

# Pacific Journal of Mathematics

**ITERATIVE TECHNIQUES FOR APPROXIMATION OF FIXED  
POINTS OF CERTAIN NONLINEAR MAPPINGS IN BANACH  
SPACES**

ROBERT L THELE

## ITERATIVE TECHNIQUES FOR APPROXIMATION OF FIXED POINTS OF CERTAIN NONLINEAR MAPPINGS IN BANACH SPACES

R. L. THELE

Let  $D$  be a closed convex subset of a Banach space  $X$ , let  $T: D \rightarrow D$  be nonexpansive (that is,  $\|Tx - Ty\| \leq \|x - y\|$  for every  $x, y \in D$ ), and let  $F_\lambda = \lambda T + (1 - \lambda)I$ , where  $\lambda \in (0, 1)$  and  $I$  denotes the identity on  $D$ . Several authors have found conditions under which the sequences of iterates  $\{T^n x\}$ , or the sequences  $\{F_\lambda^n x\}$ , converge strongly or weakly to fixed points of  $T$  for all  $x \in D$ . In this paper we establish conditions under which the sequences  $\{F_{1/2}^n x\}$  converge strongly to fixed points of  $T$  for all  $x$  in a neighborhood of the fixed point set of  $T$ ; furthermore, our theorems hold for classes of mappings  $T$  more general than the class of nonexpansive mappings.

We complement these results by proving theorems under which local convergence of iterates entails global convergence; thus by combining our results in these two areas we obtain new theorems regarding the global convergence of iterates. Finally, we give an example of a class of mappings satisfying the various conditions of our theorems.

1. Local and global convergence of iterates. Let  $D$  be a convex subset of the Banach space  $X$ , and let  $T: D \rightarrow D$ . Adopting the terminology of Furi and Vignoli [6] we say that the sequence  $\{T^n x_0\}$  of iterates of  $x_0 \in D$  is *stable* if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\|T^n x - T^n x_0\| < \varepsilon$  for every  $n = 1, 2, \dots$  whenever  $x \in D$  and  $\|x - x_0\| < \delta$ . We say that  $T$  has *stable iterates* if the sequence  $\{T^n x\}$  of iterates of  $x$  is stable for every  $x \in D$ . Finally, if  $x \in X$  and  $B \subset X$  we define  $d(x, B) = \inf \{\|x - y\| : y \in B\}$ .

**THEOREM 1.** *Let  $D$  be a convex subset of a Banach space  $X$  and suppose that  $T: D \rightarrow D$  has stable iterates. Let  $A$  be a nonempty subset of  $D$ .*

(i) *If there exists  $\rho > 0$  such that  $\{T^n x\}$  has a cluster point in  $A$  whenever  $x \in D$  and  $d(x, A) < \rho$ , then  $\{T^n x\}$  has a cluster point in  $A$  for every  $x \in D$ .*

(ii) *If there exists  $\rho > 0$  such that  $\{T^n x\}$  has its limit in  $A$  whenever  $x \in D$  and  $d(x, A) < \rho$ , then  $\{T^n x\}$  converges to some point of  $A$  for every  $x \in D$ .*

*Proof.* To prove the first statement, let  $x \in D$  and  $x_0 \in A$ . For

each  $\lambda \in [0, 1]$  let  $y_\lambda = \lambda x + (1 - \lambda)x_0$  and set  $\lambda_0 = \sup \{\lambda \in [0, 1]: \{T^n y_\lambda\}$  has a cluster point in  $A\}$ . Let  $\delta$  correspond to  $\varepsilon = \rho/3$  in the definition of the stability of  $\{T^n y_{\lambda_0}\}$ , and choose  $\lambda_1 \in [0, \lambda_0]$  such that  $\|y_{\lambda_1} - y_{\lambda_0}\| < \delta$  and  $\{T^n y_{\lambda_1}\}$  has a cluster point in  $A$ . If  $\lambda_0 = 1$ , let  $\lambda_2 = \lambda_0$ ; if  $\lambda_0 < 1$ , let  $\lambda_2 \in (\lambda_0, 1]$  be such that  $\|y_{\lambda_2} - y_{\lambda_0}\| < \delta$ . Since there exists a cluster point  $w$  in  $A$  of  $\{T^n y_{\lambda_1}\}$  and a positive integer  $N$  such that  $\|T^N y_{\lambda_1} - w\| < \rho/3$ , we have that

$$\begin{aligned} \|T^N y_{\lambda_2} - w\| &\leq \|T^N y_{\lambda_2} - T^N y_{\lambda_0}\| + \|T^N y_{\lambda_1} - T^N y_{\lambda_0}\| + \|T^N y_{\lambda_1} - w\| \\ &< \rho/3 + \rho/3 + \rho/3 = \rho. \end{aligned}$$

