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HYPERSPACES OF GRAPHS ARE HILBERT CUBES

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# HYPERSPACES OF GRAPHS ARE HILBERT CUBES

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The authors prove that  $2^{\Gamma}$  is a Hilbert cube where  $\Gamma$  is any nondegenerate, finite, connected graph and  $2^{\Gamma}$  is the space of nonvoid closed subsets of  $\Gamma$  metrized with the Hausdorff metric. This extends their result that  $2^{I}$  is a Hilbert cube. They also prove corresponding theorems for local dendrons Das well as for the space of subcontinua C(D) of D.

1. Introduction. In [9] the authors outlined their proof that  $2^{I}$ , the space of nonvoid, closed subsets of I = [0, 1] metrized with the Hausdorff metric, is a Hilbert cube Q and announced the main results concerning graphs in this paper. Here we give the complete proof, assuming that  $2^{I}$  is a Hilbert cube, that  $2^{\Gamma}$  is a Hilbert cube for any finite, connected graph  $\Gamma$ . We also prove that if D is any local dendron, then  $2^{D}$  is a Hilbert cube and prove some results about the space of subcontinua C(D) of a local dendron D that extend the results of [13].

In [10] the authors give a complete proof that  $2^{t}$  is a Hilbert cube. This settled a conjecture raised by Wojdyslawski [16] in 1938 where he also asked if  $2^{x}$  is a Hilbert cube for any nondegenerate Peano space X. The first author and D. W. Curtis have announced the proof of this latter conjecture in [5] as well as the theorem that says that C(X) is always a Q-factor for any Peano space X, and C(X) is a Hilbert cube iff X is a nondegenerate Peano space that contains no free arcs. These results are strongly dependent upon the results of this paper. The complete proofs of the  $2^{x}$  and C(X) results appear in [6].

This paper assumes the  $2^{i}$  result and not the techniques of the proof. The proofs given here use some of the fundamental results of infinite-dimensional topology, but if the reader takes these results, listed in §2, as axioms, then no previous knowledge of infinite-dimensional topology is necessary for understanding this paper.

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2. Definitions and infinite-dimensional topology background. If X is a compact metric space, then the Hausdorff metric D on  $2^x$  can be defined by

 $D(A, B) = \inf \{ \varepsilon > 0 : A \subset U(B, \varepsilon) \text{ and } B \subset U(A, \varepsilon) \}$ 

where  $U(C, \varepsilon)$  is the open  $\varepsilon$ -neighborhood of  $C \subset X$ . If V is a subset

of X, then  $2_{\nu}^{x}$  is the subspace of  $2^{x}$  consisting of all members of  $2^{x}$  that contain V, and likewise for  $C_{\nu}(X)$ .

Let Q denote the countable infinite product of I with itself and define a *Hilbert cube* as any space homeomorphic ( $\approx$ ) to Q. A space X is a Q-factor if  $X \times Q \approx Q$ . A Q-manifold is a separable metric space such that each point has an open neighborhood homeomorphic to an open subset of Q.

A map is a continuous function. If X and Y are homeomorphic compact metric spaces, then a map  $f: X \to Y$  is a near-homeomorphism if for each  $\varepsilon > 0$  there exists a homeomorphism  $h: X \to Y$  such that  $d(f, h) < \varepsilon$ . We say that  $f: X \to Y$  stabilizes to a near-homeomorphism if  $f \times id: X \times Q \to Y \times Q$  is a near-homeomorphism. By a graph we will mean a 1-dimensional polyhedron with a specific triangulation.

R. D. Anderson's notion of Z-set [1] is extensively used in this paper and is one of the fundamental concepts in infinite-dimensional topology. There have been various definitions of Z-sets in the literature [1], [2], [4], and [7]. The following is the most convenient formulation for this paper.

DEFINITION 2.1. A closed subset A of a Q-factor X is a Z-set in X if for each  $\varepsilon > 0$  there exists a map  $f: X \to X \setminus A$  such that  $d(f, id) < \varepsilon$ .

We list below two well-known properties of Z-sets, the proofs of which are very easy. All spaces below are Q-factors.

2.2. Z-set Properties.

- (a) If A is a Z-set in X, then  $A \times Y$  is a Z-set in  $X \times Y$ .
- (b) Any finite union of Z-sets is a Z-set.

One of the important theorems in infinite-dimensional topology is the following theorem of Anderson. See [11] and [14] for generalizations.

2.3. First Sum Theorem [1]. If A, B, and  $A \cap B$  are Hilbert cubes (Q-factors) and  $A \cap B$  is a Z-set in A and in B, then  $A \cup B$  is a Hilbert cube (Q-factor).

If X and Y are disjoint spaces, A a closed subset of X, and  $f: A \to Y$  a map, then the *adjunction space* of f, denoted  $X \bigcup_f Y$ , is  $(X \cup Y)/R$ , where R is the equivalence relation on  $X \cup Y$  generated by aRf(a) for each  $a \in A$ . We say X is *attached* to Y by f. If  $g: X \to Y$  is a map, then the mapping cylinder of g, denoted  $M_g$ , is the adjunction space  $(X \times I) \bigcup_{g'} Y$  where  $g': X \times \{0\} \to Y$  is defined by g'(x, 0) = g(x). The following is one of the basic theorems in the theory of Q-factors.

