Pacific Journal of Mathematics

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Vol. 53, No. 1 March 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

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Let S be a semigroup with zero. The extended centralizer $Q(M_S)$ of a right S-set M_S is defined. Necessary and sufficient conditions are given for $Q(M_S)$ to be a regular semigroup. In particular, $Q(S_S)$ is shown to be a regular semigroup when S is regular. We also show that whenever the singular congruence on S is the identity, then $Q(S_S)$ is the injective hull of S_S and is right self injective.

1. Introduction. In [3], R. E. Johnson developed the extended centralizer $Q(M_R)$ of an R-module M and noted that $Q(M_R)$ is always a (Von Neumann) regular ring. In this paper, we analogously define the extended centralizer $Q(M_S)$ of a right S-set M_S. McMorris [4] gave an example which illustrated the fact that $Q(S_S)$ is not always a regular semigroup. We give a necessary and sufficient condition for $Q(M_S)$ to be regular and show that when S is regular, $Q(S_S)$ is also regular.

Johnson showed that the ring R is embedded in $Q(R_R)$ when the singular ideal is zero. Analogously we define the singular congruence on an S-set and show that when the singular congruence is the identity, S is embedded in $Q(S_S)$. In this case we also note that $Q(S_S)$ is the injective hull of S considered as a S-set and that, moreover, $Q(S_S)$ is self injective.

2. Preliminaries. Throughout this paper each semigroup will contain a zero (0) unless otherwise specified. Let S be a semigroup. A (centered right) *S-set* M_S is a set M, with an associative scalar operation on M by elements of S, which contains an element (necessarily unique) θ such that $\theta = \theta s = m0$ for all $m \in M$ and for all $s \in S$. The symbol θ will be called the zero of M. Since the distinction between the zero of M and the zero of S is clear from the context, we shall denote both by the same symbol 0. Note that if R is a right ideal of S then R becomes an S-set R_S under ordinary multiplication. A *sub S-set* N_S of an S-set M_S is a subset N of M such that $NS \subseteq N$. If $m, n \in M_S$ and if $E \subseteq S$ we shall say that mE is *pointwise equal* to nE when $ms = ns$ for each $s \in E$. This will be denoted as $mE \doteq nE$.

Let M_S and N_S be S-sets. A function $f: M_S \rightarrow N_S$ is an *S-homomorphism* if for each $m \in M$ and $s \in S$, $f(ms) = f(m)s$. The collection of all such S-homomorphisms will be denoted by $\mathrm{Hom}_S(M, N)$. If there exists $f \in \mathrm{Hom}_S(M, N)$ which is $1-1$ and onto, we say M_S

is S-isomorphic to N_S and write $M_S \underset{S}{\approx} N_S$.

If f is an S-homomorphism the domain of f will be denoted by D_f and the range of f by R_f. The zero map from M_S will be denoted by 0_M and the identity map on M by 1_M. If $f: M_S \to N_S$ and if $A_S \subseteq N_S$ then $f^{-1}(A) = \{m \in M: f(m) \in A\}$.

An *S-congruence* τ on M_S is an equivalence relation on M such that whenever $(m, n) \in \tau$, then $(ms, ns) \in \tau$ for all $s \in S$. The identity S-congruence on M_S will be denoted by ι_M.

If S has an identity 1 the S-set M_S is said to be *unital* when $m1 = m$ for each $m \in M$. For each semigroup S we shall define S^1 by $S^1 = S \cup \{1\}$ where 1 is a symbol not in S and where multiplication on S is extended to S^1 by defining $1x = x1 = x$ for each $x \in S^1$. With the operation so defined, S^1 is a semigroup. Note that this definition for S^1 differs from the standard one. However, with the definition given here each S-set M_S becomes a unital S^1-set by defining $m1 = m$ for each $m \in M$.

The following definitions and theorem are due to Berthiaume [1]. A sub S-set N_S of M_S is said to be *large* (*essential*) in M_S if for each $f \in \mathrm{Hom}_S(M, K)$ such that $f|N$ is $1-1$ then f is $1-1$. In this case M_S is called an *essential extension* of N_S. The following lemma characterizes large sub S-sets in terms of S-congruences.