Thus  $d(T^N y_{\lambda_2}, A) \leq \|T^N y_{\lambda_2} - w\| < \rho$ , entailing that  $\{T^{N+n} y_{\lambda_2}\}$ —and hence  $\{T^n y_{\lambda_2}\}$ —has a cluster point in  $A$ . If  $\lambda_0 < 1$ , this contradicts the definition of  $\lambda_0$ ; thus  $\lambda_0 = 1$ , and since in this case  $y_{\lambda_2} = x$ , we have that  $\{T^n x\}$  has a cluster point in  $A$ .

To prove the second statement, we let  $x \in D$  and note that by our proof of the first statement  $\{T^n x\}$  has a cluster point  $w \in A$ . Thus there exists a positive integer  $N$  such that  $\|T^N x - w\| < \rho$ , implying that  $T^{N+n} x \rightarrow w$ , whence  $T^n x \rightarrow w$ .

We remark that in the case of the second statement of the theorem above,  $A$  must contain a fixed point of  $T$ , since if  $T$  is continuous the limit of a sequence  $\{T^n x\}$  is necessarily a fixed point. In our applications of this theorem we will assume either that  $A$  is the fixed point set of  $T$  or that  $A$  is a singleton.

**COROLLARY 1.** *Let  $D$  be a convex subset of a Banach space  $X$ , and let  $T: D \rightarrow D$  possess stable iterates. Let  $x_0$  be a fixed point of  $T$  for which there exists an open neighborhood  $U$  of  $x_0$ ,  $U \subset D$ , such that  $T$  is continuously Fréchet differentiable in  $U$  and  $\|T'x_0\| < 1$ . Then  $T^n x \rightarrow x_0$ , for every  $x \in D$ .*

*Proof.* Since  $T$  is continuously Fréchet differentiable in  $U$  and  $\|T'x_0\| < 1$ , there exists a constant  $k \in (0, 1)$  and an open ball  $S(x_0, \rho)$  about  $x_0$  with radius  $\rho$ ,  $S(x_0, \rho) \subset U$ , such that if  $x \in S(x_0, \rho)$  then  $\|T'z\| < k$ . Let  $y \in S(x_0, \rho)$ . Then there exists a point  $z$  in the segment from  $x_0$  to  $y$  such that (see Fréchet [5])

$$\|Tx_0 - Ty\| \leq \|T'z\| \|x_0 - y\|.$$

But  $z \in S(x_0, \rho)$  so that  $\|T'z\| < k$ . Thus for every  $y \in S(x_0, \rho)$

$$\|Tx_0 - Ty\| \leq k \|x_0 - y\|.$$

By induction,  $\|x_0 - T^n y\| \leq k^n \|x_0 - y\|$  for every  $n = 1, 2, 3, \dots$ . Since  $k^n \rightarrow 0$ ,  $T^n y \rightarrow x_0$  for every  $y \in S(x_0, \rho)$ . By part (ii) of Theorem 1,  $T^n x \rightarrow x_0$  for every  $x \in D$ .

2. **Conditions implying local convergence of iterates.** The *modulus of convexity* of a Banach space  $X$  is the function  $\delta: [0, 2] \rightarrow [0, 1]$  defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \text{ and } \|x - y\| \geq \varepsilon \right\}.$$

It is well-known (cf. [9]) that  $\delta$  is nondecreasing and continuous except possibly at 2. Furthermore, letting  $\varepsilon_0 = \sup \{ \varepsilon \in [0, 2] : \delta(\varepsilon) = 0 \}$ ,  $X$  is uniformly convex if and only if  $\varepsilon_0 = 0$ ,  $X$  is uniformly non-square if and only if  $\varepsilon_0 < 2$ , and  $X$  is strictly convex if and only if  $\delta(2) = 1$ .

We observe that if  $x, y \in X$  satisfy the conditions

$$\|x\| \leq d, \|y\| \leq d, \text{ and } \|x - y\| \geq \varepsilon, \text{ then}$$

$$\left\| \frac{x + y}{2} \right\| \leq \left( 1 - \delta\left(\frac{\varepsilon}{d}\right) \right) d.$$

Finally, we denote by  $I$  the identity mapping on any convex subset of  $X$ .