2.4. Mapping Cylinder Theorem [11] and [14]. Let X and Y be Q-factors and let  $g: X \to Y$  be a map of X into Y, then the mapping cylinder of g,  $M_g$ , is also a Q-factor. Furthermore, if  $c: M_g \to Y$  is the map defined by c([x, t]) = g(x), then c stabilizes to a near-homeomorphism.

An important corollary of this is the following.

2.5. The Attaching Theorem [10]. Let X and Y be Q-factors and let A be a closed subset of X that is a Z-set in X. If  $f: A \rightarrow$ Y is any map, then the adjunction space  $X \bigcup_f Y$  is also a Q-factor.

A relative homeomorphism  $f: (X, A) \to (Y, B)$  is a map of the pairs where  $f | X \setminus A: X \setminus A \to Y \setminus B$  is a homeomorphism. The next remark is just a convenient alternative way of viewing adjunction spaces and will not be proved. Let all spaces below be compact metric.

REMARK 2.6. If  $f: (X, A) \to (Y, B)$  is a relative homeomorphism, then Y is homeomorphic to the adjunction space  $X \bigcup_{g} B$  where g = f | A.

The main tool of this paper is the following theorem.

2.7. Compactification Theorem [13]. Let A be a closed subset of the space X where

- (1) X is a Q-factor,
- (2) A is a Q-factor,
- (3) A is a Z-set in X, and
- (4)  $X \setminus A$  is a Q-manifold.

Then X is a Hilbert cube.

The above theorem gives us conditions as to when the Q-manifold  $X \setminus A$  can be compactified to be a Hilbert cube. We list the parts of the hypothesis because in practice the verification of each part will often be a separate result. To prove that  $2^r$  is a Hilbert cube we will use the Compactification Theorem where  $X = 2^r$  and  $A = C_w(\Gamma)$  for some vertex  $w \in \Gamma$ . In §3 we will prove that  $2^r$  is a Q-factor and in §4 we will prove that  $2^r$  and  $C_w(\Gamma)$  satisfy the other three conditions. 3.  $2^r$  is a Q-factor. All of our results will be for the more general case  $2^r_{\nu}$  where V is any set of vertices (possibly empty) of a finite, connected graph. Note that if V is empty, then  $2^r_{\nu} = 2^r$ . We first prove two lemmas.

Let  $\Gamma$  be a finite, connected, acyclic graph and let V be any subset (possibly empty) of the vertices of  $\Gamma$ . Let w be a vertex of  $\Gamma$  which separates it and let  $\Gamma_1, \dots, \Gamma_n$  be the closure of the components of  $\Gamma \setminus \{w\}$ , denoting by  $V_i$  the set  $V \cap \Gamma_i$ ,  $i = 1, \dots, n$ . Suppose that  $w \notin V$  and let  $W = V \cup \{w\}$  and for each i, let  $W_i =$  $V_i \cup \{w\}$ . Let  $X_i = \bigcup_{j=1}^i (2^{\Gamma_j}_{W_j} \times \prod_{k\neq j, k=1}^i 2^{\Gamma_k}_{V_k})$ .

LEMMA 3.1.  $X_n$  is a Q-factor if the  $2_{V_i}^{\Gamma_j}$  and  $2_{W_i}^{\Gamma_j}$  are Q-factors.

**Proof.** For i < n,  $X_{i+1} = 2_{V_i+1}^{r_{i+1}} \times X_i \cup 2_{W_{i+1}}^{r_{i+1}} \times \prod_{j=1}^{i} 2_{V_j}^{r_j}$ , and  $2_{V_{i+1}}^{r_{i+1}} \times X_i \cap 2_{W_{i+1}}^{r_{i+1}} \times \prod_{j=1}^{i} 2_{V_j}^{r_j} = 2_{W_{i+1}}^{r_{i+1}} \times X_i$ . Since  $\Gamma$  is acyclic, w is a free vertex of each  $\Gamma_i$  and thus by a direct verification of the definition of a Z-set, each  $2_{W_i}^{r_i}$  is a Z-set in  $2_{V_i}^{r_i}$  and by 2.2(b),  $X_i$  is a Z-set in  $\prod_{j=1}^{i} 2_{V_j}^{r_j}$ . Thus, by 2.2(a),  $2_{W_{i+1}}^{r_{i+1}} \times X_i$  is a Z-set in  $2_{V_{i+1}}^{r_{i+1}} \times X_i$  and in  $2_{W_{i+1}}^{r_{i+1}} \times \prod_{j=1}^{i} 2_{V_j}^{r_j}$ . Note that a finite product of Q-factors is a Q-factor. Hence, by the First Sum Theorem,  $X_{i+1}$  is a Q-factor if  $X_i$  is one and since  $X_1 = 2_{W_1}^{r_1}$  is a Q-factor by hypothesis, then  $X_n$  is a Q-factor by induction and the proof is complete.

Let  $Y_n$  be the set of all members of  $2_V^{\Gamma}$  which meet each  $\Gamma_i$ .

LEMMA 3.2.  $Y_n$  is a Q-factor if  $2_W^{\Gamma}$  and the  $2_{W_j}^{\Gamma_j}$  and  $2_{W_j}^{\Gamma_j}$  are Q-factors.