LEMMA 2.1. *N_S is large in M_S iff for every S-congruence ρ on M_S such that $\rho \neq \iota_M$ we have $\rho|N \neq \iota_N$.*

An S-set M_S is *injective* if for each $A_S \subseteq B_S$ and for each $f \in \mathrm{Hom}_S(A, M)$ there exists $f' \in \mathrm{Hom}_S(B, M)$ such that $f'|A = f$. If $M_S \subseteq N_S$ and if N_S is injective then N_S is called an *injective extension* of M_S. The following theorem due to Berthiaume [1] guarantees the existence of a minimal injective extension which is unique up to S-isomorphism.

THEOREM 2.2. *The S-set M_S is a maximal essential extension of N_S iff M_S is a minimal injective extension of N_S. Every S-set N_S has such an extension which is unique up to S-isomorphism over N_S.*

The minimal injective extension of N_S given in the above theorem is called the *injective hull* of N_S. Note that M_S is the injective hull of N_S iff N_S is essential in M_S and M_S is injective.

A semigroup S will be called *self injective* if S_S is injective.

The S-set M_S is *weakly injective* if for each right ideal R of S and for each $f \in \mathrm{Hom}_S(R, M)$ there exists $m \in M$ such that $f(s) = ms$ for each $s \in R$. In ring theory it is well-known that the corresponding concepts of "injective" and "weakly injective" are equivalent.

However, for semigroups Berthiaume proved the following lemma and gave a counterexample for the converse.

LEMMA 2.3. *If the S-set* M_S *is injective then* M_S *is weakly injective.*

3. **The singular congruence on an S-set.** The following definition is a generalization of a corresponding concept in ring theory. A sub S-set N_S of M_S is *intersection large* in M_S if for each $0 \neq m \in M$ there exists $s \in S^1$ such that $0 \neq ms \in N$. Note that N_S is intersection large in M_S if and only if the intersection of N with any nonzero sub S-set of M_S is always nonzero. Properties of intersection large S-sets are given by the following lemmas which are immediate from the definition.

LEMMA 3.1. *If* $X_S \subseteqq Y_S \subseteqq Z_S$ *are S-sets then* X_S *is intersection large in* Z_S *if and only if* X_S *is intersection large in* Y_S *and* Y_S *is intersection large in* Z_S.

LEMMA 3.2. *Let* M_S *and* N_S *be S-sets and let* $\phi \in \mathrm{Hom}_S(M, N)$. *If* A_S *is intersection large in* N_S *then* $\phi^{-1}(A)$ *is intersection large in* M_S.

Note that if N_S is intersection large in M_S then $m^{-1}N = \{s \in S: ms \in N\}$ is intersection large in S_S for all $m \in M$. In order to show this, define $\phi_m: S \to M$ by $\phi_m(s) = ms$. Then $\phi_m \in \mathrm{Hom}_S(S, M)$ and $\phi_m^{-1}(N) = m^{-1}N$ is intersection large in S_S by the lemma.

The class of all intersection large sub S-sets of the S-set M_S will be denoted by $\mathscr{P}(M_S)$. This class is closed under finite intersections since $A \cap B = 1_A^{-1}(B)$ where $A, B \in \mathscr{P}(M_S)$.

Let $\mathscr{P} = \mathscr{P}(S_S)$ and for each S-set define

$$\psi = \psi(M_S) = \{(m_1, m_2) \in M \times M: m_1 D \mp m_2 D \text{ for some } D \in \mathscr{P}\} .$$

It is easily seen from the properties noted above that ψ is an S-congruence on M_S which is a two-sided congruence if $M = S$. The S-congruence ψ is called the *singular congruence* or *$\mathscr{P}$-torsion congruence* on M_S. When $\psi = \iota_M$ we say that M_S is *$\mathscr{P}$-torsion free.*

Feller and Gantos [2] showed that every large sub S-set of an S-set M_S is intersection large in M_S. The converse is not generally true. For, consider the semilattice $S = \{0, e, 1\}$ which has $0 < e < 1$ under the natural partial ordering. The right ideal eS is clearly intersection large in S. Define $f: S \to S$ by $f(x) = \begin{Bmatrix} e \text{ if } x \in \{e, 1\} \\ 0 \text{ if } x = 0 \end{Bmatrix}$. Then $f \in \mathrm{Hom}_S(S, S)$ and $f \mid eS$ is $1 - 1$. However, f is not $1 - 1$. Therefore, eS is not large in S_S.

