

# Pacific Journal of Mathematics

**A FACTORIZATION THEOREM FOR  $p$ -CONSTRAINED  
GROUPS**

WILLIAM H. SPECHT

## A FACTORIZATION THEOREM FOR $p$ -CONSTRAINED GROUPS

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**Suppose that  $G$  is a finite  $p$ -constrained group. For some prime  $p \geq 5$  let  $S$  be a Sylow  $p$ -subgroup. Assume that  $G$  admits a group of automorphisms  $A$  such that  $(|A|, |G|) = 1$  and the fixed point subgroup of  $A$  does not involve  $\text{PSL}(2, p)$ . In this paper it is shown that under these conditions**

$$G = O_{p'}(G)N(Z(J(S))) .$$

Thompson proved in [8] that if  $G$  is a strongly  $p$ -solvable group and  $O_p(G) = 1$ , then  $G = N(J(S))C(Z(S))$ , where  $S$  is a Sylow  $p$ -subgroup of  $G$ . Since his paper several other stronger results of this type have been proved by Glauberman [1], [2]. Specifically he proved his  $ZJ$ -theorem which states that if  $G$  is  $p$ -constrained and  $p$ -stable then  $G = O_{p'}(G)N(Z(J(S)))$ . This implies Thompson's conclusion by the Frattini argument. Recently Glauberman has proved that

$$G = N(J(S))C(Z(S))$$

for all  $p$ , provided that  $G$  is  $p$ -solvable and admits a group of automorphisms  $A$  such that  $(|G|, |A|) = 1$  and  $A$  has no fixed points of order  $p$ .

In this paper our goal is a theorem related to these results.

**THEOREM A.** *Let  $G$  be a  $p$ -constrained group with  $p \geq 5$  and  $S$  a Sylow  $p$ -subgroup of  $G$ . Suppose that  $G$  admits a group of automorphisms  $A$  such that  $(|A|, |G|) = 1$  and the fixed point subgroup of  $A$  does not involve  $\text{PSL}(2, p)$ . Then  $G = O_{p'}(G)N(Z(J(S)))$ .*

Using Glauberman's  $ZJ$ -theorem, Theorem A is a corollary of Theorem B.

**THEOREM B.** *Let  $G$  be a  $p$ -constrained group with  $p \geq 5$ . Suppose that  $A$  is a group of automorphisms of  $G$  such that  $(|G|, |A|) = 1$  and that the fixed point subgroup of  $A$  does not involve  $\text{PSL}(2, p)$ . Then  $G$  is  $p$ -stable.*

All the groups in this paper are finite. The notation, except for the definition of  $p$ -stability, is standard and can be found in [3]. If  $P$  is a  $p$ -group  $J(P) = \langle A \cong P \mid A \text{ is abelian and of maximal order} \rangle$ . For simplicity we will write  $Z(J(P)) = ZJ(P)$ . If  $K$  is a group, we

say  $G$  involves  $K$  if a section of  $G$  is isomorphic to  $K$ .

### 1. Assumed results and definitions.

DEFINITION 1.1. Let  $G$  be a group with  $O_p(G) \cong 1$ . Let  $S$  be a Sylow  $p$ -subgroup of  $G$  and set  $P = S \cap O_{p',p}(G)$ .  $G$  is  $p$ -constrained if  $C_G(P) \cong O_{p',p}(G)$ .

DEFINITION 1.2. Let  $G$  be a group and suppose that  $S$  is a Sylow  $p$ -subgroup.  $G$  is  $p$ -stable if for any  $R \cong S$  such that  $RO_p(G) \leq G$  and for any  $A \cong N_S(R)$  with the property that  $[R, A, A] = 1$ , we have

$$AC(R)/C(R) \cong O_p(N(R)/C(R)).$$

This definition of  $p$ -stability is taken from Gorenstein-Walter [4]. It is weaker than the definition given in Gorenstein [3]. However this definition is sufficient for Glauberman's  $ZJ$ -theorem as a check of the proof [3] will indicate.

The principal tools of this paper are two theorems of Thompson. We state these for want of an available reference.