*Proof.* If  $F: \prod_{i=1}^{n} 2_{V_{i}}^{\Gamma_{i}} \to 2_{V}^{\Gamma}$  is defined by  $F(A_{1}, \dots, A_{n}) = A_{1} \cup \dots \cup A_{n}$ , then  $F: (\prod_{i=1}^{n} 2_{V_{i}}^{\Gamma_{i}}, X_{n}) \to (Y_{n}, 2_{W}^{\Gamma})$  is a relative homeomorphism and hence  $Y_{n}$  is homeomorphic to the adjunction space  $\prod_{i=1}^{n} 2_{V_{i}}^{\Gamma_{i}} \bigcup_{f} 2_{W}^{\Gamma}$  where  $f = F \mid X_{n}$ . Since each of  $\prod_{i=1}^{n} 2_{V_{i}}^{\Gamma_{i}}$ ,  $X_{n}$ , and  $2_{W}^{\Gamma}$  is a Q-factor and since  $X_{n}$  is a Z-set in  $\prod_{i=1}^{n} 2_{V_{i}}^{\Gamma_{i}}$ , then  $Y_{n}$  is a Q-factor by the Attaching Theorem.

**PROPOSITION 3.3.** If  $\Gamma$  is a finite, connected, acyclic graph and V is any subset (possibly empty) of the vertices of  $\Gamma$ , then  $2_{v}^{\Gamma}$  is a Q-factor.

**Proof.** (By induction on the number of edges in  $\Gamma$ .) If  $\Gamma$  is degenerate (no edges), this is clear, and if  $\Gamma$  has only one edge, this is shown in [10]. Now suppose that  $\Gamma$  has more than one edge and that the proposition is true for graphs with fewer edges than  $\Gamma$ . Adopt the notation of this section but allow w to belong to V. If

 $w \in V$ , then the mapping  $\prod_{i=1}^{n} 2_{V_i}^{\Gamma_i} \to 2_{V}^{\Gamma}$  given by  $(A_1, \dots, A_n) \to A_1 \cup \dots \cup A_n$  is a homeomorphism and since each of the  $2_{V_i}^{\Gamma_i}$  is a *Q*-factor by the inductive hypothesis,  $2_{V}^{\Gamma}$  is also a *Q*-factor.

If  $w \notin V$ , then by the above we have that  $2_w^r$  is a Q-factor and hence by Lemma 3.2,  $Y_n$  is a Q-factor. For  $k = 1, \dots, n-1$ , let  $Y_k$  be the subset of  $2_v^r$  composed of those members which meet at least k of the  $\Gamma_i$ 's. If  $Y_{k+1} \neq 2_v^r$ , let  $\sigma_1, \dots, \sigma_p$  be the subsets of  $\{1, \dots, n\}$  with exactly k members which contain  $\{i: 1 \leq i \leq n, V_i \neq \emptyset\}$ , and let

$$X_{\sigma_j} = igcup_{i \in \sigma_j} (2^{r_i}_{{}^Wi} imes \prod_{m \in \sigma_j \setminus \{i\}} 2^{r_m}_{{}^Vm}) \;.$$

Then exactly as in the proof of Lemma 3.1, each  $X_{\sigma_j}$  is a Q-factor and a Z-set in  $\prod_{i \in \sigma_j} 2_{Y_i}^{\Gamma_i}$ . For  $i = 1, \dots, p$ , let  $Y_{k,i}$  be the subset of  $2_r^{\Gamma}$ , composed of those members that are contained in  $\bigcup_{j \in \sigma_i} \Gamma_j$  and which meet each  $\Gamma_i$ ,  $j \in \sigma_i$ ; let  $Y_k^j = (\bigcup_{j=1}^i Y_{k,j}) \cup Y_{k+1}$ , and let  $Y_k^0$ denote  $Y_{k+1}$ . Then  $Y_k = Y_k^p$  and  $f_{k,i}: (\prod_{j \in \sigma_i} 2_{Y_j}^{\Gamma_j}, X_{\sigma_i}) \to (Y_k^i, Y_k^{i-1})$  defined by  $f_{k,i}(A_1, \dots, A_k) = A_1 \cup \dots \cup A_k$  is a relative homeomorphism and hence  $Y_k^i \approx \prod_{j \in \sigma_i} 2_{Y_j}^{\Gamma_j} \bigcup_{g} Y_k^{i-1}$ , where  $g = f_{k,i} | X_{\sigma_i}$ . Thus, by induction we have that  $Y_k = Y_k^p$  is a Q-factor if  $Y_{k+1} = Y_k^0$  is one. Thus, since  $Y_k$  is a Q-factor we have by induction that  $Y_1 = 2_r^{\Gamma}$  is a Q-factor.

THEOREM 3.4. If  $\Gamma$  is a finite, connected graph and V is any subset (possibly empty) of the vertices of  $\Gamma$ , then  $2_{V}^{\Gamma}$  is a Q-factor.

*Proof.* As this is a topological result, new vertices may be introduced in  $\Gamma$  at will and therefore, one may assume without loss of generality that for some connected, acyclic graph  $\Gamma_0$  and some collection  $v_1, w_1, \dots, v_n, w_n$  of free vertices of  $\Gamma_0$ , that  $\Gamma = \Gamma_0/R$  where R is the equivalence relation on  $\Gamma_0$  generated by  $v_i R w_i$  for  $i = 1, \dots, n$ . For  $1 \leq k \leq n$ , let  $R_k$  be the equivalence relation on  $\Gamma_0$  generated by  $v_i R w_i$  for  $i = 1, \dots, k$ , and let  $\Gamma_k = \Gamma_0/R_k$ . Since  $R_{k-1} \subset R_k$ , we have a natural map  $\varphi_k \colon \Gamma_{k-1} \to \Gamma_k$  induced by the identity map on  $\Gamma_0$ .

