THE EXTENDED CENTRALIZER OF AN S-SET

C. V. HINKLE
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C. V. HINKLE, JR.

Let S be a semigroup with zero. The extended centralizer \( Q(M_s) \) of a right S-set \( M_s \) is defined. Necessary and sufficient conditions are given for \( Q(M_s) \) to be a regular semigroup. In particular, \( Q(S_S) \) is shown to be a regular semigroup when S is regular. We also show that whenever the singular congruence on S is the identity, then \( Q(S_S) \) is the injective hull of \( S_s \) and is right self injective.

1. Introduction. In [3], R. E. Johnson developed the extended centralizer \( Q(M_R) \) of an R-module \( M \) and noted that \( Q(M_R) \) is always a (Von Neumann) regular ring. In this paper, we analogously define the extended centralizer \( Q(M_s) \) of a right S-set \( M_s \). McMorris [4] gave an example which illustrated the fact that \( Q(S_S) \) is not always a regular semigroup. We give a necessary and sufficient condition for \( Q(M_s) \) to be regular and show that when S is regular, \( Q(S_S) \) is also regular.

Johnson showed that the ring \( R \) is embedded in \( Q(R_R) \) when the singular ideal is zero. Analogously we define the singular congruence on an S-set and show that when the singular congruence is the identity, \( S \) is embedded in \( Q(S_S) \). In this case we also note that \( Q(S_S) \) is the injective hull of \( S \) considered as a S-set and that, moreover, \( Q(S_S) \) is self injective.

2. Preliminaries. Throughout this paper each semigroup will contain a zero (0) unless otherwise specified. Let S be a semigroup. A (centered right) S-set \( M_s \) is a set \( M \), with an associative scalar operation on \( M \) by elements of S, which contains an element (necessarily unique) \( \theta \) such that \( \theta = \theta s = m0 \) for all \( m \in M \) and for all \( s \in S \). The symbol \( \theta \) will be called the zero of \( M \). Since the distinction between the zero of \( M \) and the zero of S is clear from the context, we shall denote both by the same symbol 0. Note that if \( R \) is a right ideal of S then \( R \) becomes an S-set \( R_s \) under ordinary multiplication. A sub S-set \( N_s \) of an S-set \( M_s \) is a subset \( N \) of \( M \) such that \( NS \subseteq N \). If \( m, n \in M_s \) and if \( E \subseteq S \) we shall say that \( mE \) is pointwise equal to \( nE \) when \( ms = ns \) for each \( s \in E \). This will be denoted as \( mE = nE \).

Let \( M_s \) and \( N_s \) be S-sets. A function \( f : M_s \rightarrow N_s \) is an S-homomorphism if for each \( m \in M \) and \( s \in S \), \( f(ms) = f(m)s \). The collection of all such S-homomorphisms will be denoted by \( \text{Hom}_s(M, N) \). If there exists \( f \in \text{Hom}_s(M, N) \) which is \( 1 - 1 \) and onto, we say \( M_s \)
is $S$-isomorphic to $N_s$ and write $M_s \sim N_s$.

If $f$ is an $S$-homomorphism the domain of $f$ will be denoted by $D_f$ and the range of $f$ by $R_f$. The zero map from $M_s$ will be denoted by $0_s$ and the identity map on $M$ by $1_M$. If $f : M_s \rightarrow N_s$ and if $A_s \subseteq N_s$ then $f^{-1}(A) = \{ m \in M : f(m) \in A \}$.

An $S$-congruence $\tau$ on $M_s$ is an equivalence relation on $M$ such that whenever $(m, n) \in \tau$, then $(ms, ns) \in \tau$ for all $s \in S$. The identity $S$-congruence on $M_s$ will be denoted by $\tau_M$.

If $S$ has an identity $1$ the $S$-set $M_s$ is said to be unital when $m1 = m$ for each $m \in M$. For each semigroup $S$ we shall define $S'$ by $S' = S \cup \{1\}$ where $1$ is a symbol not in $S$ and where multiplication on $S$ is extended to $S'$ by defining $1x = x1 = x$ for each $x \in S'$. With the operation so defined, $S'$ is a semigroup. Note that this definition for $S'$ differs from the standard one. However, with the definition given here each $S$-set $M_s$ becomes a unital $S'$-set by defining $m1 = m$ for each $m \in M$.

