

Pacific Journal of Mathematics

A FACTORIZATION THEOREM FOR p -CONSTRAINED GROUPS

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Suppose that G is a finite p -constrained group. For some prime $p \geq 5$ let S be a Sylow p -subgroup. Assume that G admits a group of automorphisms A such that $(|A|, |G|) = 1$ and the fixed point subgroup of A does not involve $\text{PSL}(2, p)$. In this paper it is shown that under these conditions

$$G = O_p(G)N(Z(J(S))).$$

Thompson proved in [8] that if G is a strongly p -solvable group and $O_p(G) = 1$, then $G = N(J(S))C(Z(S))$, where S is a Sylow p -subgroup of G . Since his paper several other stronger results of this type have been proved by Glauberman [1], [2]. Specifically he proved his ZJ -theorem which states that if G is p -constrained and p -stable then $G = O_p(G)N(Z(J(S)))$. This implies Thompson's conclusion by the Frattini argument. Recently Glauberman has proved that

$$G = N(J(S))C(Z(S))$$

for all p , provided that G is p -solvable and admits a group of automorphisms A such that $(|G|, |A|) = 1$ and A has no fixed points of order p .

In this paper our goal is a theorem related to these results.

THEOREM A. *Let G be a p -constrained group with $p \geq 5$ and S a Sylow p -subgroup of G . Suppose that G admits a group of automorphisms A such that $(|A|, |G|) = 1$ and the fixed point subgroup of A does not involve $\text{PSL}(2, p)$. Then $G = O_p(G)N(Z(J(S)))$.*

Using Glauberman's ZJ -theorem, Theorem A is a corollary of Theorem B.

THEOREM B. *Let G be a p -constrained group with $p \geq 5$. Suppose that A is a group of automorphisms of G such that $(|G|, |A|) = 1$ and that the fixed point subgroup of A does not involve $\text{PSL}(2, p)$. Then G is p -stable.*

All the groups in this paper are finite. The notation, except for the definition of p -stability, is standard and can be found in [3]. If P is a p -group $J(P) = \langle A \subseteq P \mid A \text{ is abelian and of maximal order} \rangle$. For simplicity we will write $Z(J(P)) = ZJ(P)$. If K is a group, we

say G involves K if a section of G is isomorphic to K .

1. Assumed results and definitions.

DEFINITION 1.1. Let G be a group with $O_p(G) \cong 1$. Let S be a Sylow p -subgroup of G and set $P = S \cap O_{p',p}(G)$. G is p -constrained if $C_G(P) \cong O_{p',p}(G)$.

DEFINITION 1.2. Let G be a group and suppose that S is a Sylow p -subgroup. G is p -stable if for any $R \cong S$ such that $RO_p(G) \trianglelefteq G$ and for any $A \cong N_S(R)$ with the property that $[R, A, A] = 1$, we have

$$AC(R)/C(R) \cong O_p(N(R)/C(R)).$$

This definition of p -stability is taken from Gorenstein-Walter [4]. It is weaker than the definition given in Gorenstein [3]. However this definition is sufficient for Glauberman's ZJ -theorem as a check of the proof [3] will indicate.

The principal tools of this paper are two theorems of Thompson. We state these for want of an available reference.

DEFINITION 1.3. Let p be a prime. We say that (G, M) is a quadratic pair for p if G is a group and

- (i) M is an irreducible $F_p G$ -module,
- (ii) G acts faithfully on M , and
- (iii) $G = \langle Q \rangle$, where $Q = \{g \in G - \{1\} \mid M(g-1)^2 = 0\}$.

THEOREM 1.4 (Central Product Theorem, Thompson). *Suppose that (G, M) is a quadratic pair for p and $p \geq 5$. Then for some natural number n , the following hold.*

- (i) $G = G_1 G_2 \cdots G_n$, $[G_i, G_j] = 1$ ($1 \leq i < j \leq n$),
- (ii) $G_i/Z(G_i)$ is simple and (G_i, M_i) is a quadratic pair for $i = 1, 2, \dots, n$,
- (iii) $Q = \bigcup_{i=1}^n (Q \cap G_i)$,
- (iv) M and $M_1 \otimes \cdots \otimes M_n$ are isomorphic $F_p G$ -modules.