The theorem is true for  $\Gamma_0$  by Proposition 3.3. Suppose the theorem is true for  $\Gamma_{k-1}$ , let X be any subset of the vertices of  $\Gamma_k$  and let  $X' = \varphi_k^{-1}(X)$ . Let  $f_k: 2_{X'}^{\Gamma_k-1} \rightarrow 2_X^{\Gamma_k}$  be the map induced by  $\varphi_k$  and observe that  $f_k$  carries  $2_{X'\cup\{v_k,w_k\}}^{\Gamma_k}$  homeomorphically onto  $2_{X\cup\{v_k,w_k\}}^{\Gamma_k}$ ). Thus, if  $\varphi_k(\{v_k,w_k\}) \in X$ , then  $2_X^{\Gamma_k}$  is a Q-factor. If  $\varphi_k(\{v_k,w_k\}) \notin X$ , let  $Y_1 = X' \cup \{v_k\}, Y_2 = X' \cup \{w_k\}$ , and  $Y_3 = X' \cup \{v_k,w_k\}$ . Then  $2_{X'}^{\Gamma_k-1}, 2_{Y_1}^{\Gamma_k-1}, i = 1, 2, 3$ , and  $2_{\varphi_k}^{\Gamma_k}(v_3)$  are Q-factors and  $2_{Y_3}^{\Gamma_k-1} = 2_{Y_1}^{\Gamma_k-1} \cap 2_{Y_2}^{\Gamma_k-1}$ . Moreover, since  $v_k$  and  $w_k$  are free vertices,  $2_{Y_3}^{\Gamma_k-1}$  is a Z-set in each of them and thus by the First Sum Theorem  $2_{Y_1}^{\Gamma_k-1} \cup 2_{Y_2}^{\Gamma_k-1}$  is a Q-factor. Also, since each of  $2_{Y_k}^{\Gamma_k-1}$ , i = 1, 2, is a Z-set in  $2_{X'}^{\Gamma_k-1}$ , their

union is also a Z-set by 2.2(b). Moreover,  $f_k: (2_{X'}^{\Gamma_{k-1}}, 2_{Y_1}^{\Gamma_{k-1}} \cup 2_{Y_2}^{\Gamma_{k-1}}) \rightarrow (2_X^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_{k-1}} \bigcup_{g_k} 2_{\varphi_k}^{\Gamma_{k-1}}) \rightarrow (2_X^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}, 2_{\varphi_k}^{\Gamma_k}))$  where  $g_k = f_k | 2_{Y_1}^{\Gamma_{k-1}} \cup 2_{Y_2}^{\Gamma_{k-1}}$ , and thus by the Attaching Theorem  $2_X^{\Gamma_k}$  is a Q-factor and the theorem follows.

4.  $2^r$  is a Hilbert cube. In this section we verify the last three conditions of the Compactification Theorem.

LEMMA 4.1. If  $\Gamma$  is a finite, connected graph and V is any set of vertices (possibly empty) of  $\Gamma$ , then  $C_v(\Gamma)$  is a Q-factor.

*Proof.* First we show that  $C_{\nu}(\Gamma)$  is contractible. Let  $\Gamma$  be endowed with a convex metric, i.e., one for which there always exists a point half way between any two given points. Then the function  $F: C_{\nu}(\Gamma) \times I \to C_{\nu}(\Gamma)$  defined by F(A, t) is equal to the closed  $t\delta$ -neighborhood of A in  $\Gamma$ , where  $\delta$  is the diameter of  $\Gamma$ , is a contraction of  $C_{\nu}(\Gamma)$  to the point  $\Gamma \in C_{\nu}(\Gamma)$ .

Next, in [8], R. Duda proves that  $C(\Gamma)$  is a polyhedron and since it is contractible we have by [11] that  $C(\Gamma)$  is a Q-factor. If  $V \neq \emptyset$ , then  $C_r(\Gamma)$  is geometrically easier to classify than  $C(\Gamma)$  and although it was not specifically dealt with in [8], it is a subpolyhedron of  $C(\Gamma)$ , and since it is contractible, it is a Q-factor. For a considerably more general result see [6].

LEMMA 4.2. If  $\Gamma$  is a finite, connected, nondegenerate graph, w is a vertex of  $\Gamma$ , and V is a collection (possibly empty) of vertices of  $\Gamma$ , then  $C_{V \cup \{w\}}(\Gamma)$  is a Z-set in  $2_{V}^{\Gamma}$ .

*Proof.* We will first prove the result for the case that  $w \in V$  by constructing for each  $\varepsilon > 0$  a map  $f: 2_{V}^{\Gamma} \rightarrow 2_{V}^{\Gamma} \backslash C_{V}(\Gamma)$  that is within  $\varepsilon$  of the identity. Let  $w_{i}$ ,  $i = 1, \dots, n$ , be the vertices of  $\Gamma$  which are joined to w by edges  $E_{i} = [w, w_{i}]$  and assume, for the metric on  $\Gamma$ , that each  $E_{i}$  is isometric with [0, 1] so that for each  $0 < \varepsilon \leq 1$  the open  $\varepsilon$ -ball about w,  $U(w, \varepsilon)$ , is precisely the set  $\{(1 - t)w + tw_{i}: 0 \leq t < \varepsilon, i = 1, \dots, n\}$ . Let  $V(w, \varepsilon)$  be the closure in  $\Gamma$  of  $U(w, \varepsilon)$  and let  $\operatorname{Bd} U(w, \varepsilon) = V(w, \varepsilon) \backslash U(w, \varepsilon)$ . For a fixed  $0 < \varepsilon < 1$ , and for  $A \in 2_{V}^{\Gamma}$ , let

 $f(A) = [A \setminus U(w, \varepsilon/2)] \cup \{w\} \cup \operatorname{Bd} U(w, \varepsilon/2)$ .

