

Pacific Journal of Mathematics

A GENERALIZATION OF $(-1, 1)$ RINGS

ERWIN KLEINFELD

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A ring is defined to be a division ring in case the equations $ax = b$, and $ya = b$, have unique solutions for x and y whenever $a \neq 0$. It is shown that division rings of characteristic $\neq 2, 3$ which satisfy the identities (i) $(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x$, (ii) $(x, y, z) + (y, z, x) + (z, x, y) = 0$, and (iii) $((x, y), y, y) = 0$, are associative.

Main section. We consider rings R of characteristic different from 2 and 3 which satisfy the identities

- (1) $(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x$,
- (2) $(x, y, z) + (y, z, x) + (z, x, y) = 0$,
- (3) $((x, y), y, y) = 0$,

where $(a, b, c) = (ab)c - a(bc)$, and $(a, b) = ab - ba$. The first of these identities holds in right alternative rings and has been investigated in combination with others [5, 6]. In an arbitrary ring it can be verified that $((x, y), z) + ((y, z), x) + ((z, x), y) = (x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (z, y, x) - (y, x, z)$. If the ring is third-power associative, then $(x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z) = 0$, so that $((x, y), z) + ((y, z), x) + ((z, x), y) = 2(x, y, z) + 2(y, z, x) + 2(z, x, y)$. We have shown that in a third-power associative ring of characteristic $\neq 2$, identity (2) is equivalent to Lie-admissibility. Next we will show that a ring which satisfies (1) and (2) is Jordan-admissible if and only if it satisfies (3). Suppose A is a ring of characteristic $\neq 2$ which satisfies (1) and (2) and which is Jordan-admissible. That means that A under a new product $xy + yx$ must satisfy the Jordan identity, so that $(xy + yx)x^2 + x^2(xy + yx) - x(x^2y + yx^2) - (x^2y + yx^2)x = 0$, which implies that $-(x^2, y, x) + (x, y, x^2) - (y, x^2, x) + (y, x, x^2) + (x, x^2, y) - (x^2, x, y) = 0$. Since we have $(yx, x, x) - (y, x^2, x) + (y, x, x^2) = (y, x, x)x$, in every ring, while (1) implies $(yx, x, x) = (y, x, x)x$, we see that $(y, x^2, x) = (y, x, x^2)$. Also (1) implies the two identities $(x^2, y, x) + (x, x, (y, x)) = x(x, y, x) + (x, y, x)x$, and $(x^2, x, y) + (x, x, (x, y)) = x(x, x, y) + (x, x, y)x$. Adding these and use of (2) shows that $(x^2, y, x) + (x^2, x, y) = x(x, y, x) + x(x, x, y) + (x, y, x)x + (x, x, y)x = -x(y, x, x) - (y, x, x)x$. In an arbitrary ring one may verify the identity $(xy, x, x) - (x, yx, x) + (x, y, x^2) = x(y, x, x) + (x, y, x)x$, while (1) implies that $(xy, x, x) = x(y, x, x)$, so that $(x, y, x^2) = (x, yx, x) + (x, y, x)x$. Also in an arbitrary ring $(x^2, x, y) - (x, x^2, y) + (x, x, xy) = x(x, x, y)$, while (1) implies $(x^2, x, y) + (x, x, (x, y)) = x(x, x, y) +$

$(x, x, y)x$. Subtracting the next to the last from the last equation it follows that $(x, x^2, y) = (x, x, yx) + (x, x, y)x$. Putting all of these substitutions in the Jordan-admissible relation, we find that $(y, x, x)x = x(y, x, x)$, as all other terms cancel. But in that case, since we have already verified that $(xy, x, x) = x(y, x, x)$, and $(yx, x, x) = (y, x, x)x$, we must have $((x, y), x, x) = 0$. This is equivalent to identity (3). Similarly one can work through this argument backwards to show that (1), (2), and (3) imply Jordan-admissibility. Of course (3) also holds in every right alternative ring. All three identities are valid in $(-1, 1)$ rings, which have been investigated in detail by Maneri [7] and Hentzel [1, 2].