The following definitions and theorem are due to Berthiaume [1]. A sub $S$-set $N_s$ of $M_s$ is said to be large (essential) in $M_s$ if for each $f \in \text{Hom}_S(M, K)$ such that $f|N$ is 1 -- 1 then $f$ is 1 -- 1. In this case $M_s$ is called an essential extension of $N_s$. The following lemma characterizes large sub $S$-sets in terms of $S$-congruences.

**Lemma 2.1.** $N_s$ is large in $M_s$ iff for every $S$-congruence $\rho$ on $M_s$ such that $\rho \neq \tau_M$ we have $\rho|N \neq \tau_N$.

An $S$-set $M_s$ is injective if for each $A_s \subseteq B_s$ and for each $f \in \text{Hom}_S(A, M)$ there exists $f' \in \text{Hom}_S(B, M)$ such that $f'|A = f$. If $M_s \subseteq N_s$ and if $N_s$ is injective then $N_s$ is called an injective extension of $M_s$. The following theorem due to Berthiaume [1] guarantees the existence of a minimal injective extension which is unique up to $S$-isomorphism.

**Theorem 2.2.** The $S$-set $M_s$ is a maximal essential extension of $N_s$ iff $M_s$ is a minimal injective extension of $N_s$. Every $S$-set $N_s$ has such an extension which is unique up to $S$-isomorphism over $N_s$.

The minimal injective extension of $N_s$ given in the above theorem is called the injective hull of $N_s$. Note that $M_s$ is the injective hull of $N_s$ iff $N_s$ is essential in $M_s$ and $M_s$ is injective.

A semigroup $S$ will be called self injective if $S_s$ is injective.

The $S$-set $M_s$ is weakly injective if for each right ideal $R$ of $S$ and for each $f \in \text{Hom}_S(R, M)$ there exists $m \in M$ such that $f(s) = ms$ for each $s \in R$. In ring theory it is well-known that the corresponding concepts of “injective” and “weakly injective” are equivalent.
However, for semigroups Berthiaume proved the following lemma and gave a counterexample for the converse.

**Lemma 2.3.** If the S-set $M_s$ is injective then $M_s$ is weakly injective.

3. The singular congruence on an S-set. The following definition is a generalization of a corresponding concept in ring theory. A sub S-set $N_s$ of $M_s$ is intersection large in $M_s$ if for each $0 \neq m \in M$ there exists $s \in S^1$ such that $0 \neq ms \in N$. Note that $N_s$ is intersection large in $M_s$ if and only if the intersection of $N$ with any nonzero sub S-set of $M_s$ is always nonzero. Properties of intersection large S-sets are given by the following lemmas which are immediate from the definition.

**Lemma 3.1.** If $X_s \subseteq Y_s \subseteq Z_s$ are S-sets then $X_s$ is intersection large in $Z_s$ if and only if $X_s$ is intersection large in $Y_s$ and $Y_s$ is intersection large in $Z_s$.

**Lemma 3.2.** Let $M_s$ and $N_s$ be S-sets and let $\phi \in \text{Hom}_S(M, N)$. If $A_s$ is intersection large in $N_s$ then $\phi^{-1}(A)$ is intersection large in $M_s$.

Note that if $N_s$ is intersection large in $M_s$ then $m^{-1}N = \{s \in S: ms \in N\}$ is intersection large in $S_s$ for all $m \in M$. In order to show this, define $\phi_m: S \to M$ by $\phi_m(s) = ms$. Then $\phi_m \in \text{Hom}_S(S, M)$ and $\phi_m^{-1}(N) = m^{-1}N$ is intersection large in $S_s$ by the lemma.

The class of all intersection large sub S-sets of the S-set $M_s$ will be denoted by $\mathcal{P}(M_s)$. This class is closed under finite intersections since $A \cap B = 1^{-1}(B)$ where $A, B \in \mathcal{P}(M_s)$.

Let $\mathcal{P} = \mathcal{P}(S_s)$ and for each S-set define

$$\psi = \psi(M_s) = \{(m_1, m_2) \in M \times M: m_1D = m_2D \text{ for some } D \in \mathcal{P}\}.$$  

It is easily seen from the properties noted above that $\psi$ is an S-congruence on $M_s$ which is a two-sided congruence if $M = S$. The S-congruence $\psi$ is called the singular congruence or $\mathcal{P}$-torsion congruence on $M_s$. When $\psi = \iota_M$ we say that $M_s$ is $\mathcal{P}$-torsion free.

