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**ITERATIVE TECHNIQUES FOR APPROXIMATION OF FIXED
POINTS OF CERTAIN NONLINEAR MAPPINGS IN BANACH
SPACES**

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ITERATIVE TECHNIQUES FOR APPROXIMATION OF FIXED POINTS OF CERTAIN NONLINEAR MAPPINGS IN BANACH SPACES

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Let D be a closed convex subset of a Banach space X , let $T: D \rightarrow D$ be *nonexpansive* (that is, $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in D$), and let $F_\lambda = \lambda T + (1 - \lambda)I$, where $\lambda \in (0, 1)$ and I denotes the identity on D . Several authors have found conditions under which the sequences of iterates $\{T^n x\}$, or the sequences $\{F_\lambda^n x\}$, converge strongly or weakly to fixed points of T for all $x \in D$. In this paper we establish conditions under which the sequences $\{F_{1/2}^n x\}$ converge strongly to fixed points of T for all x in a neighborhood of the fixed point set of T ; furthermore, our theorems hold for classes of mappings T more general than the class of nonexpansive mappings.

We complement these results by proving theorems under which local convergence of iterates entails global convergence; thus by combining our results in these two areas we obtain new theorems regarding the global convergence of iterates. Finally, we give an example of a class of mappings satisfying the various conditions of our theorems.

1. Local and global convergence of iterates. Let D be a convex subset of the Banach space X , and let $T: D \rightarrow D$. Adopting the terminology of Furi and Vignoli [6] we say that the sequence $\{T^n x_0\}$ of iterates of $x_0 \in D$ is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T^n x - T^n x_0\| < \varepsilon$ for every $n = 1, 2, \dots$ whenever $x \in D$ and $\|x - x_0\| < \delta$. We say that T has *stable iterates* if the sequence $\{T^n x\}$ of iterates of x is stable for every $x \in D$. Finally, if $x \in X$ and $B \subset X$ we define $d(x, B) = \inf \{\|x - y\|: y \in B\}$.

THEOREM 1. *Let D be a convex subset of a Banach space X and suppose that $T: D \rightarrow D$ has stable iterates. Let A be a nonempty subset of D .*

(i) *If there exists $\rho > 0$ such that $\{T^n x\}$ has a cluster point in A whenever $x \in D$ and $d(x, A) < \rho$, then $\{T^n x\}$ has a cluster point in A for every $x \in D$.*

(ii) *If there exists $\rho > 0$ such that $\{T^n x\}$ has its limit in A whenever $x \in D$ and $d(x, A) < \rho$, then $\{T^n x\}$ converges to some point of A for every $x \in D$.*

Proof. To prove the first statement, let $x \in D$ and $x_0 \in A$. For

each $\lambda \in [0, 1]$ let $y_\lambda = \lambda x + (1 - \lambda)x_0$ and set $\lambda_0 = \sup \{\lambda \in [0, 1]: \{T^n y_\lambda\} \text{ has a cluster point in } A\}$. Let δ correspond to $\varepsilon = \rho/3$ in the definition of the stability of $\{T^n y_{\lambda_0}\}$, and choose $\lambda_1 \in [0, \lambda_0]$ such that $\|y_{\lambda_1} - y_{\lambda_0}\| < \delta$ and $\{T^n y_{\lambda_1}\}$ has a cluster point in A . If $\lambda_0 = 1$, let $\lambda_2 = \lambda_0$; if $\lambda_0 < 1$, let $\lambda_2 \in (\lambda_0, 1]$ be such that $\|y_{\lambda_2} - y_{\lambda_0}\| < \delta$. Since there exists a cluster point w in A of $\{T^n y_{\lambda_1}\}$ and a positive integer N such that $\|T^N y_{\lambda_1} - w\| < \rho/3$, we have that

$$\begin{aligned} \|T^N y_{\lambda_2} - w\| &\leq \|T^N y_{\lambda_2} - T^N y_{\lambda_0}\| + \|T^N y_{\lambda_1} - T^N y_{\lambda_0}\| + \|T^N y_{\lambda_1} - w\| \\ &< \rho/3 + \rho/3 + \rho/3 = \rho. \end{aligned}$$

Thus $d(T^N y_{\lambda_2}, A) \leq \|T^N y_{\lambda_2} - w\| < \rho$, entailing that $\{T^{N+n} y_{\lambda_2}\}$ —and hence $\{T^n y_{\lambda_2}\}$ —has a cluster point in A . If $\lambda_0 < 1$, this contradicts the definition of λ_0 ; thus $\lambda_0 = 1$, and since in this case $y_{\lambda_2} = x$, we have that $\{T^n x\}$ has a cluster point in A .

To prove the second statement, we let $x \in D$ and note that by our proof of the first statement $\{T^n x\}$ has a cluster point $w \in A$. Thus there exists a positive integer N such that $\|T^N x - w\| < \rho$, implying that $T^{N+n} x \rightarrow w$, whence $T^n x \rightarrow w$.

