq 0$. A straightforward transfinite induction argument shows that $R^{(\alpha)}\bar{R}^{\alpha} = 0$ for every ordinal $\alpha > 0$. In particular, $\bar{R}^{\mu+1} = R\bar{R}^{\mu} = R^{(\mu)}\bar{R}^{\mu} = 0$.

The ultimate vanishing of transfinite left powers is a much weaker condition than left T-nilpotence, as one can see by observing that $\bar{E}^{\omega} = 0$, where E is the ring of even integers and ω is the first infinite ordinal.

2. **T-nilpotence and radical theory.** We turn now to an examination of the behavior of T-nilpotent rings as it affects radical and semi-simple classes. We denote the lower radical class by $L(\ )$, but in other respects largely conform to the conventions and usage of Divinsky's book [4], where all undefined radical-theoretic terms are explained.

The results of this section provide answers, as special cases, to some questions involving nilpotent rings which were raised and partly answered in an earlier paper of the author [7]. As some of the ideas of that paper are germane to the present discussion, we begin by recalling the basic facts.

In what follows, R^+ is the additive group of a ring R, G^0 the zeroring on an abelian group G. Let $\mathscr{T}$ be a radical class of abelian groups. Then $\mathscr{T}^* = \{R \mid R^+ \in \mathscr{T}\}$ is a radical class of rings. Such classes are called *A-radical classes* and are characterized by their property of containing, together with any member R, all rings S for which $S^+ \cong R^+$.

PROPOSITION 2.1. *Let R be a left T-nilpotent ring, $\mathscr{S}$ a semi-simple class containing $(R^+)^0$. Then $R \in \mathscr{S}$.*

Proof. If $R^2 = 0$, then $R = (R^+)^0 \in \mathscr{S}$. If $R^2 \neq 0$, then $R^{(1)} \neq 0$ and $R^{(2)} \neq R^{(1)}$. For any $s \in R^{(2)}\backslash R^{(1)}$ we have $sR^2 = (sR)R \subseteq R^{(1)}R = 0$. Hence the correspondence $r \mapsto sr$ defines a ring homomorphism from R onto sR. Also $0 \neq sR = ((sR)^+)^0 \lhd (R^+)^0$, so $sR \in \mathscr{S}$, i.e., R has a nonzero homomorphic image in $\mathscr{S}$. If $0 \neq I \lhd R$, then $(I^+)^0 \lhd (R^+)^0$, so $(I^+)^0 \in \mathscr{S}$ and as above, I has a nonzero homomorphic image in $\mathscr{S}$. Thus $R \in \mathscr{S}$.

PROPOSITION 2.2. *Let R be a left T-nilpotent ring and $\mathscr{R}$ a radical class containing R. Then $(R^+)^0 \in \mathscr{R}$.*

Proof. By Proposition 2.1, $\mathscr{R}((R^+)^0) \neq 0$. Now $\mathscr{T} = \{G \mid G^0 \in \mathscr{R}\}$ is a radical class of abelian groups ([7], Proposition 1.1) and $\mathscr{R}((R^+)^0)^+ = \mathscr{T}(R^+)$ is a fully invariant subgroup of R^+. Thus $\mathscr{R}((R^+)^0) = (I^+)^0$ for some $I \lhd R$. It follows that $((R/I)^+)^0 = (R^+)^0/\mathscr{R}((R^+)^0)$ is $\mathscr{R}$-semi-simple, whence by Proposition 2.1, so is R/I. But then $R/I = 0$, so $(R^+)^0 = (I^+)^0 \in \mathscr{R}$.

PROPOSITION 2.3. *Let R be a left T-nilpotent ring, $\mathscr{R}$ a radical class containing $(R^+)^0$. Then $R \in \mathscr{R}$.*

Proof. If $R^2 = 0$, then $R = (R^+)^0 \in \mathscr{R}$. Otherwise, as in the proof of Proposition 2.1, we can select $s \in R^{(2)} \backslash R^{(1)}$ with $0 \neq sR \subseteqq R^{(1)} = (0\colon R)$. Since $R^{(1)}$ is a zeroring, we have $sR \lhd R^{(1)} \lhd R$ and $(sR)^2 = 0$. Moreover, sR is a homomorphic image of $(R^+)^0$ *via* the correspondence $r \mapsto sr$, and so $sR \in \mathscr{R}$. Let V be a nonzero homomorphic image of R. Then $(V^+)^0$, as a homomorphic image of $(R^+)^0$, belongs to $\mathscr{R}$. As above, either $V^2 = 0$ or there is a chain $0 \neq vV \lhd V^{(1)} \lhd V$ for some $v \in V$, where $vV \in \mathscr{R}$. Thus $\mathscr{R}(V) \neq 0$ [1], i.e., R has no nonzero $\mathscr{R}$-semi-simple homomorphic images and therefore belongs to $\mathscr{R}$.