**THEOREM 2.** *Let  $D$  be a convex subset of a uniformly nonsquare Banach space  $X$ . Suppose that  $T: D \rightarrow D$  has a nonempty fixed point set  $A$  and that  $T$  satisfies the following conditions: There exist  $\rho > 0$ ,  $c > 0$ , and  $s \geq 1$  with  $(1 - \delta(c/s))s < 1$  such that if  $x \in D$  and  $d(x, A) < \rho$  then*

- (i)  $\|Tx - x\| \geq cd(x, A)$ , and
- (ii)  $\|Tx - u\| \leq s\|x - u\|$  for every  $u \in A$ .

*Then setting  $F = 1/2(I + T)$ ,  $d(F^n x, A) \rightarrow 0$  for every  $x \in D$  for which  $d(x, A) < \rho$ .*

*Proof.* We observe that if  $x \notin A$  then

$$cd(x, A) \leq \|Tx - x\| \leq \|Tx - u\| + \|x - u\| \leq (1 + s)\|x - u\|$$

for every  $u \in A$ . Thus  $cd(x, A) \leq (1 + s)d(x, A)$ , so that if  $T$  is not the identity then  $c \leq 1 + s$ . Therefore  $c/s \leq 1 + 1/s \leq 2$ , and moreover if  $c/s = 2$ , then  $s = 1$  and  $c = 2$ .

Let  $x \in D$  satisfy  $0 < d(x, A) < \rho$ , and for arbitrary  $r > 1$  let  $u_{x,r} \in A$  satisfy  $\|x - u_{x,r}\| \leq \text{minimum} \{ \rho, rd(x, A) \}$ . Thus  $\|Tx - u_{x,r}\| \leq s\|x - u_{x,r}\|$ .

Let  $d = s\|x - u_{x,r}\|$  and  $\varepsilon = \|Tx - x\|$ . Since  $\|x - u_{x,r}\| \leq d$ ,  $\|Tx - u_{x,r}\| \leq d$ , and  $\|(x - u_{x,r}) - (Tx - u_{x,r})\| = \varepsilon$  we obtain

$$\|Fx - u_{x,r}\| = \frac{1}{2} \|(x - u_{x,r}) + (Tx - u_{x,r})\|$$

$$\leq (1 - \delta(\varepsilon/d))d.$$

Now

$$\frac{\varepsilon}{d} = \frac{\|Tx - x\|}{s\|u_{x,r} - x\|} \geq \frac{cd(x, A)}{srd(x, A)} = \frac{c}{sr},$$

and thus since  $\delta$  is nondecreasing

$$1 - \delta(\varepsilon/d) \leq 1 - \delta\left(\frac{c}{sr}\right).$$

Therefore,

$$\begin{aligned} d(Fx, A) &\leq \|Fx - u_{x,r}\| \leq (1 - \delta(\varepsilon/d))d \leq \left(1 - \delta\left(\frac{c}{sr}\right)\right)d \\ &= \left(1 - \delta\left(\frac{c}{sr}\right)\right)s\|x - u_{x,r}\| \leq \left(1 - \delta\left(\frac{c}{sr}\right)\right)srd(x, A), \end{aligned}$$

for every  $r > 1$ .

Let  $\eta \equiv \lim_{r \rightarrow 1^+} (1 - \delta(c/sr))sr$ . Then  $d(Fx, A) \leq \eta d(x, A)$  whenever  $d(x, A) < \rho$ . If  $c/s < 2$  then  $\delta$  is continuous at  $c/s$  and  $\eta = (1 - \delta(c/s))s < 1$ . If  $c/s = 2$  then  $c = 2$  and  $s = 1$ , and since  $X$  is uniformly nonsquare,  $\eta = 1 - \lim_{\varepsilon \rightarrow 2^-} \delta(\varepsilon) < 1$ . By induction,  $d(F^n x, A) \leq \eta^n d(x, A)$  whenever  $d(x, A) < \rho$ , implying that  $d(F^n x, A) \rightarrow 0$  whenever  $d(x, A) < \rho$ .

**COROLLARY 2.** *If the hypotheses of Theorem 2 are satisfied and if in addition  $A$  is compact, then the sequence  $\{F^n x\}$  has a cluster point in  $A$  whenever  $d(x, A) < \rho$ .*

*Proof.* Since whenever  $d(x, A) < \rho$  we have  $d(F^n x, A) \rightarrow 0$ , we can select a sequence  $\{a_n\} \subset A$  such that  $\|F^n x - a_n\| \rightarrow 0$ . The sequence  $\{a_n\}$  has a cluster point  $a \in A$  which is then a cluster point of  $\{F^n x\}$ .