We remark that in the case of the second statement of the theorem above, A must contain a fixed point of T , since if T is continuous the limit of a sequence $\{T^n x\}$ is necessarily a fixed point. In our applications of this theorem we will assume either that A is the fixed point set of T or that A is a singleton.

COROLLARY 1. *Let D be a convex subset of a Banach space X , and let $T: D \rightarrow D$ possess stable iterates. Let x_0 be a fixed point of T for which there exists an open neighborhood U of x_0 , $U \subset D$, such that T is continuously Fréchet differentiable in U and $\|T'x_0\| < 1$. Then $T^n x \rightarrow x_0$, for every $x \in D$.*

Proof. Since T is continuously Fréchet differentiable in U and $\|T'x_0\| < 1$, there exists a constant $k \in (0, 1)$ and an open ball $S(x_0, \rho)$ about x_0 with radius ρ , $S(x_0, \rho) \subset U$, such that if $x \in S(x_0, \rho)$ then $\|T'z\| < k$. Let $y \in S(x_0, \rho)$. Then there exists a point z in the segment from x_0 to y such that (see Fréchet [5])

$$\|Tx_0 - Ty\| \leq \|T'z\| \|x_0 - y\|.$$

But $z \in S(x_0, \rho)$ so that $\|T'z\| < k$. Thus for every $y \in S(x_0, \rho)$

$$\|Tx_0 - Ty\| \leq k \|x_0 - y\|.$$

By induction, $\|x_0 - T^n y\| \leq k^n \|x_0 - y\|$ for every $n = 1, 2, 3, \dots$. Since $k^n \rightarrow 0$, $T^n y \rightarrow x_0$ for every $y \in S(x_0, \rho)$. By part (ii) of Theorem 1, $T^n x \rightarrow x_0$ for every $x \in D$.

2. Conditions implying local convergence of iterates. The modulus of convexity of a Banach space X is the function $\delta: [0, 2] \rightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \text{ and } \|x - y\| \geq \varepsilon \right\}.$$

It is well-known (cf. [9]) that δ is nondecreasing and continuous except possibly at 2. Furthermore, letting $\varepsilon_0 = \sup \{\varepsilon \in [0, 2] : \delta(\varepsilon) = 0\}$, X is uniformly convex if and only if $\varepsilon_0 = 0$, X is uniformly non-square if and only if $\varepsilon_0 < 2$, and X is strictly convex if and only if $\delta(2) = 1$.

We observe that if $x, y \in X$ satisfy the conditions

$$\|x\| \leq d, \|y\| \leq d, \text{ and } \|x - y\| \geq \varepsilon, \text{ then}$$

$$\left\| \frac{x + y}{2} \right\| \leq \left(1 - \delta\left(\frac{\varepsilon}{d}\right) \right) d.$$

Finally, we denote by I the identity mapping on any convex subset of X .

THEOREM 2. *Let D be a convex subset of a uniformly nonsquare Banach space X . Suppose that $T: D \rightarrow D$ has a nonempty fixed point set A and that T satisfies the following conditions: There exist $\rho > 0$, $c > 0$, and $s \geq 1$ with $(1 - \delta(c/s))s < 1$ such that if $x \in D$ and $d(x, A) < \rho$ then*

- (i) $\|Tx - x\| \geq cd(x, A)$, and
- (ii) $\|Tx - u\| \leq s\|x - u\|$ for every $u \in A$.

Then setting $F = 1/2(I + T)$, $d(F^n x, A) \rightarrow 0$ for every $x \in D$ for which $d(x, A) < \rho$.

Proof. We observe that if $x \notin A$ then

$$cd(x, A) \leq \|Tx - x\| \leq \|Tx - u\| + \|x - u\| \leq (1 + s)\|x - u\|$$

for every $u \in A$. Thus $cd(x, A) \leq (1 + s)d(x, A)$, so that if T is not the identity then $c \leq 1 + s$. Therefore $c/s \leq 1 + 1/s \leq 2$, and moreover if $c/s = 2$, then $s = 1$ and $c = 2$.

Let $x \in D$ satisfy $0 < d(x, A) < \rho$, and for arbitrary $r > 1$ let $u_{x,r} \in A$ satisfy $\|x - u_{x,r}\| \leq \text{minimum}\{\rho, rd(x, A)\}$. Thus $\|Tx - u_{x,r}\| \leq s\|x - u_{x,r}\|$.