DEFINITION 1.3. Let  $p$  be a prime. We say that  $(G, M)$  is a quadratic pair for  $p$  if  $G$  is a group and

- (i)  $M$  is an irreducible  $F_p G$ -module,
- (ii)  $G$  acts faithfully on  $M$ , and
- (iii)  $G = \langle Q \rangle$ , where  $Q = \{g \in G - \{1\} \mid M(g-1)^2 = 0\}$ .

THEOREM 1.4 (Central Product Theorem, Thompson). *Suppose that  $(G, M)$  is a quadratic pair for  $p$  and  $p \geq 5$ . Then for some natural number  $n$ , the following hold.*

- (i)  $G = G_1 G_2 \cdots G_n$ ,  $[G_i, G_j] = 1$  ( $1 \leq i < j \leq n$ ),
- (ii)  $G_i/Z(G_i)$  is simple and  $(G_i, M_i)$  is a quadratic pair for  $i = 1, 2, \dots, n$ ,
- (iii)  $Q = \bigcup_{i=1}^n (Q \cap G_i)$ ,
- (iv)  $M$  and  $M_1 \otimes \cdots \otimes M_n$  are isomorphic  $F_p G$ -modules.

THEOREM 1.5 (Thompson). *Suppose  $(G, M)$  is a quadratic pair for  $p \geq 5$ , and  $\bar{G} = G/Z(G)$  is simple. Then for some natural number  $e$  and  $q = p^e$ ,  $\bar{G}$  is isomorphic to one of the following groups:*

$$A_n(q), B_n(q), C_n(q), D_n(q), G_2(q), F_4(q), E_6(q), \\ E_7(q), {}^2A_n(q), {}^2D_n(q), {}^3D_4(q), {}^2E_6(q).$$

Any group from the above list will be called a simple group of

quadratic type.

## 2. $p$ -Constrained groups which are not $p$ -stable.

**LEMMA 2.1.** *Suppose that  $G$  acts on a vector space  $V$  over  $GF(p)$  and assume that  $G$  is generated by elements which act quadratically. If  $G$  is not a  $p$ -group, then  $G$  contains a normal subgroup  $H$  such that  $G/H$  is a simple group of quadratic type.*

*Proof.* Let  $W$  be a nontrivial composition factor of  $V$  under  $G$ . Then  $\bar{G} = G/C_G(W)$  acts faithfully and irreducibly on  $W$ . Since  $(\bar{G}, W)$  is a quadratic pair, Theorem 1.5 implies the result.

**THEOREM 2.2.** *A  $p$ -constrained group  $G$  with  $O_p(G) = 1$  which is not  $p$ -stable has a composition factor of quadratic type.*

*Proof.* Since  $G$  is not  $p$ -stable there exists  $R \trianglelefteq G$ ,  $R \cong S$  a Sylow  $p$ -subgroup,  $A \cong N_s(R)$  with the property that  $[R, A, A] = 1$ , and  $AC(R)/C(R) \not\subseteq O_p(N(R)/C(R))$ . Since  $R \trianglelefteq G$ ,  $\Phi(R) \trianglelefteq G$ . Consider  $\bar{G} = G/\Phi(R)$ .  $\bar{G}$  satisfies the hypotheses of the theorem so by induction  $\Phi(R) = 1$  and  $R$  is elementary abelian. Let  $L = C(R)\langle x \mid [R, x, x] = 1 \rangle$ . By assumption  $C(R) \subset L \not\subseteq O_p(G \text{ mod } C(R))$  and by definition  $L \trianglelefteq G$ . Lemma 2.1 implies that there exists  $K \trianglelefteq L$  such that  $L/K$  is simple of quadratic type.

## 3. Automorphisms of semisimple groups.

**DEFINITION 3.1.** A semisimple group is the direct product of simple groups. The simple factors are called the components.

**LEMMA 3.2.** *Suppose that  $G$  is a semisimple group with no abelian components. If  $K \trianglelefteq G$  and  $K$  is simple, then  $K$  is equal to one of the components.*

*Proof.* Standard result, [5].