We note two important consequences of Theorem 2:

**REMARK 1.** If the mapping of Theorem 2 (or Corollary 2) has a unique fixed point  $u$  then one may conclude that  $F^n x \rightarrow u$  for every  $x \in D$  for which  $\|x - u\| < \rho$ .

**REMARK 2.** If condition (ii) of Theorem 2 holds for  $s = 1$  and if  $X$  is uniformly nonsquare then one need only verify that condition (i) holds for some  $c \in (\varepsilon_0, 2]$ .

By applying Theorem 1 to Corollary 2 we obtain:

**COROLLARY 3.** *If the hypotheses of Theorem 2 are satisfied, and*

if in addition  $A$  is compact and  $F$  has stable iterates, then the sequence  $\{F^n x\}$  has a cluster point in  $A$  for every  $x \in D$ .

REMARK 3. In a uniformly nonsquare space, for each  $c \in (\varepsilon_0, 2]$  there always exists  $s > 1$  such that  $(1 - \delta(c/s))s < 1$ .

*Proof.* Since  $c > \varepsilon_0$ ,  $\lim_{\varepsilon \rightarrow c^-} \delta(\varepsilon) > 0$ . Thus  $\lim_{s \rightarrow 1^+} (1 - \delta(c/s))s = 1 - \lim_{\varepsilon \rightarrow c^-} \delta(\varepsilon) < 1$ . Therefore, there exists  $s > 1$  such that  $(1 - \delta(c/s))s < 1$ .

THEOREM 3. Let  $D$  be a convex subset of a uniformly convex Banach space  $X$ . Let  $T: D \rightarrow D$  possess a nonempty compact fixed point set  $A$ . Suppose that there exists a neighborhood  $U$  in  $D$  of  $A$  such that if  $x \in U$  then  $\|Tx - x\| \geq cd(x, A)$  for some constant  $c \in (0, 2]$ , and such that  $T$  is continuously Fréchet differentiable in  $U$  with  $\|T'x\| \leq 1$  if  $x \in A$ . Then there exists  $\rho > 0$  such that if  $x \in D$  and  $d(x, A) < \rho$  then  $d(F^n x, A) \rightarrow 0$ .

*Proof.* By the remark above there exists  $s > 1$  such that  $(1 - \delta(c/s))s < 1$ . Let  $u \in A$ . Since  $T$  has a continuous Fréchet derivative in a neighborhood of  $u$  and  $\|T'u\| \leq 1$ , there exists a neighborhood  $U_u$  in  $D$  of  $u$  such that if  $x \in U_u$  then  $\|Tx - u\| = \|Tx - Tu\| \leq s\|x - u\|$ . Letting  $V = U \cap \bigcup_{u \in A} U_u$  and choosing  $\rho > 0$  such that if  $d(x, A) < \rho$  then  $x \in V$ , the hypotheses of Theorem 2 are satisfied. Therefore  $d(F^n x, A) \rightarrow 0$ , for each  $x \in D$  with  $d(x, A) < \rho$ .

3. Some examples. Let  $D$  be a closed convex subset of a Banach space  $X$ . We consider first mappings  $T: D \rightarrow D$  satisfying the condition

$$(1) \quad \|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \\ + c[\|x - Ty\| + \|y - Tx\|]$$

where  $a$ ,  $b$ , and  $c$  are nonnegative constants such that  $a + 2b + 2c = 1$ . In particular if  $b = c = 0$ ,  $T$  is a nonexpansive mapping, while if  $b = 1/2$ ,  $T$  is of a class of mappings investigated by Kannan [10]. A general fixed point theorem in uniformly convex spaces for mappings satisfying condition (1) has recently been proved by Goebel, Kirk, and Shimi in [8]. We now obtain the following application of Theorem 2 to mappings of this type:

THEOREM 4. Let  $D$  be a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space  $X$  and let  $T: D \rightarrow D$  be a continuous mapping satisfying condition (1) above with  $b \neq 0$ .

Then  $T$  has a unique fixed point  $u$ , and  $F^n x \rightarrow u$ , for every  $x \in D$ .

*Proof.* By the fixed point theorem of [9]  $T$  has at least one fixed point. If  $Tu = u$  and  $Tv = v$  and  $u \neq v$ , then by (1)  $\|u - v\| \leq (a + 2c)\|u - v\|$ , which implies that  $b = 0$ , a contradiction. Thus  $T$  has a unique fixed point which we denote  $u$ .