Nonassociative division rings are of interest in the study of the foundation of projective geometry, as they coordinate non-Desarguesian planes. It is therefore of interest to classify those rings R which are division rings. Our main result is the following.

THEOREM. *All division rings of characteristic different from 2 and 3 which satisfy the identities (1)–(3) are associative.*

In order to prove this result we derive a number of other identities, many of which are valid in all R , not necessarily division rings. We begin with the Teichmüller identity

$$(4) \quad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z,$$

which is valid in every ring and which can be established by use of the definition of the associator (a, b, c) . Four applications of the identity (4) lead to the equations

$$\begin{aligned} (wx, y, z) - (w, xy, z) + (w, x, yz) &= w(x, y, z) + (w, x, y)z, \\ -(xy, z, w) + (x, yz, w) - (x, y, zw) &= -x(y, z, w) - (x, y, z)w, \\ (yz, w, x) - (y, zw, x) + (y, z, wx) &= y(z, w, x) + (y, z, w)x, \\ -(zw, x, y) + (z, wx, y) - (z, w, xy) &= -z(w, x, y) - (z, w, x)y, \end{aligned}$$

so that adding all the left sides we obtain, after repeated applications of (2), the sum of zero. Adding the right hand sides we obtain $(w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y))$. Thus

$$(5) \quad (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0.$$

Using (1) twice, we discover that $(ab, y, y) + (a, b, (y, y)) = a(b, y, y) + (a, y, y)b$, while $(ba, y, y) + (b, a, (y, y)) = b(a, y, y) + (b, y, y)a$. Subtracting the second equation from the first it follows that

$$(6) \quad ((a, b), y, y) = (a, (b, y, y)) - (b, (a, y, y)).$$

If we let $a = x$, and $b = y$, in (6) and use (3), then

$$(7) \quad (y, (x, y, y)) = 0 .$$

A linearization of (7), obtained by replacing y by $y + z$ and $y - z$, leads to

$$(8) \quad (y, (x, y, z)) + (y, (x, z, y)) + (z, (x, y, y)) = 0 .$$

Substituting $w = z = y$ in (5) it follows that $(y, (x, y, y)) - (y, (y, x, y)) + (y, (y, y, x)) = 0$, since $(y, y, y) = 0$, follows from (2). But $(x, y, y) + (y, x, y) + (y, y, x) = 0$, as a result of (2). Thus $2(y, (y, x, y)) = 0$, so that

$$(9) \quad (y, (y, x, y)) = 0 .$$

Linearizing (9), it follows that

$$(10) \quad (z, (y, x, y)) + (y, (z, x, y)) + (y, (y, x, z)) = 0 .$$

Substituting $x = z = y$, and $w = x$, in (1) it follows that

$$(11) \quad (xy, y, y) = (x, y, y)y .$$

Expanding $((a, b, c), y, y) = ((ab)c, y, y) - (a(bc), y, y)$ and applying (1) repeatedly to break up products, we establish that

$$(12) \quad ((a, b, c), y, y) = ((a, y, y), b, c) + (a, (b, y, y), c) + (a, b, (c, y, y)) .$$

Substituting $z = x$, and $w = y$, in (5) it follows that $2(y, (x, y, x)) - 2(x, (y, x, y)) = 0$. Because the characteristic is different from 2 this implies

$$(13) \quad (y, (x, y, x)) = (x, (y, x, y)) .$$

Substituting $z = y$, and $w = x$, in (5), it follows that $(x, (x, y, y)) - (x, (y, y, x)) + (y, (y, x, x)) - (y, (x, x, y)) = 0$. However, (y, x, x) may be replaced by $-(x, y, x) - (x, x, y)$, because of (2) and $-(y, y, x)$ by $(y, x, y) + (x, y, y)$, so that $(x, (x, y, y)) + (x, (y, x, y)) + (x, (x, y, y)) - (y, (x, y, x)) - (y, (x, x, y)) - (y, (x, x, y)) = 0$. But the second and fourth terms cancel as a consequence of (13), so that the remaining terms become $2(x, (x, y, y)) - 2(y, (x, x, y)) = 0$. Since the characteristic of R is different from 2 this implies $(x, (x, y, y)) = (y, (x, x, y))$. Substituting $z = x$ in (8), it follows that $(x, (x, y, y)) = -(y, (x, y, x)) - (y, (x, x, y)) = (y, (y, x, x))$, since $(y, x, x) + (x, y, x)(x, x, y) = 0$, as a consequence of (2). We can now combine the previous equations to obtain $(y, (x, x, y)) = (y, (y, x, x))$. Since $(y, x, x) + (x, y, x) + (x, x, y) = 0$, this implies