It is clear that  $[A \setminus U(w, \varepsilon/2)] \cup \{w\} \in 2_{\nu}^{r} \setminus C_{\nu}(\Gamma)$  but this assignment of A would not be continuous basically for the reason that one may have two points  $x \in U(w, \varepsilon/2)$  and  $y \notin U(w, \varepsilon/2)$  that are very close together. Including the set Bd  $U(w, \varepsilon/2)$  in the image under f of A establishes the continuity of f, which is within  $\varepsilon$  of the identity map because in  $2_v^r$  the distance between  $\{w\}$  and Bd  $U(w, \varepsilon/2)$  is  $\varepsilon/2 < \varepsilon$ . Thus, since f is continuous and the image of f misses  $C_v(\Gamma)$ ,  $C_v(\Gamma)$ is a Z-set in  $2_v^r$ .

We will now modify these techniques to prove the theorem in the case  $w \notin V$ : Let  $W = V \cup \{w\}$ . If the above map f were defined on  $2_{V}^{\Gamma}$  it would not be within  $\varepsilon$  of the identity, as is seen by comparing f(A) and A for sets A with no points close to w. Since our main technique of mapping  $2_{V}^{\Gamma}$  off  $C_{W}(\Gamma)$  is to delete an open set about w, we will phase out this process so that we will be deleting open sets about w only from those members of  $2_{V}^{\Gamma}$  that contain points close to w.

For  $0 \leq a \leq 1$  we denote the point  $(1 - a)w + aw_i \in [w, w_i]$  simply by  $[a]_i$ . For  $A \in 2_V^r$ , let  $a_i \in [0, 1]$  be the number such that  $[a_i]_i$  is the point of  $A \cap E_i$  nearest to w, if  $A \cap E_i \neq \emptyset$ . If  $0 \leq a_i \leq \varepsilon$ , let  $a'_i = \max\{0, 2a_i - \varepsilon\}$  observing that if  $0 \leq a_i \leq \varepsilon/2$ , then  $a'_i = 0$ ; and if  $a_i = \varepsilon$ , then  $a'_i = a_i$ . For  $A \in 2_V^r$ , let

$$f(A) = egin{cases} A \cup \{ [a_i']_i \colon 1 \leq i \leq n, \, 0 \leq a_i \leq arepsilon \} \ A \cup \{ [(2\delta/arepsilon)a_i' + (1-2\delta/arepsilon)a_i]_i \colon 1 \leq i \leq n, \, 0 \leq a_i \leq arepsilon \} \ ext{ if } 0 \leq \delta \leq arepsilon/2 \end{cases}$$

where  $\delta = \delta(A) = D(A, 2_w^r)$ , which in this case is the minimum distance between points of A and w. Then f is a well-defined function since it is uniquely defined for elements  $A \in 2_v^r$ , where  $\delta = \varepsilon/2$ . Also, f is phased back to the identity at  $\delta = 0$ , that is, if  $\delta(A) = 0$ , then f(A) = A; and this establishes the continuity of f. Also observe that if  $\delta(A) = \varepsilon/2$ , then  $w \in f(A)$  and if  $\delta(A) \ge \varepsilon$ , then f(A) = A. Let  $\alpha(A) = \max\{0, \varepsilon/2 - \delta(A)\}$  and define g on  $f(2_v^r)$  by

$$gf(A) = egin{cases} [f(A) ackslash U(w, \, lpha(A))] \cup \operatorname{Bd} U(w, \, lpha(A)) & ext{if } \delta(A) < arepsilon/2 \ f(A) & ext{if } \delta(A) \geqq arepsilon/2 \end{cases}$$

The continuity of g follows since  $\alpha$  is continuous and since for  $A \in 2_{\nu}^{\Gamma}$  where  $\delta(A)$  is less than  $\varepsilon/2$  but close to  $\varepsilon/2$ , then Bd  $U(w, \alpha(A))$  is close to  $\{w\}$ , and hence gf(A) is close to f(A). Furthermore, the composition  $gf\colon 2_{\nu}^{\Gamma}\to 2_{\nu}^{\Gamma}$  is within  $\varepsilon$  of the identity and  $gf(2_{\nu}^{\Gamma})\cap C_{W}(\Gamma) = \emptyset$  and thus,  $C_{W}(\Gamma)$  is a Z-set in  $2_{\nu}^{\Gamma}$ .

The next lemma will be the inductive step for the main theorem of this section. Let  $L_1, \dots, L_m$  be a finite collection of finite, connected graphs, let W be a collection of vertices from  $\bigcup_{i=1}^m L_i$ where W contains at least one vertex of each  $L_i$ , and let  $K = (\bigcup_{i=1}^m L_i)/W$  be the quotient space obtained by taking the disjoint union of the  $L_i$  and identifying all the vertices in W. Let  $p: \bigcup_{i=1}^m L_i \to K$  be the quotient map and let w = p(W). LEMMA 4.3. If each  $2_{V_i}^{L_i}$  is a Hilbert cube for each collection  $V_i$ (possibly empty) of vertices of  $L_i$ , then  $2_V^{\kappa}$  is a Hilbert cube for each set of vertices V (possibly empty) of K.