The following proposition gives a sufficient condition for the converse to be true.

PROPOSITION 3.3. *Let M_S be a right S-set such that M_S is $\mathscr{P}$-torsion free. Then $\mathscr{P}(M_S)$ is the set of large sub S-sets of M_S.*

Proof. Let $A_S \in \mathscr{P}(M_S)$ and let $f \in \mathrm{Hom}_S(M, B)$ such that $f \mid A$ is $1 - 1$ where B_S is an S-set. Suppose $f(x_1) = f(x_2)$. Let $D = x_1^{-1}A \cap x_2^{-1}A = \{s \in S: x_1 s \in A \text{ and } x_2 s \in A\}$. Then $D \in \mathscr{P}(S)$ and since $f(x_1) = f(x_2)$, we have $f(x_1 s) = f(x_2 s)$ for all $s \in D$. However, $x_1 s$, $x_2 s \in A$ for all $s \in D$ and $f \mid A$ is $1 - 1$. Thus $x_1 D \doteqdot x_2 D$ and it follows that $x_1 = x_2$ since M_S is $\mathscr{P}$-torsion free.

It was noted in §2 that an injective S-set M_S is always weakly injective but that a weakly injective S-set is not necessarily injective. In the following proposition we show that the two concepts are equivalent whenever M_S is $\mathscr{P}$-torsion free.

PROPOSITION 3.4. *Let M_S be a weakly injective S-set such that M_S is $\mathscr{P}$-torsion free. Then M_S is injective.*

Proof. Let $A_S \subseteqq B_S$ and let $f \in \mathrm{Hom}_S(A, M)$. Let M^* be the injective hull of M_S. Then M_S is large in M_S^* and hence is intersection large in M_S^*. Also, by Lemma 2.1 we see that M_S^* is $\mathscr{P}_I$-torsion free since $\psi(M_S^*) \mid M_S = \psi(M_S) = \iota_M$. Thus, since M_S^* is injective, there exists $f' \in \mathrm{Hom}_S(B, M^*)$ such that $f' \mid A = f$. We claim that $f' \in \mathrm{Hom}_S(B, M)$. Let $b \in B$ and let $f'(b) = n$. By the note following Lemma 3.2 we have $D = n^{-1}M \in \mathscr{P}(S)$. Define $\phi: n^{-1}M \to M$ by $\phi(s) = ns$. Thus we have $\phi \in \mathrm{Hom}_S(n^{-1}M, M)$ and since M_S is weakly injective there exists $m \in M$ such that $\phi(s) = ms$ for each $s \in n^{-1}M$. Therefore, $mD \doteqdot nD$ and since $\psi_I(M_S^*) = \iota_{M^*}$ it follows that $n = m \in M$.

4. The extended centralizer of an S-set. The construction of the extended centralizer Q of an S-set M_S is similar to that given by Johnson [3] for rings over modules and is outlined as follows:

Let $\mathscr{P} = \mathscr{P}(M_S)$ be the class of intersection large sub S-sets of the S-set M_S. Let $F = \bigcup_{D \in \mathscr{P}} \mathrm{Hom}_S(D, M)$ and define multiplication on S by $fg = h$ where $h: D_g \cap g^{-1}(D_f) \to M$ by $h(x) = f(g(x))$. Then under this multiplication F is a semigroup. Define a binary relation ω on the semigroup F by $(f, g) \in \omega$ if there exists $D \in \mathscr{P}$ such that $f \mid D = g \mid D$. Then ω is a two-sided congruence on F. The semigroup $Q = Q(M_S) = F/\omega$ is called the *extended centralizer of* M_S. The elements of Q will be denoted by $\bar{f}$ where $f \in F$.

