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GENERALIZED LERCH ZETA FUNCTION

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The purpose of this paper is to establish certain properties of the generalized Lerch zeta function $\theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n+a)^{-\nu} z^{(n+a)b}$. The main result yields another infinite series representation for θ . A generalization of Hardy's relation follows as an immediate corollary.

1. Introduction. The function $\Phi(z, \nu, a)$ defined by the power series

$$(1) \quad \Phi(z, \nu, a) = \sum_{n=0}^{\infty} (n+a)^{-\nu} z^n,$$

for $|z| < 1$, $0 < a \leq 1$ and arbitrary ν , is called Lerch's zeta function. For $z = 1$, this function becomes Hurwitz' zeta function

$$(2) \quad \Phi(1, \nu, a) = \zeta(\nu, a) = \sum_{n=0}^{\infty} (n+a)^{-\nu}, \operatorname{Re} \nu > 1 \text{ and } 0 < a \leq 1.$$

Lerch's function has been extensively investigated in [1], [2], [3], [5, v. 1, p. 27], [7], [8], and [12]. One important result

$$(3) \quad \Phi(z, \nu, a) = \Gamma(1-\nu) z^{-a} (\log 1/z)^{\nu-1} + z^{-a} \sum_{r=0}^{\infty} \zeta(\nu-r, a) \frac{(\log z)^r}{r!},$$

for $|\log z| < 2\pi$, $0 < a \leq 1$, $\nu \neq 1, 2, 3, \dots$, which transforms Lerch's series into another series, is derived in Erdélyi [5, v. 1, p. 29] by using Lerch's transformation formula and Hurwitz' series for the Hurwitz zeta function. Hardy's relation (see Hardy [7] and Mellin [10]) follows immediately from (3):

$$(4) \quad \lim_{z \rightarrow 1} \{\Phi(z, \nu, a) - \Gamma(1-\nu) (\log 1/z)^{\nu-1} z^{-a}\} = \zeta(\nu, a).$$

The purpose of this paper is to establish certain properties of the function $\theta(z, \nu, a, b)$ where

$$(5) \quad \theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n+a)^{-\nu} z^{(n+a)b}, \text{ for } |z| < 1, 0 < a \leq 1, 0 < b.$$

It is appropriate to call θ the generalized Lerch zeta function because

$$\theta(z, \nu, a, 1) = z^a \Phi(z, \nu, a).$$

Using an approach which is more direct than the above mentioned derivation of equation (3), we will establish

$$(6) \quad \begin{aligned} & \theta(z, \nu, a, b) \\ &= b^{-1} \Gamma\left(\frac{1-\nu}{b}\right) (\log 1/z)^{(\nu-1)/b} + \sum_{r=0}^{\infty} \zeta(\nu - br, a) \frac{(\log z)^r}{r!}, \end{aligned}$$

where $\nu \neq 1, 1+b, 1+2b, \dots, 0 < a, b \leq 1$. Formula (6) is valid for unrestricted z if $0 < b < 1$, and for $|\log z| < 2\pi$ if $b = 1$. In the latter case equation (6) becomes equation (3). Furthermore, from (6) we immediately obtain the following generalization of Hardy's relation:

$$(7) \quad \lim_{z \rightarrow 1} \left\{ \theta(z, \nu, a, b) - b^{-1} \Gamma\left(\frac{1-\nu}{b}\right) (\log 1/z)^{(\nu-1)/b} \right\} = \zeta(\nu, a),$$

for $0 < b \leq 1$.

2. Derivation of formula (6). Consider the function

$$f(x) = x^{-\nu} e^{-\beta x^b}, \quad \operatorname{Re} \beta > 0, b > 0.$$

The Mellin transform of $f(x)$ with respect to the parameter s is

$$g(s) = b^{-1} \beta^{(\nu-s)/b} \Gamma\left(\frac{s-\nu}{b}\right), \quad \operatorname{Re} s > \operatorname{Re} \nu, \operatorname{Re} \beta > 0, b > 0.$$

For Mellin transform theory see [4, v. 1, Ch. 2], [10], [11], and [13, p. 46]; for tables see [6, v. 1, p. 303]. Writing $f(x)$ in its Mellin inversion integral form with $x = n + a$, we obtain by summing on n and interchanging the order of summation and integration

$$(8) \quad \sum_{n=0}^{\infty} (n+a)^{-\nu} e^{-\beta(n+a)^b} = \frac{\beta^{\nu/b}}{2\pi i b} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \beta^{-s/b} \zeta(s, a) \Gamma\left(\frac{s-\nu}{b}\right) ds,$$

where $\sigma_0 > \max\{1, \operatorname{Re} \nu\}$. The left hand side of (8) is $\theta(e^{-\beta}, \nu, a, b)$. To evaluate the right hand side integral we will use the residue theorem.