Feller and Gantos [2] showed that every large sub S-set of an S-set $M_s$ is intersection large in $M_s$. The converse is not generally true. For, consider the semilattice $S = \{0, e, 1\}$ which has $0 < e < 1$ under the natural partial ordering. The right ideal $eS$ is clearly intersection large in $S$. Define $f: S \to S$ by $f(x) = \begin{cases} e \text{ if } x \in \{e, 1\} \\ 0 \text{ if } x = 0 \end{cases}$. Then $f \in \text{Hom}_S(S, S)$ and $f|eS$ is $1 - 1$. However, $f$ is not $1 - 1$. Therefore, $eS$ is not large in $S_s$. 

However, for semigroups Berthiaume proved the following lemma and gave a counterexample for the converse.
The following proposition gives a sufficient condition for the converse to be true.

**Proposition 3.3.** Let $M_S$ be a right $S$-set such that $M_S$ is $\mathcal{P}$-torsion free. Then $\mathcal{P}(M_S)$ is the set of large sub $S$-sets of $M_S$.

**Proof.** Let $A_S \in \mathcal{P}(M_S)$ and let $f \in \text{Hom}_S(M, B)$ such that $f|A$ is $1 - 1$ where $B_S$ is an $S$-set. Suppose $f(x_1) = f(x_2)$. Let $D = x_1^{-1}A \cap x_2^{-1}A = \{s \in S: x_1s \in A$ and $x_2s \in A\}$. Then $D \in \mathcal{P}(S)$ and since $f(x_1) = f(x_2)$, we have $f(x_1s) = f(x_2s)$ for all $s \in D$. However, $x_1s, x_2s \in A$ for all $s \in D$ and $f|A$ is $1 - 1$. Thus $x_1D = x_2D$ and it follows that $x_1 = x_2$ since $M_S$ is $\mathcal{P}$-torsion free.

It was noted in §2 that an injective $S$-set $M_S$ is always weakly injective but that a weakly injective $S$-set is not necessarily injective. In the following proposition we show that the two concepts are equivalent whenever $M_S$ is $\mathcal{P}$-torsion free.

**Proposition 3.4.** Let $M_S$ be a weakly injective $S$-set such that $M_S$ is $\mathcal{P}$-torsion free. Then $M_S$ is injective.

**Proof.** Let $A_S \subseteq B_S$ and let $f \in \text{Hom}_S(A, M)$. Let $M^*$ be the injective hull of $M_S$. Then $M_S$ is large in $M^*_S$ and hence is intersection large in $M^*_S$. Also, by Lemma 2.1 we see that $M^*_S$ is $\mathcal{P}$-torsion free since $\psi(M^*_S)|M_S = \psi(M_S) = \iota_M$. Thus, since $M^*_S$ is injective, there exists $f' \in \text{Hom}_S(B, M^*)$ such that $f'|A = f$. We claim that $f' \in \text{Hom}_S(B, M)$. Let $b \in B$ and let $f'(b) = n$. By the note following Lemma 3.2 we have $D = n^{-1}M \in \mathcal{P}(S)$. Define $\phi: n^{-1}M \rightarrow M$ by $\phi(s) = ns$. Thus we have $\phi \in \text{Hom}_S(n^{-1}M, M)$ and since $M_S$ is weakly injective there exists $m \in M$ such that $\phi(s) = ms$ for each $s \in n^{-1}M$. Therefore, $mD = nD$ and since $\psi_1(M^*_S) = \iota_{M^*}$ it follows that $n = m \in M$.