THEOREM 1.5 (Thompson). *Suppose (G, M) is a quadratic pair for $p \geq 5$, and $\bar{G} = G/Z(G)$ is simple. Then for some natural number e and $q = p^e$, \bar{G} is isomorphic to one of the following groups:*

$$A_n(q), B_n(q), C_n(q), D_n(q), G_2(q), F_4(q), E_6(q), \\ E_7(q), {}^2A_n(q), {}^2D_n(q), {}^3D_4(q), {}^2E_6(q).$$

Any group from the above list will be called a simple group of

quadratic type.

2. p -Constrained groups which are not p -stable.

LEMMA 2.1. *Suppose that G acts on a vector space V over $GF(p)$ and assume that G is generated by elements which act quadratically. If G is not a p -group, then G contains a normal subgroup H such that G/H is a simple group of quadratic type.*

Proof. Let W be a nontrivial composition factor of V under G . Then $\bar{G} = G/C_G(W)$ acts faithfully and irreducibly on W . Since (\bar{G}, W) is a quadratic pair, Theorem 1.5 implies the result.

THEOREM 2.2. *A p -constrained group G with $O_p(G) = 1$ which is not p -stable has a composition factor of quadratic type.*

Proof. Since G is not p -stable there exists $R \trianglelefteq G$, $R \cong S$ a Sylow p -subgroup, $A \cong N_S(R)$ with the property that $[R, A, A] = 1$, and $AC(R)/C(R) \not\subseteq O_p(N(R)/C(R))$. Since $R \trianglelefteq G$, $\Phi(R) \trianglelefteq G$. Consider $\bar{G} = G/\Phi(R)$. \bar{G} satisfies the hypotheses of the theorem so by induction $\Phi(R) = 1$ and R is elementary abelian. Let $L = C(R)\langle x \mid [R, x, x] = 1 \rangle$. By assumption $C(R) \subset L \not\subseteq O_p(G \text{ mod } C(R))$ and by definition $L \trianglelefteq G$. Lemma 2.1 implies that there exists $K \trianglelefteq L$ such that L/K is simple of quadratic type.

3. Automorphisms of semisimple groups.

DEFINITION 3.1. A semisimple group is the direct product of simple groups. The simple factors are called the components.

LEMMA 3.2. *Suppose that G is a semisimple group with no abelian components. If $K \trianglelefteq G$ and K is simple, then K is equal to one of the components.*

Proof. Standard result, [5].

We prove now a basic lemma about automorphisms of a semisimple group with isomorphic nonabelian components. Let G be the direct product of t copies of the simple group H . Define $H_i = \{(1, 1, \dots, x_i, \dots, 1) \mid x_i \in H\}$ for $1 \leq i \leq t$. Then G is the direct product of the H_i 's. Two subgroups of $\text{Aut}(G)$ are readily available. The first is $L = \prod \text{Aut}(H_i)$ where the action is the natural one. The second is K , the group of permutations of the H_i 's.

LEMMA 3.3. $\text{Aut}(G)$ permutes the set $\{H_i\}$.

Proof. This is an immediate consequence of Lemma 3.2.

THEOREM 3.4. Suppose that G is the direct product of t copies of the simple nonabelian group H . Define K and L as above. Then $L \trianglelefteq \text{Aut}(G)$, $L \cap K = 1$, $LK = \text{Aut}(G)$ and $K \cong \text{Sym}(t)$.

Proof. By Lemma 3.3 we know that every $\sigma \in \text{Aut}(G)$ permutes the set $\{H_i, \dots, H_i\}$. In particular there is a homomorphism

$$\Psi: \text{Aut}(G) \longrightarrow \text{Sym}(t).$$

Clearly $L = \ker(\Psi)$, and $K \cong \Psi(K) \cong \text{Sym}(t)$. The result follows.

4. Automorphisms of a group with a quadratic factor. The main result of this section is the following.

THEOREM 4.1. Let G be a group with a composition factor of quadratic type. If $A \subseteq \text{Aut}(G)$ and $(|A|, |G|) = 1$, then the fixed point subgroup of A involves $\text{PSL}(2, p)$.

We proceed via a series of lemmas.

LEMMA 4.2. Suppose H is a simple nonabelian group of quadratic type with respect to the prime $p \geq 5$. If $A \subseteq \text{Aut}(H)$ and $(|A|, |H|) = 1$, then the fixed point subgroup of A involves $\text{PSL}(2, p)$.