If we denote

$$(9) \quad h(s) = \beta^{-s/b} \zeta(s, a) \Gamma\left(\frac{s-\nu}{b}\right),$$

then $h(s)$ has a first order pole at $s = 1$ with residue $\beta^{-1/b} \Gamma([1 - \nu/b])$, and an infinite set of first order poles at $s = \nu - br$ with residues

$$\frac{(-1)^r}{r!} b \beta^{r-\nu/b} \zeta(\nu - br, a), \quad r = 0, 1, 2, \dots$$

Consider the contour integral

$$(10) \quad \int_C h(s) ds,$$

where the path of integration C is indicated in Figure 1 below, such that the half-circle C' of radius d separates the poles $s = \nu - Nb$ and $s = \nu - (N + 1)b$. Then $h(s)$ is one-valued and analytic inside and on C except at the points $s = 1, s = \nu - rb$ ($r = 0, 1, 2, \dots, N$).

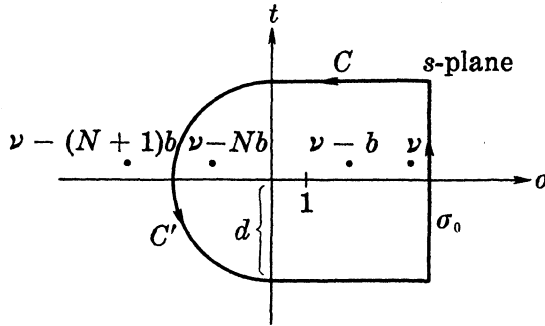


FIG. 1

Now we let N tend to infinity through positive integers.

To investigate the contributions along individual segments of the contour C we will need the following well-known properties of the gamma function and Hurwitz' zeta function, which can be found in Erdélyi [5] and/or Whittaker and Watson [14]:

$$(11) \quad \Gamma(s) = (2\pi/s)^{1/2} e^{-s} e^{s \log s} \left(1 + O\left(\frac{1}{s}\right) \right) \text{ as } |s| \longrightarrow \infty, |\arg s| < \pi .$$

$$(12) \quad \frac{\Gamma(s + \alpha)}{\Gamma(s + \beta)} = s^{\alpha - \beta} \left(1 + O\left(\frac{1}{s}\right) \right) \text{ as } |s| \longrightarrow \infty, |\arg s| < \pi .$$

$$(13) \quad |\Gamma(\sigma + it)| = O(|t|^{\sigma - 1/2} e^{-\pi|t|/2}) \text{ as } |t| \longrightarrow \infty \\ \text{with } \sigma \text{ fixed } (\sigma, t \text{ real}) .$$

$$(14) \quad \zeta(s, a) = 2(2\pi)^{s-1} \Gamma(1 - s) \sum_{n=1}^{\infty} n^{s-1} \sin\left(2\pi na + \frac{\pi s}{2}\right), \\ \text{Re } s < 0, 0 < a \leq 1 .$$

$$(15) \quad \zeta(\sigma + it, a) = o(|t|) \text{ as } |t| \longrightarrow \infty \text{ with } 0 \leq \sigma \text{ fixed } (\sigma, t \text{ real}) .$$

It is clear that the contributions to the integral (10) along the horizontal lines of length σ_0 (see Figure 1) vanish as $d \rightarrow \infty$ because of (13) and (15). To find the contribution along the half-circle C' , it is sufficient to investigate the behavior of $h(s)$ on the quarter-circle of radius d for $\pi/2 < \arg s = \phi < \pi$, since the modulus of $h(s)$ on the quarter-circle for $-\pi < \phi < -\pi/2$ is the same by Schwarz's reflection principle. From (11), (12), and (14) we obtain

$$\begin{aligned}
 h(s) &= O\{|s|^{-\nu/b} e^{|s|\cos\phi(\log 2\pi e - b^{-1}\log \beta e)} \\
 &\quad \times e^{|s|\sin\phi(\pi/2 - \phi/b)} e^{|s|\cos\phi(b^{-1}\log(|s|/b) - \log|s|)}\} \\
 (16) \qquad &\text{as } |s| \longrightarrow \infty \text{ in } \frac{\pi}{2} < \phi < \pi, s = |s|e^{i\phi}.
 \end{aligned}$$