4. The extended centralizer of an $S$-set. The construction of the extended centralizer $Q$ of an $S$-set $M_S$ is similar to that given by Johnson [3] for rings over modules and is outlined as follows:

Let $\mathcal{P} = \mathcal{P}(M_S)$ be the class of intersection large sub $S$-sets of the $S$-set $M_S$. Let $F = \bigcup_{D \in \mathcal{P}} \text{Hom}_S(D, M)$ and define multiplication on $S$ by $fg = h$ where $h: D_g \cap g^{-1}(D_f) \rightarrow M$ by $h(x) = f(g(x))$. Then under this multiplication $F$ is a semigroup. Define a binary relation $\omega$ on the semigroup $F$ by $(f, g) \in \omega$ if there exists $D \in \mathcal{P}$ such that $f|D = g|D$. Then $\omega$ is a two-sided congruence on $F$. The semigroup $Q = Q(M_S) = F/\omega$ is called the extended centralizer of $M_S$. The elements of $Q$ will be denoted by $\bar{f}$ where $f \in F$.

In ring theory the extended centralizer is always (von Neumann) regular. An example given by McMorris in [4] shows that this is not
The case for semigroups. We can however give a necessary and sufficient condition for $Q$ to be regular in terms of splitting $S$-homomorphisms, which were studied by Feller and Gantos in [2]. Recall that an $S$-homomorphism $f$ which maps an $S$-set $M_s$ onto an $S$-set $N_s$ is said to split if there exists $g \in \text{Hom}_s(N, M)$ such that $fg = 1_N$.

**Theorem 4.1.** The semigroup $Q(M_s)$ is regular if and only if each equivalence class $\bar{f}$ of $Q(M_s) = F/\omega$ contains an element which splits.

**Proof.** Assume first that $Q(M_s) = Q$ is regular and let $\bar{f} \in Q$. Then there exists $\bar{g} \in Q$ such that $\bar{f} \bar{g} \bar{f} = \bar{f}$. Hence if $E = \{x \in D_{\bar{f}} : fgf(x) = f(x)\}$ then $E \in \mathcal{P}$. Let $f' = f|E$ and $g' = g|E_{\bar{f}}$. Let $y \in D_{\bar{f}}$, and let $x' = g(y)$. Since $y$ is also an element of $E$, there exists $x \in E$ such that $f(x) = y$ and we see that $y = f(x) = fgf(x) = fg(y) = f(x')$. Furthermore, $f'(x') = f(x) = fgf(x) = fgf(x')$ and it follows that $x' \in E$. Therefore, $y = fg(y) = f(x') = f'(x') = f'g'(y)$ and we see that $f'$ splits.

Conversely, for $f \in Q$ there exists $f' \in \bar{f}$ such that $f'$ splits. Hence, there exists $g' : R_{\bar{f}} \to D_{\bar{f}}$ such that $f'g' = t_{R_{\bar{f}}}$. By Zorn’s lemma there is a maximal sub $S$-set $N_s$ of $M_s$ such that $D_x \cap N = 0$. It easily follows that $D = D_x \cup N \in \mathcal{P}$ in this case. The $S$-homomorphism $g'$ can be extended to an $S$-homomorphism $g \in \text{Hom}_s(D, M)$ by defining $g(x) = 0$ if $x \in N$ and $g(x) = g'(x)$ if $x \in D_x$. Hence we have $g \in F$. Let $x \in D_{\bar{f}}$. Then $f'g'f'(x) = f'g'f(x) = 1_{R_{\bar{f}}}f'(x) = f'(x)$. Therefore, $\bar{f} \bar{g} \bar{f} = \bar{f} \bar{g} \bar{f} = \bar{f}' = \bar{f}$ and it follows that $Q$ is regular.

In the case where $M = S$ we have the following theorem.