Proof. By the main result from Steinberg [7], $\text{Aut}(H) = M$ contains a normal series $H \subseteq \tilde{H} \subseteq \tilde{M} \subseteq M$. Furthermore by the same theorem there are groups F and E , F the field automorphisms and E the graph automorphisms, such that $M = \tilde{H}EF$. Since every simple group of quadratic type is a finite Chevalley group they must all involve $\text{PSL}(2, p)$. Thus $(|A|, |H|) = 1$ and $p \geq 5$ imply that $(|A|, 2.3, p) = 1$. By order considerations Steinberg's theorem implies that $A \cap \tilde{H} = 1$ and $A \subseteq \tilde{M}$, where $\tilde{M} = \tilde{H}F$.

Now let $N = F \cap \tilde{H}A$. Then

$$\tilde{H}N = \tilde{H}(F \cap \tilde{H}A) = \tilde{H}F \cap \tilde{H}A = M \cap \tilde{H}A = \tilde{H}A.$$

Since $A \cap \tilde{H} = N \cap \tilde{H} = 1$, $(|\tilde{H}|, |A|) = 1$ and N is solvable; the Schur-Zassenhaus theorem implies that A is conjugate to N in M . If we prove the result for a conjugate of A it is certainly true for A . Therefore we may assume that $A = N \subseteq F$.

Now the field automorphisms have a fixed point subgroup which

contains the corresponding Chevalley group over the prime field $GF(p)$. In particular this subgroup involves $\text{PSL}(2, p)$. Since $A \subseteq F$, certainly the fixed point subgroup of A involves $\text{PSL}(2, p)$ as desired.

LEMMA 4.3. *Let G be the direct product of t copies of H , a simple group of quadratic type with respect to the prime $p \geq 5$. Suppose that $A \subseteq \text{Aut}(G)$ and that $(|A|, |G|) = 1$. Then the fixed point subgroup of A involves $\text{PSL}(2, p)$.*

Proof. We adopt the notation presented in §3. Let A^* be the subgroup of A stabilizing H_1 . Then $A^*/C_{A^*}(H_1)$ is a subgroup of $\text{Aut}(H_1) \cong \text{Aut}(H)$. Therefore by Lemma 4.2 there exists subgroups U_1 and V_1 contained in the fixed point subgroup of A^* on H_1 such that $V_1/U_1 \cong \text{PSL}(2, p)$.

Now let T be a transversal of A^* in A . Suppose that t and u are distinct elements of T . By Lemma 3.3 $H_1^t = H_j$ and $H_1^u = H_j$ for some i and j . If $i = j$, then $H_1^t = H_1^u$ and tu^{-1} stabilizes H_1 contrary to assumption. Thus $i \neq j$ and $[H_1^t, H_1^u] = 1$. This fact implies that the set $V = \{\prod_{t \in T} x^t \mid x \in V_1\}$ is a group. Furthermore it implies that the elements of V are fixed by A . If $U = \{\prod_{t \in T} x^t \mid x \in U_1\}$, then $V/U \cong V_1/U_1 \cong \text{PSL}(2, p)$ and we conclude that the fixed point subgroup of A involves $\text{PSL}(2, p)$.

As a consequence of Lemma 4.3 we get the following corollary.

COROLLARY 4.4. *Suppose that X is the direct product of t copies of a simple group of quadratic type with respect to the prime $p \geq 5$. Assume that G is a group, $A \subseteq \text{Aut}(G)$ and G contains a factor isomorphic to X that is normalized by A . If $(|A|, |G|) = 1$, then the fixed point subgroup of A involves $\text{PSL}(2, p)$.*

Proof. Suppose that $K \trianglelefteq L \trianglelefteq G$ and that A normalizes $L/K = X$. By Lemma 4.3 there exist subgroups S and T such that $K \trianglelefteq S \trianglelefteq T \trianglelefteq L$, $T/S \cong \text{PSL}(2, p)$ and A fixes T/K . Suppose that q is a prime divisor of $|\text{PSL}(2, p)|$ and let Q be a Sylow q -subgroup of T normalized by A . Then since $Q = C_q(A)[Q, A]$ and A fixes T/S , $Q = C_q(A)(Q \cap S)$. Pick such a Q for each prime divisor of $|\text{PSL}(2, p)|$ and call this set of Sylow subgroups \mathcal{S} . Then

$$T = \langle C_q(A) \mid Q \in \mathcal{S} \rangle S$$

and consequently $C_G(A)$ involves $\text{PSL}(2, p)$.