We prove now a basic lemma about automorphisms of a semisimple group with isomorphic nonabelian components. Let  $G$  be the direct product of  $t$  copies of the simple group  $H$ . Define  $H_i = \{(1, 1, \dots, x_i, \dots, 1) \mid x_i \in H\}$  for  $1 \leq i \leq t$ . Then  $G$  is the direct product of the  $H_i$ 's. Two subgroups of  $\text{Aut}(G)$  are readily available. The first is  $L = \prod \text{Aut}(H_i)$  where the action is the natural one. The second is  $K$ , the group of permutations of the  $H_i$ 's.

LEMMA 3.3.  $\text{Aut}(G)$  permutes the set  $\{H_i\}$ .

*Proof.* This is an immediate consequence of Lemma 3.2.

THEOREM 3.4. Suppose that  $G$  is the direct product of  $t$  copies of the simple nonabelian group  $H$ . Define  $K$  and  $L$  as above. Then  $L \trianglelefteq \text{Aut}(G)$ ,  $L \cap K = 1$ ,  $LK = \text{Aut}(G)$  and  $K \cong \text{Sym}(t)$ .

*Proof.* By Lemma 3.3 we know that every  $\sigma \in \text{Aut}(G)$  permutes the set  $\{H_1, \dots, H_t\}$ . In particular there is a homomorphism

$$\Psi: \text{Aut}(G) \longrightarrow \text{Sym}(t).$$

Clearly  $L = \ker(\Psi)$ , and  $K \cong \Psi(K) \cong \text{Sym}(t)$ . The result follows.

4. Automorphisms of a group with a quadratic factor. The main result of this section is the following.

THEOREM 4.1. Let  $G$  be a group with a composition factor of quadratic type. If  $A \subseteq \text{Aut}(G)$  and  $(|A|, |G|) = 1$ , then the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .

We proceed via a series of lemmas.

LEMMA 4.2. Suppose  $H$  is a simple nonabelian group of quadratic type with respect to the prime  $p \geq 5$ . If  $A \subseteq \text{Aut}(H)$  and  $(|A|, |H|) = 1$ , then the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .

*Proof.* By the main result from Steinberg [7],  $\text{Aut}(H) = M$  contains a normal series  $H \subseteq \tilde{H} \subseteq \tilde{M} \subseteq M$ . Furthermore by the same theorem there are groups  $F$  and  $E$ ,  $F$  the field automorphisms and  $E$  the graph automorphisms, such that  $M = \tilde{H}EF$ . Since every simple group of quadratic type is a finite Chevalley group they must all involve  $\text{PSL}(2, p)$ . Thus  $(|A|, |H|) = 1$  and  $p \geq 5$  imply that  $(|A|, 2, 3, p) = 1$ . By order considerations Steinberg's theorem implies that  $A \cap \tilde{H} = 1$  and  $A \subseteq \tilde{M}$ , where  $\tilde{M} = \tilde{H}F$ .

Now let  $N = F \cap \tilde{H}A$ . Then

$$\tilde{H}N = \tilde{H}(F \cap \tilde{H}A) = \tilde{H}F \cap \tilde{H}A = M \cap \tilde{H}A = \tilde{H}A.$$

Since  $A \cap \tilde{H} = N \cap \tilde{H} = 1$ ,  $(|\tilde{H}|, |A|) = 1$  and  $N$  is solvable; the Schur-Zassenhaus theorem implies that  $A$  is conjugate to  $N$  in  $M$ . If we prove the result for a conjugate of  $A$  it is certainly true for  $A$ . Therefore we may assume that  $A = N \subseteq F$ .

Now the field automorphisms have a fixed point subgroup which

contains the corresponding Chevalley group over the prime field  $GF(p)$ . In particular this subgroup involves  $\text{PSL}(2, p)$ . Since  $A \cong F$ , certainly the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$  as desired.

**LEMMA 4.3.** *Let  $G$  be the direct product of  $t$  copies of  $H$ , a simple group of quadratic type with respect to the prime  $p \geq 5$ . Suppose that  $A \cong \text{Aut}(G)$  and that  $(|A|, |G|) = 1$ . Then the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .*

*Proof.* We adopt the notation presented in §3. Let  $A^*$  be the subgroup of  $A$  stabilizing  $H_1$ . Then  $A^*/C_{A^*}(H_1)$  is a subgroup of  $\text{Aut}(H_1) \cong \text{Aut}(H)$ . Therefore by Lemma 4.2 there exists subgroups  $U_1$  and  $V_1$  contained in the fixed point subgroup of  $A^*$  on  $H_1$  such that  $V_1/U_1 \cong \text{PSL}(2, p)$ .