$$(14) \quad (y, (x, y, x)) = -2(y, (y, x, x)) = -2(y, (x, x, y)) .$$

Substituting $y = a + b$, $x = y$, and $y = a - b$, $x = y$, in (14) it follows that

$$(15) \quad (a, (y, b, y)) + (b, (y, a, y)) = -2(a, (b, y, y)) - 2(b, (a, y, y)) .$$

Substituting $x = xy$ in (3), we derive $((xy, y), y, y) = 0$. Substituting $x = (x, y)$ in (11) it follows that $((x, y)y, y, y) = ((x, y), y, y)y = 0$, using (3). But $(xy, y) - (x, y)y = (xy)y - y(xy) - (xy)y + (yx)y = (y, x, y)$. Consequently $((y, x, y), y, y) = ((xy, y), y, y) - ((x, y)y, y, y) = 0$. We have shown

$$(16) \quad ((y, x, y), y, y) = 0 .$$

Substituting $x = (z, x, y) + (y, x, z)$ in (3), we see that $((z, x, y) + (y, x, z), y, y) = 0$. But then use of (10) leads to $0 = -((z, (y, x, y)), y, y) = (((y, x, y), z), y, y)$. Substituting $a = (y, x, y)$, $b = z$, in (6), it follows that $((y, x, y), z, y, y) = ((y, x, y), (z, y, y)) - (z, ((y, x, y), y, y)) = ((y, x, y), (z, y, y))$, because of (16). Thus $((y, x, y), (z, y, y)) = 0$. Permuting x and z in this last identity we obtain

$$(17) \quad ((y, z, y), (x, y, y)) = 0 .$$

From (8) it follows that $(z, (x, y, y)) = -(y, (x, y, z)) - (y, (x, z, y)) = ((x, y, z) + (x, z, y), y)$. Substituting $x = (x, y, z) + (x, z, y)$ in (3), it follows that $((x, y, z) + (x, z, y), y, y) = 0$. Thus $((z, (x, y, y), y, y)) = 0$. Substituting $a = z$, $b = (x, y, y)$ in (6), it follows that $((z, (x, y, y)), y, y) = (z, ((x, y, y), y, y)) - ((x, y, y), (z, y, y))$. Hence $0 = (z, ((x, y, y), y, y)) - ((x, y, y), (z, y, y))$, so that

$$(18) \quad (z, ((x, y, y), y, y)) = ((x, y, y), (z, y, y)) .$$

Substitute $a = c = y$, and $b = x$, in (12). Then $((y, x, y), y, y) = (y, (x, y, y), y)$. As a consequence of (16) it follows that

$$(19) \quad (y, (x, y, y), y) = 0 .$$

In (15) substitute $a = z$, $b = (x, y, y)$. Then $(z, (y, (x, y, y), y)) + ((x, y, y), (y, z, y)) = -2(z, ((x, y, y), y, y)) - 2((x, y, y), (z, y, y))$. Because of (17) and (19) the left side of the last identity is zero. Thus $-2(z, ((x, y, y), y, y)) - 2((x, y, y), (z, y, y)) = 0$. Using characteristic different from 2, this implies

$$(20) \quad (z, ((x, y, y), y, y)) + ((x, y, y), (z, y, y)) = 0 .$$

Comparing identities (18) and (20), and using characteristic different from 2 once more, we obtain

$$(21) \quad (z, ((x, y, y), y, y)) = 0 .$$

Substituting $x = xy$ in (21), we see that $(z, ((xy, y, y), y, y)) = 0$. But (11) implies that $(xy, y, y) = (x, y, y)y$, so that $(z, ((x, y, y)y, y, y)) = 0$. Substituting $x = (x, y, y)$ in (11), it follows that $((x, y, y)y, y, y) =$