If  $x \in D$ , then since  $Tu = u$

$$(2) \quad \begin{aligned} \|Tx - u\| &\leq a\|x - u\| + b\|x - Tx\| + c[\|x - u\| + \|u - Tx\|] \\ &\leq (a + b + c)\|x - u\| + (b + c)\|u - Tx\|. \end{aligned}$$

By combining terms we obtain for every  $x \in D$

$$\|Tx - u\| \leq \|x - u\|.$$

If  $x \in D$  we have by inequality (2) above that

$$(1 - c)\|Tx - u\| \leq (a + c)\|x - u\| + b\|x - Tx\|.$$

Thus

$$\begin{aligned} (1 - c)[\|x - u\| - \|x - Tx\|] &\leq (1 - c)\|Tx - u\| \\ &\leq (a + c)\|x - u\| + b\|x - Tx\|. \end{aligned}$$

Collecting terms we obtain

$$(1 + b - c)\|x - Tx\| \geq (1 - a - 2c)\|x - u\|.$$

Since  $1 + b - c > 0$  and  $1 - a - 2c > 0$  we have for every  $x \in D$

$$\|x - Tx\| \geq \frac{1 - a - 2c}{1 + b - c} \|x - u\|.$$

The conditions of Theorem 2 are now satisfied (for  $s = 1$  and for every  $\rho > 0$ ), and thus in view of Remarks 1 and 2 above  $F^n x \rightarrow u$  for every  $x \in D$ .

As another example we consider strongly pseudo-contractive mappings. If  $D$  is a convex subset of a Banach space  $X$  and  $C \subset D$ , a mapping  $T: D \rightarrow D$  is said to be *strongly pseudo-contractive* relative to  $C$  [7] if for each  $x \in X$  and  $r > 0$  there exists a number  $\alpha_r(x) < 1$  such that  $\|x - y\| \leq \alpha_r(x) \|(1 + r)(x - y) - r(Tx - Ty)\|$ , for every  $y \in C$ . It is easily seen that if  $T$  has a fixed point  $u \in C$ , then  $u$  is the only fixed point of  $T$ . Conditions for the existence of fixed points for such mappings are given in [7]. The following theorem gives conditions under which strongly pseudo-contractive mappings satisfy condition (i) of Theorem 2.

**THEOREM 5.** *Let  $D$  be a convex subset of a Banach space  $X$*

and let  $T: D \rightarrow D$  be strongly pseudo-contractive relative to  $C$ . If  $T$  has a fixed point  $u \in C$ , and if for some  $r > 0$   $\limsup_{x \rightarrow u} \alpha_r(x) < 1$ , then there exists  $c > 0$  and an open ball  $S(u, \varepsilon)$  of radius  $\varepsilon$  about  $u$  such that if  $x \in D \cap S(u, \varepsilon)$  then  $\|x - Tx\| \geq c \|x - u\|$ .

*Proof.* Since  $\limsup_{x \rightarrow u} \alpha_r(x) < 1$ , there exists an open ball  $S(u, \varepsilon)$  of radius  $\varepsilon$  about  $u$  and a constant  $k \in (0, 1)$  such that if  $x \in D \cap S(u, \varepsilon)$  then  $\alpha_r(x) \leq k$ . Let  $c = (1 - k)/(kr)$ . Then  $(1 - \alpha_r(x))/(\alpha_r(x)r) \geq c$  for each  $x \in D \cap S(u, \varepsilon)$ . Since  $Tu = u$ , for each  $x \in D \cap S(u, \varepsilon)$

$$\begin{aligned} \|x - u\| &\leq \alpha_r(x) \|(1 + r)(x - u) - r(Tx - u)\| \\ &= \alpha_r(x) \|r(x - Tx) + (x - u)\| \\ &\leq \alpha_r(x)r \|x - Tx\| + \alpha_r(x) \|x - u\|, \end{aligned}$$

yielding

$$\frac{1 - \alpha_r(x)}{\alpha_r(x)r} \|x - u\| \leq \|x - Tx\|.$$

Thus

$$c \|x - u\| \leq \|x - Tx\|$$

for every  $x \in D \cap S(u, \varepsilon)$ .

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Received December 18, 1972 and in revised form March 1, 1974. This paper is essentially a portion of the author's University of Iowa doctoral dissertation, written under the supervision of Professor W. A. Kirk.

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