**Theorem 4.2.** If $S$ is a regular semigroup then $Q(S)$ is regular.

**Proof.** By the previous theorem it is sufficient to show that each equivalence class $\bar{f}$ of $Q(S_S)$ contains an element which splits. Let $\bar{f} \in Q$ and let $\mathcal{F} = \{(D_a, g_a) : D_a$ is a right ideal of $S$ in $D_f$ such that $f_a = f|D_a$ splits on $D_a$ and $g_a : R_a = f_a(D_a) \to D_a$ such that $f_a g_a = 1_{R_a}\}$. The set $\mathcal{F}$ is nonempty since $((0), 0) \in \mathcal{F}$ where the zero in the second coordinate is the zero map. Define a partial order $\preceq$ on $\mathcal{F}$ by $(D_a, g_a) \preceq (D_b, g_b)$ iff $D_a \subseteq D_b$ and $g_b|R_a = g_a$. By an application of Zorn’s lemma, $\mathcal{F}$ contains a maximal element $(D_M, g_M)$. To complete the proof it is sufficient to show that $D_M \in \mathcal{P}(S_S)$. Suppose this is not true. Then $D_M$ is not intersection large in $D_f$. Hence there exists $e \in D_f$ such that $eS^1 \neq 0$ and $eS^1 \cap D_M = 0$. Since $S$ is regular, we may assume that $e^2 = e$. Let $x = f(e)$ then $xe = f(e)e = f(e^2) = f(e) = x$. We now consider two cases.
Case 1. Suppose $x \in S \cap R_M \neq 0$. Then there exists $s \in S$ such that $0 \neq xes \in R_M$. Consider $esS \subseteq eS$. Let $D' = D_M \cup esS$ and let $f' = f|D'$. Then $f'(D') = R_M$. If $y \in R_M$ then $f'g_M(y) = f_Mg_M(y) = y$. Hence $(D', g_M) \in \mathcal{F}$ and $(D_M, g_M) \prec (D', g_M)$ which contradicts the maximality of $(D_M, g_M)$.

Case 2. Suppose $x \in S \cap R_M = 0$. Let $x'$ be an inverse of $x$ and define $g': R' = R_M \cup xeS \to D' = D_M \cup eS$ by $g'(y) = \{g_M(y) \text{ if } y \in R_M\} \cup \{ex'y \text{ if } y \in xeS\}$. Note that $g' \in \text{Hom}(R', D')$. Now let $f' = f|D'$ and let $y \in R'$. If $y \in R_M$ then $f'g'(y) = f_Mg_M(y) = 1_{R_M}(y) = y$. On the other hand, if $y \in xeS$, say $y =xes$, then $f'g'(y) = f'g'(xes) = f'(ex'xes) = xe'xes = xes = y$. Hence it follows that $f'g' = 1_{R'}$. Thus, $(D', g') \in \mathcal{F}$ and clearly $(D_M, g_M) \prec (D', g')$ which again contradicts the maximality of $(D_M, g_M)$.

Therefore, $D_M$ must be intersection large in $S$ and the theorem follows.

An $S$-set $M_S$ is intersection uniform if every nonzero sub $S$-set of $M$ is intersection large.

**Theorem 4.3.** The semigroup $Q = Q(M_S)$ is a right cancellative semigroup with zero if and only if $M_S$ is intersection uniform.

**Proof.** Suppose that $Q$ is a right cancellative semigroup with zero and let $N_S$ be a nonzero sub $S$-set of $M_S$. Using Zorn's lemma to find a maximal sub $S$-set $N'$ of $M$ such that $N \cap N' = 0$, define a function $f$ on $N \cup N'$ by $f(x) = x$ if $x \in N$ and $f(x) = 0$ if $x \in N'$. Then $f^2 = f$ and $f \in F$. If $\bar{f} = 0$ then there exists $D \in \mathcal{P}(M_S)$ such that $D \supseteq D_f$ and $f(D) = 0$ which implies that $D \supseteq N'$. Hence $N' \in \mathcal{P}(M_S)$. But this is impossible since $N \cap N' = 0$. Thus we have $\bar{f} \neq 0$. Since $1_M\bar{f} = \bar{f} = \bar{f}\bar{f}$ and since each nonzero element of $Q$ is right cancellable, it follows that $\bar{1}_M = \bar{f}$. Therefore, there exists $D \in \mathcal{P}$ such that $D \supseteq N$ and it follows that $N \in \mathcal{P}(M_S)$. The proof of the converse is immediate.

5. The injective hull of a $\mathcal{P}$-torsion free semigroup. Throughout this section we shall consider the semigroup $S$ as an $S$-set over itself. For $s \in S$ define $\phi_s: S \to S$ by $\phi_s(t) = st$. Then $\phi_s \in F$ and it easily follows that the map $\phi: S \to Q$ by $\phi(s) = \phi_s$ is a representation of $S$ in $Q = Q(S_S)$. Note also that we can regard $Q$ as a centered right $S$-set by defining $f\phi = \bar{f}\phi_s$ for each $f \in Q$ and for each $s \in S$. The following lemmas are easy consequences of the above remarks.