PROPOSITION 2.4. *Let R be a left T-nilpotent ring, $\mathscr{S}$ a semi-simple class containing R. Then $(R^+)^0 \in \mathscr{S}$.*

Proof. Let $\mathscr{R}$ be the radical class whose semi-simple class is $\mathscr{S}$. As in the proof of Proposition 2.2, $\mathscr{R}((R^+)^0) = (I^+)^0$ for some $I \lhd R$. By Proposition 2.3, $I \in \mathscr{R}$, so $I = 0$ and $\mathscr{R}((R^+)^0) = 0$.

Thus radical theory distinguishes between left T-nilpotent rings solely on the basis of their additive groups. The following theorem summarizes the situation.

THEOREM 2.5. *Let $\mathscr{R}$ be a radical class, $\mathscr{S}$ the corresponding semi-simple class. If R is a left T-nilpotent ring, then*

(i) $R \in \mathscr{R} \Leftrightarrow (R^+)^0 \in \mathscr{R}$;

(ii) $R \in \mathscr{S} \Leftrightarrow (R^+)^0 \in \mathscr{S}$.

Furthermore, there is an A-radical class $\mathscr{T}^*$ such that $\mathscr{R}(R) = \mathscr{T}^*(R)$ for every left T-nilpotent ring R.

Proof. The foregoing propositions take care of (i) and (ii).

Let $\mathscr{T} = L(\{V^+ \mid V \in \mathscr{R}$ and V is left T-nilpotent$\})$. Then $\mathscr{T} = L(\{V^+ \mid V \in \mathscr{R}$ and $V^2 = 0\})$. For any left T-nilpotent ring R, $\mathscr{T}^*(R)^+$ belongs to $\mathscr{T}$, so if G is a nonzero homomorphic image of $\mathscr{T}^*(R)^+$, G has a nonzero subgroup S^+, where $S \in \mathscr{R}$ and $S^2 = 0$. This implies that every nonzero homomorphic image of $[\mathscr{T}^*(R)^+]^0$ has a nonzero $\mathscr{R}$-ideal and hence that $[\mathscr{T}^*(R)^+]^0 \in \mathscr{R}$. But $\mathscr{T}^*(R)$ is left T-nilpotent, so by Proposition 2.3, $\mathscr{T}^*(R) \in \mathscr{R}$ and thus $\mathscr{T}^*(R) \subseteqq \mathscr{R}(R)$. On the other hand, $\mathscr{R}(R)$ is left T-nilpotent, so $\mathscr{R}(R)^+ \in \mathscr{T}$, whence $\mathscr{R}(R) \in \mathscr{T}^*$ and $\mathscr{R}(R) \subseteqq \mathscr{T}^*(R)$.

COROLLARY 2.6. *Let $\mathscr{M}$ be a nonempty class of left T-nilpotent rings, $\mathscr{M}^0 = \{(R^+)^0 \mid R \in \mathscr{M}\}$. Then $L(\mathscr{M}) = L(\mathscr{M}^0)$.*

Corollary 2.6 answers, in particular, a question which we raised in [7]: For which classes $\mathscr{M}$ of nilpotent rings is $L(\mathscr{M})$ determined by the zerorings it contains?

Using an argument similar to that in the proof of Proposition 2.3, one can show that if $\mathscr{M}$ is a nonvoid *homomorphically closed* class of left T-nilpotent rings, then $L(\mathscr{M}) = L(\{R \in \mathscr{M} \mid R^2 = 0\})$. For nilpotent rings, this was proved by Sadiq Zia and Wiegandt [15].

Some further consequences of Theorem 2.5 are described in the next few results.