$((x, y, y), y, y)y$, so that

$$(22) \quad (z, ((x, y, y), y, y)y) = 0 .$$

By expansion $(a, bc) + (b, ca) + (c, ab) = -(a, b, c) - (b, c, a) - (c, a, b) = 0$, as a consequence of (2). Thus we have established

$$(23) \quad (a, bc) + (b, ca) + (c, ab) = 0 .$$

Substituting $a = ((x, y, y), y, y)$, $b = y$, $c = z$, in (23), it follows that $((x, y, y), y, y, yz) + (y, z((x, y, y), y, y)) + (z, ((x, y, y), y, y)y) = 0$. Substituting $z = yz$ in (21) implies that the first term of the last identity must vanish. The third term vanishes because of (22). That leaves

$$(24) \quad (y, z((x, y, y), y, y)) = 0 .$$

We are now ready to prove the following.

LEMMA 1. *If R is a division ring then for every element y in R either $(y, R) = 0$, or $(R, y, y) = 0$.*

Proof. It follows directly from (24) that either $((R, y, y), y, y) = 0$, or $(y, R) = 0$. Suppose that $((R, y, y), y, y) = 0$. Then $((x^2, y, y), y, y) = 0$. Substituting $w = x$, $z = y$, in (1), it follows that $(x^2, y, y) = x(x, y, y) + (x, y, y)x$. Thus $0 = (x(x, y, y) + (x, y, y)x, y, y)$. Substituting $w = (x, y, y)$, and $z = y$ in (1), it follows that $((x, y, y)x, y, y) = (x, y, y)^2 + ((x, y, y), y, y)x = (x, y, y)^2$, as a consequence of $((R, y, y), y, y) = 0$. Also substituting $w = x$, $x = (x, y, y)$, $z = y$, in (1), it follows that $(x(x, y, y), y, y) = x((x, y, y), y, y) + (x, y, y)^2 = (x, y, y)^2$. Thus $0 = (x(x, y, y) + (x, y, y)x, y, y) = 2(x, y, y)^2$. Then characteristic different from 2 implies that $(x, y, y)^2 = 0$, and since R is a division ring, $(x, y, y) = 0$. This completes the proof of the lemma.

LEMMA 2. *If R is a division ring then for all elements x, y, z in R we have the identity $(z, (x, y, y)) = 0$.*

Proof. From (8) it follows that $(z, (x, y, y)) = -(y, (x, y, z)) - (y, (x, z, y))$. Then Lemma 1 suffices to show that either the left hand side or the right hand side vanishes for every fixed y and arbitrary x and z in R . This completes the proof of the lemma.

LEMMA 3. *If R is a division ring then the identity $(x, (x, y, y)z) = 0$, holds for all elements x, y, z in R .*

Proof. Substituting x^2 for x in Lemma 2, it follows that

$(z, (x^2, y, y)) = 0$. But substituting $w = x$, $z = y$, in (1), it follows that $(x^2, y, y) = x(x, y, y) + (x, y, y)x = 2x(x, y, y)$, as a result of substituting $z = x$ in Lemma 2. Thus $0 = (z, 2x(x, y, y))$, so that $(z, x(x, y, y)) = 0$. Substituting $a = z$, $b = x$, $c = (x, y, y)$ in (23), it follows that $(z, x(x, y, y)) + (x, (x, y, y)z) + ((x, y, y), zx) = 0$. We have just shown that the first term of the last equation is zero. Substituting zx for z in Lemma 2 implies that the third term is zero as well. Hence $(x, (x, y, y)z) = 0$. This completes the proof of the lemma.