*Proof.* To apply the Compactification Theorem, we have that  $2_{V}^{\kappa}$  is a Q-factor by 3.4,  $C_{W}(K)$  is a Q-factor by 4.1 where  $W = V \cup \{w\}$ , and  $C_{W}(K)$  is a Z-set in  $2_{V}^{\kappa}$ , by 4.2. It remains to be shown that  $2_{V}^{\kappa} \setminus C_{W}(K)$  is a Q-manifold.

If  $A \in 2_{V}^{\kappa} \setminus C_{W}(K)$ , then A has a component missing w. If A is connected, then it has an open neighborhood U in  $2_{V}^{\kappa}$  homeomorphic to an open set of  $2_{V_i}^{L_i}$ , for some i and some collection  $V_i$  of vertices of  $L_i$ . Since  $2_{V_i}^{L_i}$  is by hypothesis a Hilbert cube, U is homeomorphic to an open subset of the Hilbert cube. If A is not connected, then it has a separation into two disjoint closed nonempty subsets  $A_1$  and  $A_2$  such that  $A = A_1 \cup A_2$ . Assuming that  $w \notin A_2$ , let  $U_1$  and  $U_2$  be disjoint open sets of K containing  $A_1$  and  $A_2$ , respectively. Now, for some  $i_1, \dots, i_k$ ,  $1 \leq k \leq m$ ,  $A_2$  has an open neighborhood  $W_2$  in  $2_{A_2\cap V}^{\kappa}$  consisting of sets lying entirely within  $U_2$ , which is homeomorphic to a product  $U_{21} imes U_{22} imes \cdots imes U_{2k}$  of open sets of the Hilbert cubes  $2_{V_j}^{L_j}$ ,  $j = i_1, \dots, i_k$  where  $V_j = L_j \cap p^{-1}(A_2 \cap V)$ . On the other hand, the set  $W_1 = \{B \in 2_{V'}^{\kappa}: B \subset U_1\}$ , where  $V' = V \cap A_1$ , is an open neighborhood of  $A_1$  in  $2^{\kappa}_{V'}$  which is by 3.4 a Q-factor. Now  $U = \{B \cup C : B \in W_1, C \in W_2\}$  is an open neighborhood of A in  $2_V^K$  which is homeomorphic to  $W_{\scriptscriptstyle 1} imes W_{\scriptscriptstyle 2}$  and hence, to an open subset of the Hilbert cube  $2_{V'}^{\kappa} \times \Pi\{2_{V'}^{L_j}: j = i_1, \dots, i_k\}$ . Therefore,  $2_V^{\kappa} \setminus C_W(K)$  is a Qmanifold and the proof is complete.

THEOREM 4.4. If  $\Gamma$  is a nondegenerate, finite, connected graph and V is any set (possibly empty) of vertices of  $\Gamma$ , then  $2_v^{\Gamma}$  is a Hilbert cube.

*Proof.* Let  $\mathscr{G}$  be the class of all nondegenerate, finite, connected graphs. For each  $K \in \mathscr{G}$ , let V(K) be the number of vertices of K, E(K) the number of edges of K, and R(K) = E(K) - V(K) + 1. (R(K)) is the rank of the first homology group  $H_1(K)$ ; it is also E(K) - E(L) for each maximal acyclic subgraph L of K.) Let  $\mathscr{G}_i$  be the class of all members K of  $\mathscr{G}$  for which R(K) = i, and let  $\mathscr{G}_{ij}$  be the subclass of  $\mathscr{G}_i$  composed of all members K of  $\mathscr{G}_i$  with E(K) = j.

The theorem holds for  $\mathscr{G}_{01}$ , being the main results of [9] and [10]. Specifically,  $2^{i}$ ,  $2^{i}_{0}$ ,  $2^{i}_{1}$ , and  $2^{i}_{01}$  are all Hilbert cubes. Now fix  $(i, j) \neq (0, 1)$  and suppose that the theorem holds for each  $\mathscr{G}_{i'j'}$  with i' < i or i' = i and j' < j.

Let  $K \in \mathscr{G}_{ij}$  and let V be a set of vertices (possibly empty) of K and let w be a vertex of K which is not a free vertex of K. Construct a new complex K' by "splitting" K at w. That is, let  $v_1, \dots, v_n$  be the vertices of K which are joined to w by edges  $[w, v_i]$  of K and let  $w_1, \dots, w_n$  be abstract vertices not in K. Then  $K' = (K \setminus \bigcup_{i=1}^n [w, v_i)) \cup \bigcup_{i=1}^n [w_i, v_i]$  and K' has as vertices all vertices of K except w together with  $w_1, \dots, w_n$  and has as edges all edges of K which do not contain w together with the new edges  $[w_i, v_i]$ ,  $i = 1, \dots, n$ . Now, if w separates K, each component L of K' has E(L) < E(K) and  $R(L) \leq R(K)$ , while if w does not separate K, then  $K' \in \mathscr{G}$  and R(K') < R(K). Thus, by the induction hypothesis, each component of K' satisfies the theorem and hence by Lemma 4.3,  $2_F^{K}$  is a Hilbert cube and thus by induction the theorem is proved.