LEMMA 4.5. *Suppose G is a group with a composition factor isomorphic to K , then G contains a semisimple factor X normalized by A such that every component of X is isomorphic to K .*

Proof. Let F be the semidirect product of G and suppose that $\{F_i\}$ is a chief series of F containing G . Then there exists i such that $F_{i-1} \subset F_i \subset G$ and F_i/F_{i-1} has K as a composition factor. Since F_i/F_{i-1} is a direct product of isomorphic simple group, it is the product of copies of K .

Proof of Theorem 4.1. Theorem 4.1 is now a consequence of Lemma 4.5 and Corollary 4.4.

5. **Proof of Theorem B.** Theorem B is a consequence of the following result.

THEOREM 5.1. *Let G be a p -constrained group with $p \geq 5$. Suppose that $A \subseteq \text{Aut}(G)$ and $(|G|, |A|) = 1$. Then if G is not p -stable the fixed point subgroup of A involves $\text{PSL}(2, p)$.*

Proof. Suppose that $O_p(G) \supset 1$ and set $\bar{G} = G/O_p(G)$. \bar{G} is not p -stable and induction implies the result. Thus we may assume that $O_p(G) = 1$.

Theorem 2.2 implies that G contains a composition factor of quadratic type. Then Theorem 4.1 implies that the fixed point subgroup of A involves $\text{PSL}(2, p)$.

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Received March 2, 1973.

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Martin Bartelt, <i>Strongly unique best approximates to a function on a set, and a finite subset thereof</i>	1
S. J. Bernau, <i>Theorems of Korovkin type for L_p-spaces</i>	11
S. J. Bernau and Howard E. Lacey, <i>The range of a contractive projection on an L_p-space</i>	21
Marilyn Breen, <i>Decomposition theorems for 3-convex subsets of the plane</i>	43
Ronald Elroy Bruck, Jr., <i>A common fixed point theorem for a commuting family of nonexpansive mappings</i>	59
Aiden A. Bruen and J. C. Fisher, <i>Blocking sets and complete k-arcs</i>	73
R. Creighton Buck, <i>Approximation properties of vector valued functions</i>	85
Mary Rodriguez Embry and Marvin Rosenblum, <i>Spectra, tensor products, and linear operator equations</i>	95
Edward William Formanek, <i>Maximal quotient rings of group rings</i>	109
Barry J. Gardner, <i>Some aspects of T-nilpotence</i>	117
Juan A. Gatica and William A. Kirk, <i>A fixed point theorem for k-set-contractions defined in a cone</i>	131
Kenneth R. Goodearl, <i>Localization and splitting in hereditary noetherian prime rings</i>	137
James Victor Herod, <i>Generators for evolution systems with quasi continuous trajectories</i>	153
C. V. Hinkle, <i>The extended centralizer of an S-set</i>	163
I. Martin (Irving) Isaacs, <i>Lifting Brauer characters of p-solvable groups</i>	171
Bruce R. Johnson, <i>Generalized Lerch zeta function</i>	189
Erwin Kleinfeld, <i>A generalization of $(-1, 1)$ rings</i>	195
Horst Leptin, <i>On symmetry of some Banach algebras</i>	203
Paul Weldon Lewis, <i>Strongly bounded operators</i>	207
Arthur Larry Lieberman, <i>Spectral distribution of the sum of self-adjoint operators</i>	211
I. J. Maddox and Michael A. L. Willey, <i>Continuous operators on paranormed spaces and matrix transformations</i>	217
James Dolan Reid, <i>On rings on groups</i>	229
Richard Miles Schori and James Edward West, <i>Hyperspaces of graphs are Hilbert cubes</i>	239
William H. Specht, <i>A factorization theorem for p-constrained groups</i>	253
Robert L. Thele, <i>Iterative techniques for approximation of fixed points of certain nonlinear mappings in Banach spaces</i>	259
Tim Eden Traynor, <i>An elementary proof of the lifting theorem</i>	267
Charles Irvin Vinsonhaler and William Jennings Wickless, <i>Completely decomposable groups which admit only nilpotent multiplications</i>	273
Raymond O'Neil Wells, Jr., <i>Comparison of de Rham and Dolbeault cohomology for proper surjective mappings</i>	281
David Lee Wright, <i>The non-minimality of induced central representations</i>	301
Bertram Yood, <i>Commutativity properties in Banach $*$-algebras</i>	307