PROPOSITION 2.7. *For any radical class $\mathscr{R}$ and any left T-nilpotent ring R we have*
(i) *$\mathscr{R}(R(n \times n)) = \mathscr{R}(R)(n \times n)$ for all n;*
(ii) *$\mathscr{R}(R[x]) = \mathscr{R}(R)[x]$;*
(iii) *$\mathscr{R}(R[G]) = \mathscr{R}(R)[G]$ for any group G.*

Proof. Let $\mathscr{R}$ coincide with the A-radical class $\mathscr{T}^*$ on left T-nilpotent rings. Then $\mathscr{T}^*(R(n \times n)) = \mathscr{T}^*(R)(n \times n)$ for any ring R ([7], Proposition 1.5). In particular, if R is left T-nilpotent, so is $R(n \times n)$ (Proposition 1.4) and we have

$$\mathscr{R}(R(n \times n)) = \mathscr{T}^*(R(n \times n)) = \mathscr{T}^*(R)(n \times n) = \mathscr{R}(R)(n \times n)\,.$$

The other statements can be proved similarly.

PROPOSITION 2.8. *Let $\mathscr{R}$ be a radical class, $\mathscr{S}$ the corresponding semi-simple class. Then for a left T-nilpotent ring $\mathscr{R}$ we have*
(i) *$R \in \mathscr{R} \Leftrightarrow R^n \in \mathscr{R}$ for all positive integers n;*

(ii) $R \in \mathscr{S} \Leftrightarrow R^n \in \mathscr{S}$ *for all positive integers* n.

Proof. If $R \in \mathscr{R}$, then $(R^+)^0 \in \mathscr{R}$. By Corollary 2.3 of [7], $((R^n)^+)^0 \in \mathscr{R}$ and so $R^n \in \mathscr{R}$, for each n. Since $\mathscr{S}$ is hereditary, (ii) is clear.

PROPOSITION 2.9. *For every radical class $\mathscr{R}$ and left T-nilpotent ring R, $\mathscr{R}(R)^+$ is a pure subgroup of R^+.*

Proof. $\mathscr{R}(R)^+ = \mathscr{T}(R^+)$ for a radical class $\mathscr{T}$ of abelian groups and hence is pure by Proposition 1.1 of [6].

PROPOSITION 2.10. *If R and S are left T-nilpotent rings with R^+ and S^+ quasi-isomorphic, then R belongs to a given radical class if and only if S does.*

Proof. This follows from the closure of radical classes of abelian groups under quasi-isomorphisms ([6], Theorem 1.3).

In [7] we observed that if $\mathscr{T}$ is a radical class of abelian groups, $\mathscr{T}^0$ the class of zerorings on groups in $\mathscr{T}$, then $L(\mathscr{T}^0) \subseteqq \mathscr{B} \cap \mathscr{T}^*$, where $\mathscr{B}$ is the Baer lower (= prime) radical class, and that no examples of strict inequality are known, though equality has been demonstrated only when $\mathscr{T}$ is pure-hereditary. Despite first appearances, Theorem 2.5 provides no new examples of equality, because of the following result, which generalizes the theorem in [8].

THEOREM 2.11. *The following conditions are equivalent for a radical class $\mathscr{R} \neq \{0\}$:*

(i) *$\mathscr{R}$ consists of left T-nilpotent rings.*

(ii) *$\mathscr{R}$ consists of nilpotent rings.*

(iii) *$\mathscr{R}$ consists of zerorings.*

(iv) *$\mathscr{R}$ is the class $\mathscr{D}_P$ of zerorings on divisible P-groups for some set P of primes.*

(Here a *P-group* is a direct sum of p-groups, $p \in P$).

Proof. We need only show that (i) implies (iv). If every ring in $\mathscr{R}$ has a torsion additive group, then either $\mathscr{R} = \mathscr{D}_P$ for some P, or for some prime p, $\mathscr{R}$ contains all $\mathscr{B}$-rings with p-primary additive groups ([7], Theorem 3.3). If $\mathscr{R}$ contains a ring whose additive group is not torsion, it contains a ring R for which R^+ is torsion-free. By Theorem 2.5, $(R^+)^0 \in \mathscr{R}$ and so (see e.g. [6], Corollary 2.3) $\mathscr{R}$ contains the zeroring on the group of rational numbers. But then Theorem 4.2 of [7] implies that $\mathscr{R}$ contains all $\mathscr{B}$-rings with divisible additive groups. To complete the proof, we need only exhibit examples of non-left T-nilpotent $\mathscr{B}$-rings with primary and divisible additive groups. The algebra of Example 3, p. 19 of [4] is commutative and nil, and hence belongs to $\mathscr{B}$, but is not left T-nilpotent. By considering algebras over fields of finite and zero characteristics, we obtain our desired examples.