5.  $2^{D}$  and C(D) for local dendrons D. In this section we generalize the theorems to each *dendron*, that is, a Peano space which contains no simple closed curve, and to each *local dendron*, that is, a Peano space such that each point has a closed neighborhood which is a dendron. In particular, each dendron is a local dendron. We can express (see [13]) each dendron D as the limit of an inverse sequence  $(T_n, r_n)$ ,  $\lim (T_n, r_n)$ , where  $T_1$  is an arc and for each  $n \ge 1$ ,  $T_{n+1}$  is the union of  $T_n$  and an arc  $[a_n, b_n]$  where  $T_n \cap [a_n, b_n] = \{a_n\}$ , and where  $r_n: T_{n+1} \to T_n$  is the retraction which collapses  $[a_n, b_n]$  to  $a_n$ . The inverse sequence  $(T_n, r_n)$  induces the inverse sequence  $(2^{T_n}, r_n^*)$  where  $r_n^*: 2^{T_{n+1}} \to 2^{T_n}$  is defined by  $r_n^*(A) = r_n(A)$ . Then  $2^{D}$  is homeomorphic to  $\lim (2^{T_n}, r_n^*)$ .

The corresponding inverse limit representation for local dendrons is the same except that  $T_1$  is allowed to be a finite, connected graph. We argue this as follows. For a local dendron D there exists an  $\varepsilon > 0$  such that each closed connected subset of D with diameter less than  $\varepsilon$  is a dendron. Cover D with a finite collection of closed connected neighborhoods  $\{D_i\}$  with diameter less than  $\varepsilon/2$ . The pairwise intersections of the  $D_i$  are connected. In each nonempty intersection of elements of the  $\{D_i\}$  pick a point and then in each  $D_i$  construct a tree connecting each of the selected points contained in that  $D_{i}$ . Then the union of these trees will be a finite connected graph, a candidate for  $T_1$  in the above inverse limit presentation. Now we can add the remaining stickers to the trees in the prescribed manner to obtain the local dendron D as the  $\lim (T_n, r_n)$ . Such an inverse limit for a local dendron D will be called a *standard* inverse limit representation for D. Also, for a given finite subset V of Dwe can easily construct  $T_1$  to contain V by including it in the set of points picked in the intersections of the  $D_i$ . We will need the next result.

THEOREM 5.1. Morton Brown [3]. Let  $S = \lim (X_n, f_n)$ , where the  $X_n$  are all homeomorphic to a given compact metric space X and each  $f_n$  is a near-homeomorphism. Then S is homeomorphic to X.

LEMMA 5.2. If  $f: Q \rightarrow Q$  is a map that stabilizes to a near-homeomorphism, then f is a near-homeomorphism.

**Proof.** Define  $\alpha_n: Q \times Q \to Q$  by  $\alpha_n((x_1, x_2, \dots), (y_1, y_2, \dots)) = (x_1, \dots, x_n, y_1, x_{n+1}, y_2, x_{n+2}, y_3, \dots)$ . Then each  $\alpha_n$  is a homeomorphism and hence each  $\alpha_n \circ (f \times id) \circ \alpha_n^{-1}$  is a near-homeomorphism since  $f \times id$  is one by assumption. Furthermore,  $d(f, \alpha_n \circ (f \times id) \circ \alpha_n^{-1}) \to 0$  as  $n \to \infty$  and hence f is a uniform limit of near-homeomorphisms and thus is a near-homeomorphism.

THEOREM 5.3. If D is a nondegenerate local dendron and V is any finite subset (possibly empty) of D, then  $2_v^p$  is a Hilbert cube.

*Proof.* We follow the proof of [Theorem 2, 13] which states a corresponding result for C(D). Choose a standard inverse limit representation for D where  $V \subset T_1$ . Let  $r'_n: 2^{T_{n+1}}_{V \cup \{b_n\}} \to 2^{T_n}_{V}$  be the restriction of the map  $r_n^*$ , let  $M_{r'_n}$  be the mapping cylinder of  $r'_n$ , and let  $c_n: M_{r'_n} \to 2_V^{T_n}$  be the natural projection defined by  $c_n([A, t]) = r'_n(A)$ . Since  $2_{V \cup \{b_n\}}^{T_{n+1}}$  and  $2_{V^n}^{T_n}$  are Q-factors by 3.4, it follows by the Mapping Cylinder Theorem that  $c_n$  stabilizes to a near-homeomorphism. We will show below that  $M_{r'_{x}}$  is homeomorphic to  $2^{T_{n+1}}_{V^{n+1}}$  in such a way that the projection map  $c_n$  is topologically equivalent to  $r_n^*$ . Thus, since each of  $2_{V^n}^{T}$  and  $2_{V^{n+1}}^{T}$  is a Hilbert cube, we have by 5.2 that  $c_n$  is a near-homeomorphism and hence so is  $r_n^*$ . The proof that  $2_{V}^{p} \approx Q$  will then be complete by 5.1 since  $2_{V}^{p}$  is homeomorphic to an inverse limit of Hilbert cubes  $2_{V^n}^{V^n}$  where the bonding maps are nearhomeomorphisms. We now verify the above stated fact about  $M_{r'_{x}}$ . Define  $g_n: 2_V^{T_{n+1}} \to M_{r'_n}$  as follows where we parametrize  $[a_n, b_n]$  to be order isomorphic with [0, 1] and let sup  $(A \cap [a_n, b_n]) = d$  if it exists. Let

$$g_n(A) = egin{cases} [A], & ext{if } A \cap (a_n, b_n] = arnothing \ [(A \cap T_n) \cup (1/d(A \cap [a_n, b_n]), d)], & ext{if } A \cap (a_n, b_n] \neq arnothing \ . \end{cases}$$

Then  $g_n$  is a homeomorphism so that the following diagram is



commutative and this completes the proof.

