GENERALIZED LERCH ZETA FUNCTION

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The purpose of this paper is to establish certain properties of the generalized Lerch zeta function \( \theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)b} \). The main result yields another infinite series representation for \( \theta \). A generalization of Hardy’s relation follows as an immediate corollary.

1. Introduction. The function \( \Phi(z, \nu, a) \) defined by the power series

\[
\Phi(z, \nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^n ,
\]

for \( |z| < 1 \), \( 0 < a \leq 1 \) and arbitrary \( \nu \), is called Lerch’s zeta function. For \( z = 1 \), this function becomes Hurwitz’ zeta function

\[
\Phi(1, \nu, a) = \zeta(\nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu}, \text{Re } \nu > 1 \text{ and } 0 < a \leq 1 .
\]

Lerch’s function has been extensively investigated in [1], [2], [3], [5, v. 1, p. 27], [7], [8], and [12]. One important result

\[
\Phi(z, \nu, a) = \Gamma(1 - \nu) z^{-\nu} (\log 1/z)^{\nu-1} + z^{-\nu} \sum_{\sigma=0}^{\infty} \zeta(\nu - \sigma, a) \frac{(\log z)^\sigma}{\sigma!} ,
\]

for \( |\log z| < 2\pi \), \( 0 < a \leq 1 \), \( \nu \neq 1, 2, 3, \ldots \), which transforms Lerch’s series into another series, is derived in Erdélyi [5, v. 1, p. 29] by using Lerch’s transformation formula and Hurwitz’ series for the Hurwitz zeta function. Hardy’s relation (see Hardy [7] and Mellin [10]) follows immediately from (3):

\[
\lim_{z \to 1} \{ \Phi(z, \nu, a) - \Gamma(1 - \nu) (\log 1/z)^{\nu-1} z^{-\nu} \} = \zeta(\nu, a) .
\]

The purpose of this paper is to establish certain properties of the function \( \theta(z, \nu, a, b) \) where

\[
\theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)b} , \text{ for } |z| < 1, 0 < a \leq 1, 0 < b .
\]

It is appropriate to call \( \theta \) the generalized Lerch zeta function because

\[
\theta(z, \nu, a, 1) = z^a \Phi(z, \nu, a) .
\]

Using an approach which is more direct than the above mentioned derivation of equation (3), we will establish
\[ \theta(z, \nu, a, b) \]
\[ = b^{-1} I\left(1 - \frac{\nu}{b}\right) \left(\log \frac{1}{z}\right)^{(\nu-1)/b} + \sum_{r=0}^{\infty} \zeta(\nu - br, a) \frac{(\log z)^r}{r!} , \]

where \( \nu \neq 1, 1 + b, 1 + 2b, \ldots, 0 < a, b \leq 1 \). Formula (6) is valid for unrestricted \( z \) if \( 0 < b < 1 \), and for \( |\log z| < 2\pi \) if \( b = 1 \). In the latter case equation (6) becomes equation (3). Furthermore, from (6) we immediately obtain the following generalization of Hardy’s relation:

\[ \lim_{z \to 1} \left\{ \theta(z, \nu, a, b) - b^{-1} I\left(1 - \frac{\nu}{b}\right) \left(\log \frac{1}{z}\right)^{(\nu-1)/b} \right\} = \zeta(\nu, a) , \]

for \( 0 < b \leq 1 \).

2. Derivation of formula (6). Consider the function

\[ f(x) = x^{-\nu} e^{-x b} , \quad \text{Re} \, \beta > 0, b > 0. \]

The Mellin transform of \( f(x) \) with respect to the parameter \( s \) is

\[ g(s) = b^{-1} \beta^{(s-\nu)/b} \Gamma\left(\frac{s - \nu}{b}\right) , \quad \text{Re} \, s > \text{Re} \, \nu, \text{Re} \, \beta > 0, b > 0 . \]

For Mellin transform theory see [4, v. 1, Ch. 2], [10], [11], and [13, p. 46]; for tables see [6, v. 1, p. 303]. Writing \( f(x) \) in its Mellin inversion integral form with \( x = n + a \), we obtain by summing on \( n \) and interchanging the order of summation and integration

\[ \sum_{n=0}^{\infty} (n + a)^{-\nu} e^{-b(n + a) b} = \frac{\beta^{\nu/b}}{2\pi i b} \int_{c-i \infty}^{c+i \infty} \beta^{-s/b} \zeta(s, a) I\left(\frac{s - \nu}{b}\right) ds , \]

where \( c > \max \{1, \text{Re} \, \nu\} \). The left hand side of (8) is \( \theta(e^{-\beta}, \nu, a, b) \).

To evaluate the right hand side integral we will use the residue theorem.