In ring theory the extended centralizer is always (von Neumann) regular. An example given by McMorris in [4] shows that this is not the case for semigroups. We can however give a necessary and sufficient condition for Q to be regular in terms of splitting S-homomorphisms, which were studied by Feller and Gantos in [2]. Recall that an S-homomorphism f which maps an S-set M_S onto an S-set N_S is said to *split* if there exists $g \in \mathrm{Hom}_S(N, M)$ such that $fg = 1_N$.

THEOREM 4.1. *The semigroup $Q(M_S)$ is regular if and only if each equivalence class $\bar{f}$ of $Q(M_S) = F/\omega$ contains an element which splits.*

Proof. Assume first that $Q(M_S) = Q$ is regular and let $\bar{f} \in Q$. Then there exists $\bar{g} \in Q$ such that $\bar{f}\bar{g}\bar{f} = \bar{f}$. Hence if $E = \{x \in D_{fgf}: fgf(x) = f(x)\}$ then $E \in \mathscr{P}$. Let $f' = f \mid E$ and $g' = g \mid R_{f'}$. Let $y \in D_{g'}$ and let $x' = g(y)$. Since y is also an element of $R_{f'}$, there exists $x \in E$ such that $f(x) = y$ and we see that $y = f(x) = fgf(x) = fg(y) = f(x')$. Furthermore, $f(x') = f(x) = fgf(x) = fgf(x')$ and it follows that $x' \in E$. Therefore, $y = fg(y) = f(x') = f'(x') = f'g'(y)$ and we see that f' splits.

Conversely, for $\bar{f} \in Q$ there exists $f' \in \bar{f}$ such that f' splits. Hence, there exists $g': R_{f'} \rightarrow D_{f'}$ such that $f'g' = \iota_{R_{f'}}$. By Zorn's lemma there is a maximal sub S-set N_S of M_S such that $D_{g'} \cap N = 0$. It easily follows that $D = D_{g'} \cup N \in \mathscr{P}$ in this case. The S-homomorphism g' can be extended to an S-homomorphism $g \in \mathrm{Hom}_S(D, M)$ by defining $g(x) = 0$ if $x \in N$ and $g(x) = g'(x)$ if $x \in D_{g'}$. Hence we have $g \in F$. Let $x \in D_{f'}$. Then $f'gf'(x) = f'g'f'(x) = 1_{R_{f'}}f'(x) = f'(x)$. Therefore, $\bar{f}\bar{g}\bar{f} = \bar{f}'\bar{g}\bar{f}' = \bar{f}' = \bar{f}$ and it follows that Q is regular.

In the case where $M = S$ we have the following theorem.

THEOREM 4.2. *If S is a regular semigroup then $Q(S)$ is regular.*

Proof. By the previous theorem it is sufficient to show that each equivalence class $\bar{f}$ of $Q(S_S)$ contains an element which splits. Let $\bar{f} \in Q$ and let $\mathscr{F} = \{(D_\alpha, g_\alpha): D_\alpha$ is a right ideal of S in D_f such that $f_\alpha = f \mid D_\alpha$ splits on D_α and $g_\alpha: R_\alpha = f_\alpha(D_\alpha) \rightarrow D_\alpha$ such that $f_\alpha g_\alpha = 1_{R_\alpha}$. The set $\mathscr{F}$ is nonempty since $(\{0\}, 0) \in \mathscr{F}$ where the zero in the second coordinate is the zero map. Define a partial order $\leqq$ on $\mathscr{F}$ by $(D_\alpha, g_\alpha) \leqq (D_\beta, g_\beta)$ iff $D_\alpha \subseteqq D_\beta$ and $g_\beta \mid R_\alpha = g_\alpha$. By an application of Zorn's lemma, $\mathscr{F}$ contains a maximal element (D_M, g_M). To complete the proof it is sufficient to show that $D_M \in \mathscr{P}(S_S)$. Suppose this is not true. Then D_M is not intersection large in D_f. Hence there exists $e \in D_f$ such that $eS^1 \neq 0$ and $eS^1 \cap D_M = 0$. Since S is regular, we may assume that $e^2 = e$. Let $x = f(e)$ then $xe = f(e)e = f(e^2) = f(e) = x$. We now consider two cases.