The last result is perhaps a bit surprising, as left T-nilpotence is a rather "large" property (cf. Theorem 1.1).

3. Some abelian group properties and their effect on admissible multiplications. Wickless [18] has considered abelian groups which admit only nilpotent multiplications. In this section we shall obtain some related results for T-nilpotence. Theorem 1.1 of [18] remains valid if T-nilpotence replaces nilpotence, so that interest is centered on torsion-free groups.

In this section, all groups are abelian. We denote the type of an element x of a torsion-free group by $T(x)$. A *nil* type is the type of a height sequence $(h_1, h_2, \cdots)$ for which $0 < h_n < \infty$ for infinitely many values of n. For unexplained terms see [5].

Consideration of opposite rings easily establishes the following result.

PROPOSITION 3.1. *If every ring on a group G is left T-nilpotent, then every ring on G is right T-nilpotent.*

Thus in the sequel we can refer unambiguously to rings which admit only T-nilpotent multiplications.

We first consider chain conditions on the type set of a torsion-free group G as they affect the ring multiplications which can be defined on G. A couple of lemmas are required as preparation for our main result.

LEMMA 3.2. *Let a, b be elements with nil types in a torsion-free ring.*

(i) *Either $T(ab) > T(a)$ or $T(ab) > T(b)$.*

(ii) *If neither a nor b has infinite height at infinitely many primes, then $T(ab) > T(a)$ and $T(ab) > T(b)$.*

(iii) *If $T(ab) \not> T(a)$, then $T(a) > T(b)$.*

Proof. (i) is just Lemma 3.1 of [18]. From the proof of the latter it can be deduced that if $T(ab) = T(a)$ and neither a nor b

has infinite height at infinitely many primes, then b has zero height at almost all primes, which is impossible. (iii) is clear.

LEMMA 3.3. *Let R be a ring such that R^+ is torsion-free, all of its nonzero elements have nil types and its type set has ACC and DCC. For any sequence $a_1, a_2, \cdots$ of elements of R with $a_1 \neq 0$, there exists an index n such that $T(a_1) < T(a_1 \cdots a_n)$.*

Proof. Suppose there is a sequence $a_1, a_2, \cdots$ which fails to satisfy the stated condition. Then $a_1 \cdots a_n \neq 0$ and $T(a_1 \cdots a_n) = T(a_1)$ for all values of n. This implies, by Lemma 3.1, that $T(a_1) > T(a_2 \cdots a_n)$ for each $n > 1$. Because of *ACC*, there exists n_2 such that $T(a_2 \cdots a_{n_2})$ is maximal in $\{T(a_2 \cdots a_n) \mid n = 3, 4, \cdots\}$. If there exists $m > n_2$ with $T(a_{n_2+1} \cdots a_m) \nless T(a_2 \cdots a_{n_2})$ then $T(a_2 \cdots a_{n_2} a_{n_2+1} \cdots a_m) > T(a_2 \cdots a_{n_2})$ by Lemma 3.1, contradicting the maximality of $T(a_2 \cdots a_{n_2})$. Hence $T(a_2 \cdots a_{n_2}) > T(a_{n_2+1} \cdots a_n)$ for all $n > n_2$. By repetitions of this argument, we obtain an infinite chain

$$T(a_1) > T(a_2 \cdots a_{n_2}) > T(a_{n_2+1} \cdots a_{n_3}) > T(a_{n_3+1} \cdots a_{n_4}) > \cdots$$

contradicting our assumption of *DCC*.

THEOREM 3.4. *Let G be a torsion-free group, all of whose nonzero elements have nil types and whose type set has ACC and DCC. Then G admits only T-nilpotent multiplications.*

Proof. Suppose a ring R with $R^+ \cong G$ has a sequence $a_1, a_2, \cdots$ of elements such that $a_1 a_2 \cdots a_n \neq 0$ for all n. Then by Lemma 3.3, $T(a_1) < T(a_1 \cdots a_{n_1})$ for some n_1. Let $b_1 = a_1 \cdots a_{n_1}$, $b_2 = a_{n_1+1}$, $b_3 = a_{n_1+2}$, etc. Then

$$\begin{aligned} T(a_1) < T(a_1 \cdots a_{n_1}) &= T(b_1) < T(b_1 \cdots b_{n_2}) \\ &= T(a_1 \cdots a_{n_1} a_{n_1+1} \cdots a_{n_1+n_2-1}) \end{aligned}$$

for some n_2. Repetitions of this process lead to a violation of *ACC*, so there is no such sequence.