**Lemma 5.1.** $\psi(S_S) = \phi^{-1} \circ \phi$. 


LEMMA 5.2. For each \( f \in F \) and for each \( s \in D_f \), \( f_{\phi_8} = \phi_{f(s)} \).

When \( \gamma(S_s) = \iota_s \) we shall assume that \( S \) is embedded in \( Q = Q_{\phi}(S) \) under the identification \( s \mapsto \tilde{s} \). From Lemma 5.2 we see that \( \tilde{f}s = f(s) \) for each \( \tilde{f} \in Q_{\phi}(S) \) and for each \( s \in D_f \) under the identification described above. Thus we see that \( S_s \) is intersection large in \( Q_s \). In addition, the next lemma shows that \( Q_s \) is \( \mathcal{P} \)-torsion free.

LEMMA 5.3. If \( S \) is \( \mathcal{P} \)-torsion free then \( Q_s \) is \( \mathcal{P} \)-torsion free.

**Proof.** Let \( (\tilde{f}_1, \tilde{f}_2) \in \psi(Q_s) \). Then there exists \( E \in \mathcal{P} \) such that \( \tilde{f}_1 E = \tilde{f}_2 E \). Let \( E' = E \cap D_{f_1} \cap D_{f_2} \in \mathcal{P} \). Then for each \( s \in E' \), we have \( f_1(s) = \tilde{f}_1 s = \tilde{f}_2 s = f_2(s) \) and it follows that \( \tilde{f}_1 = \tilde{f}_2 \).

The following lemma is immediate from Lemma 2.1 and the remarks preceding the above lemma.

LEMMA 5.4. If \( S \) is \( \mathcal{P} \)-torsion free then \( S_s \) is large in \( Q_s \).

We now can show that \( Q_s \) is the injective hull of \( S \) and furthermore \( Q \) is injective as a \( Q \)-set.

**Theorem 5.5.** If \( S \) is \( \mathcal{P} \)-torsion free then \( Q_s = Q(S_s) \) is the injective hull of \( S_s \).

**Proof.** Since \( S_s \) is large in \( Q_s \) by Lemma 5.3, we need only show that \( Q_s \) is injective. By Lemma 5.3 and Proposition 3.4 it suffices to verify that \( Q_s \) is weakly injective. Let \( R \) be a right ideal of \( S \) and let \( \Phi \in \mathrm{Hom}_s(R, Q) \). Since \( S_s \) is intersection large in \( Q_s \), \( R' = \Phi^{-1}(S) \in \mathcal{P} \) and \( f = \Phi|R' \in F \). We claim that \( \Phi(r) = \tilde{r}f \) for each \( r \in R \). For each \( s \in r^{-1}R' = \{ s: rs \in R' \} \) we have \( \Phi(r)s = \Phi(rs) = f(rs) = (\tilde{r}f)s \). Thus, since \( r^{-1}R' \in \mathcal{P} \), it follows that \( (\Phi(r), \tilde{r}f) \in \psi(Q_s) \) which is the identity \( S \)-congruence on \( Q_s \). Therefore, \( \Phi(r) = \tilde{r}f \) for each \( r \in R \) and the result follows.

**Theorem 5.6.** If \( S \) is \( \mathcal{P} \)-torsion free then \( Q = Q(S_s) \) is self injective.

**Proof.** Let \( A_0 \subseteq B_0 \) be \( Q \)-sets and let \( \Phi' \in \mathrm{Hom}_0(A, B) \). Then \( \Phi' \in \mathrm{Hom}_s(A, Q) \). Since \( Q_s \) is the injective hull of \( S_s \), there exists \( \Phi \in \mathrm{Hom}_s(B, Q) \) such that \( \Phi|A = \Phi' \). We claim that \( \Phi \) is a \( Q \)-homomorphism. Let \( b \in B \) and \( \tilde{f} \in Q \). Then for each \( s \in D_f \) we have \( \Phi(b\tilde{f})s = \Phi(b\tilde{f}s) = \Phi(bf(s)) = \Phi(b)f(s) = \Phi(b)\tilde{f}s \). Thus \( (\Phi(b\tilde{f}), \Phi(b)\tilde{f}) \in \psi(Q_s) \) which is the identity congruence on \( Q \). Therefore, it follows that \( \Phi(b\tilde{f}) = \Phi(b)\tilde{f} \).
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