Case 1. Suppose $xeS \cap R_M \neq 0$. Then there exists $s \in S$ such that $0 \neq xes \in R_M$. Consider $esS \subseteq eS$. Let $D' = D_M \cup esS$ and let $f' = f \mid D'$. Then $f'(D') = R_M$. If $y \in R_M$ then $f'g_M(y) = f_M g_M(y) = y$. Hence $(D', g_M) \in \mathscr{F}$ and $(D_M, g_M) < (D', g_M)$ which contradicts the maximality of (D_M, g_M).

Case 2. Suppose $xeS \cap R_M = 0$. Let x' be an inverse of x and define $g': R' = R_M \cup xeS \rightarrow D' = D_M \cup eS$ by $g'(y) = \begin{Bmatrix} g_M(y) \text{ if } y \in R_M \\ ex'y \text{ if } y \in xeS \end{Bmatrix}$. Note that $g' \in \operatorname{Hom}(R', D')$. Now let $f' = f \mid D'$ and let $y \in R'$. If $y \in R_M$ then $f'g'(y) = f_M g_M(y) = 1_{R_M}(y) = y$. On the other hand, if $y \in xeS$, say $y = xes$, then $f'g'(y) = f'g'(xes) = f'(ex'xes) = xx'xes = xes = y$. Hence it follows that $f'g' = 1_{R'}$. Thus, $(D', g') \in \mathscr{F}$ and clearly $(D_M, g_M) < (D', g')$ which again contradicts the maximality of (D_M, g_M).

Therefore, D_M must be intersection large in S and the theorem follows.

An S-set M_S is *intersection uniform* if every nonzero sub S-set of M is intersection large.

THEOREM 4.3. *The semigroup $Q = Q(M_S)$ is a right cancellative semigroup with zero if and only if M_S is intersection uniform.*

Proof. Suppose that Q is a right cancellative semigroup with zero and let N_S be a nonzero sub S-set of M_S. Using Zorn's lemma to find a maximal sub S-set N' of M such that $N \cap N' = 0$, define a function f on $N \cup N'$ by $f(x) = x$ if $x \in N$ and $f(x) = 0$ if $x \in N'$. Then $f^2 = f$ and $f \in F$. If $\bar{f} = \bar{0}$ then there exists $D \in \mathscr{P}(M_S)$ such that $D \subseteq D_f$ and $f(D) = 0$ which implies that $D \subseteq N'$. Hence $N' \in \mathscr{P}(M_S)$. But this is impossible since $N \cap N' = 0$. Thus we have $\bar{f} \neq \bar{0}$. Since $\bar{1}_M \bar{f} = \bar{f} = \bar{f}\bar{f}$ and since each nonzero element of Q is right cancellable, it follows that $\bar{1}_M = \bar{f}$. Therefore, there exists $D \in \mathscr{P}$ such that $D \subseteq N$ and it follows that $N \in \mathscr{P}(M_S)$. The proof of the converse is immediate.

5. The injective hull of a $\mathscr{P}$-torsion free semigroup.

Throughout this section we shall consider the semigroup S as an S-set over itself. For $s \in S$ define $\phi_s: S \rightarrow S$ by $\phi_s(t) = st$. Then $\phi_s \in F$ and it easily follows that the map $\phi: S \rightarrow Q$ by $\phi(s) = \bar{\phi}_s$ is a representation of S in $Q = Q(S_S)$. Note also that we can regard Q as a centered right S-set by defining $\bar{f}s = \overline{f\phi_s}$ for each $\bar{f} \in Q$ and for each $s \in S$. The following lemmas are easy consequences of the above remarks.

LEMMA 5.1. $\psi(S_S) = \phi^{-1} \circ \phi$.

LEMMA 5.2. *For each* $f \in F$ *and for each* $s \in D_f$, $f\phi_s = \phi_{f(s)}$.