Now let  $T$  be a transversal of  $A^*$  in  $A$ . Suppose that  $t$  and  $u$  are distinct elements of  $T$ . By Lemma 3.3  $H_1^t = H_j$  and  $H_1^u = H_j$  for some  $i$  and  $j$ . If  $i = j$ , then  $H_1^t = H_1^u$  and  $tu^{-1}$  stabilizes  $H_1$  contrary to assumption. Thus  $i \neq j$  and  $[H_1^t, H_1^u] = 1$ . This fact implies that the set  $V = \{\prod_{t \in T} x^t \mid x \in V_1\}$  is a group. Furthermore it implies that the elements of  $V$  are fixed by  $A$ . If  $U = \{\prod_{t \in T} x^t \mid x \in U_1\}$ , then  $V/U \cong V_1/U_1 \cong \text{PSL}(2, p)$  and we conclude that the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .

As a consequence of Lemma 4.3 we get the following corollary.

**COROLLARY 4.4.** *Suppose that  $X$  is the direct product of  $t$  copies of a simple group of quadratic type with respect to the prime  $p \geq 5$ . Assume that  $G$  is a group,  $A \cong \text{Aut}(G)$  and  $G$  contains a factor isomorphic to  $X$  that is normalized by  $A$ . If  $(|A|, |G|) = 1$ , then the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .*

*Proof.* Suppose that  $K \trianglelefteq L \trianglelefteq G$  and that  $A$  normalizes  $L/K = X$ . By Lemma 4.3 there exist subgroups  $S$  and  $T$  such that  $K \trianglelefteq S \trianglelefteq T \trianglelefteq L$ ,  $T/S \cong \text{PSL}(2, p)$  and  $A$  fixes  $T/K$ . Suppose that  $q$  is a prime divisor of  $|\text{PSL}(2, p)|$  and let  $Q$  be a Sylow  $q$ -subgroup of  $T$  normalized by  $A$ . Then since  $Q = C_Q(A)[Q, A]$  and  $A$  fixes  $T/S$ ,  $Q = C_Q(A)(Q \cap S)$ . Pick such a  $Q$  for each prime divisor of  $|\text{PSL}(2, p)|$  and call this set of Sylow subgroups  $\mathcal{S}$ . Then

$$T = \langle C_Q(A) \mid Q \in \mathcal{S} \rangle S$$

and consequently  $C_Q(A)$  involves  $\text{PSL}(2, p)$ .

**LEMMA 4.5.** *Suppose  $G$  is a group with a composition factor isomorphic to  $K$ , then  $G$  contains a semisimple factor  $X$  normalized by  $A$  such that every component of  $X$  is isomorphic to  $K$ .*

*Proof.* Let  $F$  be the semidirect product of  $G$  and suppose that  $\{F_i\}$  is a chief series of  $F$  containing  $G$ . Then there exists  $i$  such that  $F_{i-1} \subset F_i \subset G$  and  $F_i/F_{i-1}$  has  $K$  as a composition factor. Since  $F_i/F_{i-1}$  is a direct product of isomorphic simple group, it is the product of copies of  $K$ .

*Proof of Theorem 4.1.* Theorem 4.1 is now a consequence of Lemma 4.5 and Corollary 4.4.

5. **Proof of Theorem B.** Theorem B is a consequence of the following result.

**THEOREM 5.1.** *Let  $G$  be a  $p$ -constrained group with  $p \geq 5$ . Suppose that  $A \subseteq \text{Aut}(G)$  and  $(|G|, |A|) = 1$ . Then if  $G$  is not  $p$ -stable the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .*

*Proof.* Suppose that  $O_p(G) \supset 1$  and set  $\bar{G} = G/O_p(G)$ .  $\bar{G}$  is not  $p$ -stable and induction implies the result. Thus we may assume that  $O_p(G) = 1$ .

Theorem 2.2 implies that  $G$  contains a composition factor of quadratic type. Then Theorem 4.1 implies that the fixed point subgroup of  $A$  involves  $\text{PSL}(2, p)$ .

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Received March 2, 1973.

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