In [13], it is proved that the subcontinua C(D) of a dendron D form a Q-factor which is a Hilbert cube if and only if the branch points of D are dense. We will extend this result to local dendrons D and to spaces  $C_{\nu}(D)$  where V is a finite subset of D.

LEMMA 5.4. For each local dendron D and each finite subset V (possibly empty) of D,  $C_v(D)$  is a Q-factor.

*Proof.* Choose a standard inverse limit representation,  $\lim (T_n, r_n)$ , for D where  $V \subset T_1$ . Then  $C_V(D) \approx \lim (C_V(T_n), r_n^*)$ . As in the proof of Theorem 5.3 the space  $C_V(T_{n+1})$  is naturally homeomorphic to the mapping cylinder  $M_{r'_n}$  where  $r'_n: C_{V \cup \{b_n\}}(T_{n+1}) \to C_V(T_n)$  is the restriction of  $r_n^*$ . Furthermore, the map  $r_n^*$  is topologically equivalent to the natural projection  $c_n: M_{r'_n} \to C_V(T_n)$  which stabilizes to a nearhomeomorphism. Since each space  $C_V(T_n)$  is a Q-factor by Lemma 4.1 and since each bounding map  $r_n^*$  stabilizes to a near-homeomorphism, then  $C_V(D) \approx \lim (C_V(T_n), r_n^*)$  is a Q-factor and the proof is complete.

To prove that  $C_{\nu}(D)$  is a Hilbert cube if the branch points of Dare dense, we will need Lemmas 4.1 and 5.4 together with the next two lemmas to satisfy the hypothesis of the Compactification Theorem where  $X = C_{\nu}(D)$  and  $A = C_{\nu}(T_1)$ .

LEMMA 5.5. Let D be a local dendron with a dense set of branch points, let V be a finite subset (possibly empty) of D, and let  $\lim (T_n, r_n)$  be a standard inverse limit representation for D where  $V \subset T_1$ . Then  $C_V(T_1)$  is a Z-set in  $C_V(D)$ .

*Proof.* A local dendron admits a convex metric. Using a convex metric on D, for sufficiently small  $\varepsilon > 0$ , the map f on  $C_{\nu}(D)$  defined by setting f(A) equal to the closed  $\varepsilon$ -neighborhood of A in D is a map from  $C_{\nu}(D)$  into itself where  $d(f, id) < \varepsilon$ . Since the branch points of D are dense, we also have that  $f: C_{\nu}(D) \to C_{\nu}(D) \setminus C_{\nu}(T_1)$  and hence  $C_{\nu}(T_1)$  is a Z-set in  $C_{\nu}(D)$ .

LEMMA 5.6. If D, V, and  $\lim (T_n, r_n)$  are as above, then  $C_v(D) \setminus C_v(T_1)$  is a Q-manifold.

*Proof.* Let  $A \in C_v(D) \setminus C_v(T_1)$ . It is sufficient, since  $C_v(D) \setminus C_v(T_1)$  is open in  $C_v(D)$ , to show that A has an open neighborhood in  $C_v(D)$  that is homeomorphic to an open subset of the Hilbert cube. If  $A \cap T_1$  is either empty or a single point, then V is either empty or is a single point and there exists an open set U in D containing A and a dendron  $D_1$  such that  $A \subset U \subset D_1 \subset D$ . If W is the set of all

elements of  $C_v(D)$  contained in U, then W is an open neighborhood of A in  $C_v(D)$  and is an open subset of  $C_v(D_1)$  which is a Hilbert cube by an obvious modification of West's proof [13] that  $C(D_1)$  is a Hilbert cube.

If  $A \cap T_1$  is nondegenerate, let E be the closure of some component of  $D \setminus T_1$  that contains some points of A and let F be the closure of  $D \setminus E$ . Then E is a dendron and F is a local dendron containing  $T_1$  and each has a dense set of branch points and  $E \cap F$  is one point, say q. Then  $C_q(E)$  is a Hilbert cube by modifying West's argument and  $C_W(F)$ , where  $W = V \cup \{q\}$ , is a Q-factor by Lemma 5.4 and hence  $C_q(E) \times C_W(F)$  is a Hilbert cube. The map  $\alpha: C_q(E) \times$  $C_W(F) \to C_V(D)$  defined by  $\alpha(A, B) = A \cup B$  is an embedding into  $C_V(D)$ where the image of  $\alpha$  is a closed neighborhood (not a small one) of A and thus  $C_V(D) \setminus C_V(T_1)$  is a Q-manifold.

THEOREM 5.7. If D is a local dendron and V is a finite subset (possibly empty) of D, then  $C_{\nu}(D)$  is a Q-factor, and furthermore if the branch points of D are dense, then  $C_{\nu}(D)$  is a Hilbert cube.

*Proof.* The first part of the theorem is Lemma 5.4 and the second part follows from applying Lemmas 4.1 and 5.4-5.6 to the Compactification Theorem and observing that D admits a standard inverse limit representation  $\lim (T_n, r_n)$  where  $V \subset T_1$ .

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