Using Lemma 3.2 (ii) we can similarly prove

THEOREM 3.5. *Let G be a torsion-free group, all of whose nonzero elements have nil types, none of whose nonzero elements has infinite height at infinitely many primes and whose type set has ACC. Then G admits only T-nilpotent multiplications.*

In [18], Theorem 3.2, it is shown that if a group G, in other respects like that in Theorem 3.4, has merely *ACC* on its type set, then $R \in \mathscr{B}$ for every ring R with $R^+ \cong G$. An alternative proof of this result can be obtained as follows (see [14] or [10], p. 196): Let $b_1, b_2, \cdots$ be an m-sequence in R, i.e., a sequence such that there is another sequence $c_1, c_2, \cdots$ with $b_{n+1} = b_n c_n b_n$ for each n. Then $T(b_{n+1}) \geqq 2T(b_n) > T(b_n)$, so $b_m = 0$ for some m.

To put Theorem 3.4 into perspective we need to establish whether or not ACC (with the other conditions) implies T-nilpotence.

EXAMPLE 3.6. Let $H_1, H_2, \cdots$ be height sequences with nil types such that $H_n + H_{n+1} = H_n$ for each n and let $\tau_1, \tau_2, \cdots$ be the corresponding types. Then $\tau_1 > \tau_2 > \cdots$. Let X_n be that subgroup of the rational numbers for which $T(X_n) = \tau_n$, $1 \in X_n$ and 1 has height sequence H_n in X_n. Under the usual rational number multiplication, we have $X_n X_{n+1} \subseteqq X_n$ for each n. Let R be the ring of strictly lower triangular $\aleph_0 \times \aleph_0$ matrices over the rationals for which almost all entries are zero and all entries in the nth column belong to X_n. In what follows, $[x]_{ij}$ is the matrix whose (i, j)th entry is x and whose others are zero.

If $[a]_{ij}, [b]_{jk} \in R$, then $i > j > k$ and $[a]_{ij}[b]_{jk} = [ab]_{ik}$, and as $a \in X_j \subseteqq X_{k+1}$, $b \in X_k$, we have $ab \in X_{k+1}X_k \subseteqq X_k$ and so $[ab]_{ik} \in R$. Thus R is a subring of the ring of all row-finite rational matrices. It is not right T-nilpotent since $[1]_{n+1,n}[1]_{n,n-1} \cdots [1]_{21} = [1]_{n+1,1}$ for each n. (R is of course left T-nilpotent; see [2], Example (5), p. 476). However, $R^+ = \bigoplus_{n=1}^{\infty} G_n$ where each G_n is a direct sum of copies of X_n and thus the type set of R^+ is $\{T(0)\} \cup \{\tau_n \mid n = 1, 2, \cdots\}$, which has ACC but not DCC.

The height sequences listed below satisfy all the requirements of Example 3.6:

$$(\infty, 1, \infty, 1, \infty, 1, \infty, 1, \infty, 1, \infty, 1, \infty, 1, \cdots)$$
$$(\infty, 0, 1, 0, \infty, 0, 1, 0, \infty, 0, 1, 0, \infty, 0, \cdots)$$
$$(\infty, 0, 0, 0, 1, 0, 0, 0, \infty, 0, 0, 0, 1, 0, \cdots)$$
$$\cdots .$$

It is perhaps worth noting that in Example 3.6 R^+ is completely decomposable.

The following example shows that a group satisfying the conditions of Theorem 3.4 can admit a nonnilpotent multiplication.