When $\psi(S_S) = \iota_S$ *we shall assume that* S *is embedded in* $Q = Q_{\mathscr{P}}(S)$ *under the identification* $s\langle - \rangle \bar{\phi}_s$. *From Lemma* 5.2 *we see that* $\bar{f}s = f(s)$ *for each* $\bar{f} \in Q_{\mathscr{P}}(S)$ *and for each* $s \in D_f$ *under the identification described above. Thus we see that* S_S *is intersection large in* Q_S. *In addition, the next lemma shows that* Q_S *is* $\mathscr{P}$*-torsion free.*

LEMMA 5.3. *If* S *is* $\mathscr{P}$*-torsion free then* Q_S *is* $\mathscr{P}$*-torsion free.*

Proof. Let $(\bar{f}_1, \bar{f}_2) \in \psi(Q_S)$. Then there exists $E \in \mathscr{P}$ such that $\bar{f}_1 E = \bar{f}_2 E$. Let $E' = E \cap D_{f_1} \cap D_{f_2} \in \mathscr{P}$. Then for each $s \in E'$, we have $f_1(s) = \bar{f}_1 s = \bar{f}_2 s = f_2(s)$ and it follows that $\bar{f}_1 = \bar{f}_2$.

The following lemma is immediate from Lemma 2.1 and the remarks preceding the above lemma.

LEMMA 5.4. *If* S *is* $\mathscr{P}$*-torsion free then* S_S *is large in* Q_S.

We now can show that Q_S is the injective hull of S and furthermore Q is injective as a Q-set.

THEOREM 5.5. *If* S *is* $\mathscr{P}$*-torsion free then* $Q_S = Q(S_S)$ *is the injective hull of* S_S.

Proof. Since S_S is large in Q_S by Lemma 5.3, we need only show that Q_S is injective. By Lemma 5.3 and Proposition 3.4 it suffices to verify that Q_S is weakly injective. Let R be a right ideal of S and let $\Phi \in \mathrm{Hom}_S(R, Q)$. Since S_S is intersection large in Q_S, $R' = \Phi^{-1}(S) \in \mathscr{P}$ and $f = \Phi | R' \in F$. We claim that $\Phi(r) = \bar{f}r$ for each $r \in R$. For each $s \in r^{-1}R' = \{s \colon rs \in R'\}$ we have $\Phi(r)s = \Phi(rs) = f(rs) = (\bar{f}r)s$. Thus, since $r^{-1}R' \in \mathscr{P}$, it follows that $(\Phi(r), \bar{f}r) \in \psi(Q_S)$ which is the identity S-congruence on Q_S. Therefore, $\Phi(r) = \bar{f}r$ for each $r \in R$ and the result follows.

THEOREM 5.6. *If* S *is* $\mathscr{P}$*-torsion free then* $Q = Q(S_S)$ *is self injective.*

Proof. Let $A_Q \subseteqq B_Q$ be Q-sets and let $\Phi' \in \mathrm{Hom}_Q(A, B)$. Then $\Phi' \in \mathrm{Hom}_S(A, Q)$. Since Q_S is the injective hull of S_S, there exists $\Phi \in \mathrm{Hom}_S(B, Q)$ such that $\Phi | A = \Phi'$. We claim that Φ is a Q-homomorphism. Let $b \in B$ and $\bar{f} \in Q$. Then for each $s \in D_f$ we have $\Phi(b\bar{f})s = \Phi(b\bar{f}s) = \Phi(bf(s)) = \Phi(b)f(s) = \Phi(b)\bar{f}s$. Thus $(\Phi(b\bar{f}), \Phi(b)\bar{f}) \in \psi(Q_S)$ which is the identity congruence on Q. Therefore, it follows that $\Phi(b\bar{f}) = \Phi(b)\bar{f}$.

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Received July 19, 1973. This paper represents a portion of the author's doctoral dissertation written under the supervision of Professor D. R. LaTorre. The author wishes to thank Professor LaTorre and Professor John K. Luedeman for their assistance and encouragement. Research for this work was supported by a National Science Foundation Traineeship and by a Faculty Fellowship from the Virginia Military Institute.

CLEMSON UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by Intarnational Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 53, No. 1 March, 1974