LEMMA 4. *If R is a division ring then the identity $((x, z), y, y) = 0$, holds for all elements x, y, z in R .*

Proof. Substituting $a = x$, $b = z$ in (6), it follows that $((x, z), y, y) = (x, (z, y, y)) - (z, (x, y, y)) = 0$, as a consequence of Lemma 2. This completes the proof of the lemma.

THEOREM 1. *If R is a division ring then R is either right alternative or the identity $((a, b), c) = 0$, holds for all elements a, b, c in R .*

Proof. Linearizing the identity of Lemma 3, we obtain $(w, (x, y, y)z) = -(x, (w, y, y)z)$. Substituting $w = (a, b)$ in this, it is clear that $((a, b), (x, y, y)z) = -(x, ((a, b), y, y)z) = 0$, using Lemma 4 with $x = a$, $z = b$. If R is not right alternative, then for every c in R we can find a z such that $(x, y, y)z = c$. This completes the proof of the theorem.

We are now ready to prove the main result.

THEOREM 2. *If R is a division ring then R must be associative.*

Proof. If R is right alternative then it is known [3, 8] that R must be alternative and then (2) suffices to make R associative. Then as a result of Theorem 1 we are down to the case where the identity $((a, b), c) = 0$, holds for all elements a, b, c in R . In (4) substitute $w = x$, and $x = y$. Then $(xy, y, z) - (x, y^2, z) + (x, y, yz) = x(y, y, z) + (x, y, y)z$. Substituting $w = x$, and $x = y$, in (1) leads to $(xy, y, z) + (x, y, (y, z)) = x(y, y, z) + (x, y, z)y$. Subtract the left hand sides of the last two identities to obtain

$$(25) \quad (x, y^2, z) - (x, y, zy) = -(x, y, y)z + (x, y, z)y.$$

Substituting $w = x$, $x = z$, $z = y$ in (4), we find that

$$(26) \quad (xz, y, y) - (x, zy, y) + (x, z, y^2) = x(z, y, y) + (x, z, y)y.$$

Define K to be the set of all elements k in R such that $(k, R) = 0$. Clearly K is closed under subtraction. Also Lemma 2 implies that (x, y, y) belongs to K , for all elements x, y in R . Hence $(x, y, z) + (x, z, y)$ is also an element of K , for all x, y, z in R . Adding the left hand sides of (25) and (26), it is clear that we get an element of K . Thus

$$(27) \quad (x, y, z)y + (x, z, y)y + x(z, y, y) - (x, y, y)z \text{ belongs to } K.$$

Substituting $w = x$, $x = z$, $z = y$ in (1), it follows that $(xz, y, y) = x(z, y, y) + (x, y, y)z$, so that

$$(28) \quad x(z, y, y) + (x, y, y)z \text{ belongs to } K.$$

Subtracting (28) from (27), it is clear that

$$(29) \quad (x, y, z)y + (x, z, y)y - 2(x, y, y)z \text{ belongs to } K.$$

Because of (11) we have $(x, y, y)y = (xy, y, y)$, so that Lemma 2, with xy substituted for x , implies

$$(30) \quad (x, y, y)y \text{ belongs to } K.$$

Linearizing (30), we obtain

$$(31) \quad (x, y, z)y + (x, z, y)y + (x, y, y)z \text{ belongs to } K.$$

Comparing (29) and (31), it is clear that $3(x, y, y)z$ belongs to K , so that $(x, y, y)z$ must also. Thus $(w, (x, y, y)z) = 0$, for all w, x, y, z in R . If R is not right alternative, then clearly R must be commutative. But in that case it is certainly flexible and that was enough to imply that R is alternative and hence associative [6]. This is a contradiction, so R must have been right alternative and hence associative to begin with. This completes the proof of the theorem.

We conclude with a generalization of Theorem 2. Let S be a division ring of characteristic $\neq 2, 3$, which satisfies (1), (3) and a weaker form of identity (2), namely that (2) holds whenever x, y, z lie in a subring which can be generated by two elements. Now we are dealing with a set of identities which are valid in every alternative ring. From the conclusion of Theorem 2 it follows that every subring of S generated by two elements is associative, so that S must in fact be alternative. This yields a worthwhile generalization of the main result, with little additional effort.

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