If we denote

\[ h(s) = \beta^{-s/b} \zeta(s, a) I\left(\frac{s - \nu}{b}\right) , \]

then \( h(s) \) has a first order pole at \( s = 1 \) with residue \( \beta^{\nu/b} \Gamma([1 - \nu/b]) \), and an infinite set of first order poles at \( s = \nu - br \) with residues

\[ \frac{(-1)^r}{r!} b \beta^{\nu-1/b} \zeta(\nu - br, a) , \quad r = 0, 1, 2, \ldots . \]

Consider the contour integral

\[ \int_{c} h(s) ds , \]
where the path of integration $C$ is indicated in Figure 1 below, such that the half-circle $C'$ of radius $d$ separates the poles $s = \nu - Nb$ and $s = \nu - (N + 1)b$. Then $h(s)$ is one-valued and analytic inside and on $C$ except at the points $s = 1$, $s = \nu - rb$ ($r = 0, 1, 2, \cdots, N$).

![Diagram](image)

Fig. 1

Now we let $N$ tend to infinity through positive integers.

To investigate the contributions along individual segments of the contour $C$ we will need the following well-known properties of the gamma function and Hurwitz' zeta function, which can be found in Erdélyi [5] and/or Whittaker and Watson [14]:

\begin{enumerate}
  \item [11] $\Gamma(s) = (2\pi/s)^{1/2}e^{-s}s^{s-1/2}e^{-\pi/4|\text{arg } s|}$ as $|s| \to \infty$, $|\text{arg } s| < \pi$.
  \item [12] $\frac{\Gamma(s + \alpha)}{\Gamma(s + \beta)} = s^{\alpha-\beta}(1 + O\left(\frac{1}{s}\right))$ as $|s| \to \infty$, $|\text{arg } s| < \pi$.
  \item [13] $|\Gamma(\sigma + it)| = O(|t|^{-1/2}e^{-\pi|t|^2/4})$ as $|t| \to \infty$ with $\sigma$ fixed ($\sigma, t$ real).
  \item [14] $\zeta(s, \alpha) = 2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1} \sin \left(2\pi na + \frac{\pi s}{2}\right)$, $\text{Re } s < 0, 0 < \alpha \leq 1$.
  \item [15] $\zeta(\sigma + it, \alpha) = o(|t|)$ as $|t| \to \infty$ with $0 \leq \sigma$ fixed ($\sigma, t$ real).
\end{enumerate}

It is clear that the contributions to the integral (10) along the horizontal lines of length $\sigma_0$ (see Figure 1) vanish as $d \to \infty$ because of (13) and (15). To find the contribution along the half-circle $C'$, it is sufficient to investigate the behavior of $h(s)$ on the quarter-circle of radius $d$ for $\pi/2 < \text{arg } s = \phi < \pi$, since the modulus of $h(s)$ on the quarter-circle for $-\pi < \phi < -\pi/2$ is the same by Schwarz's reflection principle. From (11), (12), and (14) we obtain
In formula (16) we have assumed \( \beta \) to be real and positive. The analytic continuation to complex \( \beta \) will be obtained later. The following three cases are possible:

(i) \( b > 1 \): Then (16) is dominated by the last exponential function and \( h(s) \) tends to infinity when \( d \to \infty \). Thus, the contribution over the semi-circle tends to infinity as \( d \to \infty \) and formula (8) is not applicable, although the series in (8) converges for \( \text{Re} \beta > 0 \).

(ii) \( b = 1 \): It is clear from (16) that the integral over the semi-circle \( C' \) vanishes as \( d \to \infty \), provided \( \beta < 2\pi \).

(iii) \( 0 < b < 1 \): The integral over the semi-circle \( C' \) vanishes as \( d \to \infty \), regardless of \( \beta \).

Hence, in cases (ii) and (iii) we obtain by (8) and the residue theorem

\[
\theta(e^{-\beta}, \nu, a, b) = \frac{1}{b} \Gamma\left(\frac{1 - \nu}{b}\right) \beta^{(\nu - 1)/b} = \sum_{r=0}^{\infty} \frac{(-1)^{r}}{r!} \zeta(\nu - br, a)\beta^{r}.
\]

The \textit{r.h.s}. series in (17) is a Taylor series around the origin and is therefore an analytic function of \( \beta \) in its circle of convergence, while the \textit{l.h.s}. expression is valid only when \( \text{Re} \beta > 0 \) and \( \nu \) arbitrary; or \( \text{Re} \beta = 0 \), \( \text{Im} \beta \neq 0 \), and \( \text{Re} \nu > 0 \); or \( \beta = 0 \) and \( \text{Re} \nu > 1 \). Therefore, (17) represents the analytic continuation with respect to \( \beta \) of the \textit{l.h.s}. of (17) valid for \( \text{Re} \beta > 0 \) into the \textit{r.h.s}. which is valid for unrestricted \( \beta \) when \( 0 < b < 1 \), or for \( |\beta| < 2\pi \) when \( b = 1 \). If \( b > 1 \), (17) is not valid.

Formula (6) is obtained from (17) by setting

\[
z = e^{-\beta}, \quad \beta = \log 1/z.
\]

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