EXAMPLE 3.7. Let $\{N_1, N_2, \cdots\}$ be a partition of the natural numbers such that $|N_i| = \aleph_0$ for each i, and for each i, n, let τ_{in} be the type of the height sequence $(h_1, h_2, \cdots)$, where

$$h_k = \begin{cases} n, & \text{if } k \in N_i \\ 0, & \text{if } k \notin N_i \end{cases},$$

X_{in} that group of rational numbers in which 1 exists and has the height sequence $(h_1, h_2, \cdots)$ corresponding to τ_{in}. For each i, we define the ring R_i as follows:

$$R_i^+ = X_{i1} \oplus \cdots \oplus X_{ii}$$

$$\langle a \rangle_{ik} \langle b \rangle_{ij} = \begin{cases} \langle ab \rangle_{i,j+k}, & \text{if } j + k \leqq i \\ 0, & \text{if } j + k > i \end{cases},$$

where $\langle a \rangle_{ik}$ is the element of R_i whose nth component is $\delta_{kn} a$. Clearly $R_i^{i+1} = 0 \neq R_i^i$. Let $R = \bigoplus_{i=1}^{\infty} R_i$ (ring direct sum). Since any two types τ_{in}, τ_{jm}, with $i \neq j$, are incomparable, the type set of R^+ is

$$\{T(0)\} \cup \{\tau_{in_i} \mid i = 1, 2, \cdots; n_i = 1, \cdots, i\} \cup \{\tau_0\}$$

where τ_0 is the type of the height sequence $(0, 0, \cdots)$. This has both chain conditions, but R is not nilpotent.

Note that *DCC* on its own does not imply anything in particular. (See the example on pp. 253-254 of [18].)

We conclude this section with a result of a somewhat different character, derived from the material in § 2.

THEOREM 3.8. *Let $G(\neq 0)$ be an indecomposable torsion-free group which is homogeneous of type τ and in which every proper pure subgroup is completely decomposable. If R is a nilpotent ring with $R^+ \cong G$, then $R^2 = 0$.*

Proof. As shown in the proof of [8], Theorem or [9], Proposition 2.2, $(R/R^2)^+$ is not a torsion group, so if $R^2 \neq 0$, then $(R^2)_*^+$, the smallest pure subgroup of R^+ containing $(R^2)^+$, is completely decomposable. By Proposition 2.8, $R^2 \in L(\{R\})$, and since $(R^2)_*^+/(R^2)^+$ is a torsion group and $(R^2)_*^+$ is the additive group of an ideal (which we denote by $(R^2)_*$) of R, Corollary 1.2 of [6] and Theorem 2.5 jointly imply that $(R^2)_* \in L(\{R\})$. Thus $(R^2)_*$ has a nonzero subring isomorphic to a homomorphic image of R. But then $\mathrm{Hom}(R^+, (R^2)_*^+) \neq 0$, so R^+ has a homomorphic image which is rational of type τ. But this is impossible ([5], Proposition 86.5), so $R^2 = 0$.

Of course, in the last result only nonnil types are relevant, as any homogeneous group of nil type admits only the trivial multiplication. This observation appears to have first been made (in a slightly more general form) by Szele ([17], Hilfssatz 3).

Any homogeneous indecomposable group of rank 2 satisfies the conditions of Theorem 3.8. Fuchs ([5], pp. 125–128) gives a construction which yields groups of every type except that of the rationals, and every finite rank, which satisfy the conditions.

4. **Idealizer conditions.** The *idealizer* of a subring S of a ring R is the largest subring of R in which S is an ideal. Left and right idealizers are defined analogously. A ring is said to satisfy the *idealizer condition* (etc.) if ever subring is properly contained in its idealizer (etc.).

Szász ([16], Theorem 6) has shown that a ring which is left and right T-nilpotent satisfies the idealizer condition and the same argument shows that left T-nilpotent rings satisfy the left idealizer condition. We give here an example to show that left T-nilpotent rings can fail to satisfy the right idealizer condition and hence the idealizer condition.

EXAMPLE 4.1. We make use of Bass' Example (5), p. 476 of [2]. Let R be the ring of strictly lower triangular $\aleph_0 \times \aleph_0$ matrices over a field which have almost all entries zero, $S = \{(a_{ij}) \in R \mid a_{n1} = 0 \forall n\}$. Then S is a left ideal of R. If $(x_{ij}) \in R \backslash S$, let $x_{k1} \neq 0$. Then $[1]_{lk} \in S$ for each $l > k$ (notation as in Example 3.6) and $[1]_{lk}(x_{ij})$ has $(l, 1)$th entry $x_{k1} \neq 0$ and so does not belong to S. Hence S is its own right idealizer.

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