In formula (16) we have assumed β to be real and positive. The analytic continuation to complex β will be obtained later. The following three cases are possible:

(i) $b > 1$: Then (16) is dominated by the last exponential function and $h(s)$ tends to infinity when $d \rightarrow \infty$. Thus, the contribution over the semi-circle tends to infinity as $d \rightarrow \infty$ and formula (8) is not applicable, although the series in (8) converges for $\text{Re } \beta > 0$.

(ii) $b = 1$: It is clear from (16) that the integral over the semi-circle C' vanishes as $d \rightarrow \infty$, provided $\beta < 2\pi$.

(iii) $0 < b < 1$: The integral over the semi-circle C' vanishes as $d \rightarrow \infty$, regardless of β .

Hence, in cases (ii) and (iii) we obtain by (8) and the residue theorem

$$(17) \quad \theta(e^{-\beta}, \nu, a, b) - \frac{1}{b} \Gamma\left(\frac{1-\nu}{b}\right) \beta^{(\nu-1)/b} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \zeta(\nu - br, a) \beta^r.$$

The *r.h.s.* series in (17) is a Taylor series around the origin and is therefore an analytic function of β in its circle of convergence, while the *l.h.s.* expression is valid only when $\text{Re } \beta > 0$ and ν arbitrary; or $\text{Re } \beta = 0$, $\text{Im } \beta \neq 0$, and $\text{Re } \nu > 0$; or $\beta = 0$ and $\text{Re } \nu > 1$. Therefore, (17) represents the analytic continuation with respect to β of the *l.h.s.* of (17) valid for $\text{Re } \beta > 0$ into the *r.h.s.* which is valid for unrestricted β when $0 < b < 1$, or for $|\beta| < 2\pi$ when $b = 1$. If $b > 1$, (17) is not valid.

Formula (6) is obtained from (17) by setting

$$z = e^{-\beta}, \quad \beta = \log 1/z.$$

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REFERENCES

1. T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math., **1** (1951), 161-167.
2. ———, *Remark on the Hurwitz zeta function*, Proc. Amer. Math. Soc., **2** (1951), 690-693.
3. E. W. Barnes, *On certain functions defined by Taylor series of finite radius of convergence*, Proc. London Math. Soc., **4** (1906), 284-316.
4. G. Doetsch, *Handbuch der Laplace Transformation*, 3 vols., Birkhauser, Basel, 1950-1956.
5. A. Erdélyi et al., *Higher Transcendental Functions*, vol. 1, McGraw Hill, New York, 1953.

6. A. Erdélyi *et al.*, *Tables of Integral Transforms*, 2 vols., McGraw Hill, New York, 1954.
7. G. H. Hardy, *Method for determining the behavior of certain classes of power series near a singular point on the circle of convergence*, Proc. London Math. Soc., **3** (1905), 381-389.
8. M. Lerch, *Note sur la fonction $\sum_{k=0}^{\infty} e^{2\pi i k x} (w+k)^{-s}$* , Acta Math., **11** (1887), 19-24.
9. G. G. MacFarlane, *The application of Mellin transforms to the summation of slowly convergent series*, Philos. Mag., (7) **40** (1949), 188-197.
10. H. Mellin, *Abriss einer einheitlichen theorie der gamma und der hypergeometrischen funktionen*, Math. Ann., **68** (1910), 305-337.
11. ———, *Über den zusammenhang zwischen den linearen differentialund differenzen-gleichungen*, Acta Math., **25** (1902), 139-164.
12. F. Oberhettinger, *Note on the Lerch zeta function*, Pacific J. Math., **6** (1956), 117-120.
13. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon, Oxford, 1937.
14. E. T. Whittaker and G. N. Watson, *A course in Modern Analysis*, 4th ed., Cambridge University Press, London, 1927.

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