Let $d = s\|x - u_{x,r}\|$ and $\varepsilon = \|Tx - x\|$. Since $\|x - u_{x,r}\| \leq d$, $\|Tx - u_{x,r}\| \leq d$, and $\|(x - u_{x,r}) - (Tx - u_{x,r})\| = \varepsilon$ we obtain

$$\|Fx - u_{x,r}\| = \frac{1}{2} \|(x - u_{x,r}) + (Tx - u_{x,r})\|$$

$$\leq (1 - \delta(\varepsilon/d))d.$$

Now

$$\frac{\varepsilon}{\bar{d}} = \frac{\|Tx - x\|}{s\|u_{x,r} - x\|} \geq \frac{cd(x, A)}{srd(x, A)} = \frac{c}{sr},$$

and thus since δ is nondecreasing

$$1 - \delta(\varepsilon/d) \leq 1 - \delta\left(\frac{c}{sr}\right).$$

Therefore,

$$\begin{aligned} d(Fx, A) &\leq \|Fx - u_{x,r}\| \leq (1 - \delta(\varepsilon/d))d \leq \left(1 - \delta\left(\frac{c}{sr}\right)\right)d \\ &= \left(1 - \delta\left(\frac{c}{sr}\right)\right)s\|x - u_{x,r}\| \leq \left(1 - \delta\left(\frac{c}{sr}\right)\right)srd(x, A), \end{aligned}$$

for every $r > 1$.

Let $\eta \equiv \lim_{r \rightarrow 1^+} (1 - \delta(c/sr))sr$. Then $d(Fx, A) \leq \eta d(x, A)$ whenever $d(x, A) < \rho$. If $c/s < 2$ then δ is continuous at c/s and $\eta = (1 - \delta(c/s))s < 1$. If $c/s = 2$ then $c = 2$ and $s = 1$, and since X is uniformly nonsquare, $\eta = 1 - \lim_{\varepsilon \rightarrow 2^-} \delta(\varepsilon) < 1$. By induction, $d(F^n x, A) \leq \eta^n d(x, A)$ whenever $d(x, A) < \rho$, implying that $d(F^n x, A) \rightarrow 0$ whenever $d(x, A) < \rho$.

COROLLARY 2. *If the hypotheses of Theorem 2 are satisfied and if in addition A is compact, then the sequence $\{F^n x\}$ has a cluster point in A whenever $d(x, A) < \rho$.*

Proof. Since whenever $d(x, A) < \rho$ we have $d(F^n x, A) \rightarrow 0$, we can select a sequence $\{a_n\} \subset A$ such that $\|F^n x - a_n\| \rightarrow 0$. The sequence $\{a_n\}$ has a cluster point $a \in A$ which is then a cluster point of $\{F^n x\}$.

We note two important consequences of Theorem 2:

REMARK 1. If the mapping of Theorem 2 (or Corollary 2) has a unique fixed point u then one may conclude that $F^n x \rightarrow u$ for every $x \in D$ for which $\|x - u\| < \rho$.

REMARK 2. If condition (ii) of Theorem 2 holds for $s = 1$ and if X is uniformly nonsquare then one need only verify that condition (i) holds for some $c \in (\varepsilon_0, 2]$.

By applying Theorem 1 to Corollary 2 we obtain:

COROLLARY 3. *If the hypotheses of Theorem 2 are satisfied, and*

if in addition A is compact and F has stable iterates, then the sequence $\{F^n x\}$ has a cluster point in A for every $x \in D$.

REMARK 3. In a uniformly nonsquare space, for each $c \in (\varepsilon_0, 2]$ there always exists $s > 1$ such that $(1 - \delta(c/s))s < 1$.

Proof. Since $c > \varepsilon_0$, $\lim_{\varepsilon \rightarrow c^-} \delta(\varepsilon) > 0$. Thus $\lim_{s \rightarrow 1^+} (1 - \delta(c/s))s = 1 - \lim_{\varepsilon \rightarrow c^-} \delta(\varepsilon) < 1$. Therefore, there exists $s > 1$ such that $(1 - \delta(c/s))s < 1$.

THEOREM 3. Let D be a convex subset of a uniformly convex Banach space X . Let $T: D \rightarrow D$ possess a nonempty compact fixed point set A . Suppose that there exists a neighborhood U in D of A such that if $x \in U$ then $\|Tx - x\| \geq cd(x, A)$ for some constant $c \in (0, 2]$, and such that T is continuously Fréchet differentiable in U with $\|T'x\| \leq 1$ if $x \in A$. Then there exists $\rho > 0$ such that if $x \in D$ and $d(x, A) < \rho$ then $d(F^n x, A) \rightarrow 0$.

Proof. By the remark above there exists $s > 1$ such that $(1 - \delta(c/s))s < 1$. Let $u \in A$. Since T has a continuous Fréchet derivative in a neighborhood of u and $\|T'u\| \leq 1$, there exists a neighborhood U_u in D of u such that if $x \in U_u$ then $\|Tx - u\| = \|Tx - Tu\| \leq s\|x - u\|$. Letting $V = U \cap \bigcup_{u \in A} U_u$ and choosing $\rho > 0$ such that if $d(x, A) < \rho$ then $x \in V$, the hypotheses of Theorem 2 are satisfied. Therefore $d(F^n x, A) \rightarrow 0$, for each $x \in D$ with $d(x, A) < \rho$.

3. Some examples. Let D be a closed convex subset of a Banach space X . We consider first mappings $T: D \rightarrow D$ satisfying the condition

$$(1) \quad \|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|)$$

where a , b , and c are nonnegative constants such that $a + 2b + 2c = 1$. In particular if $b = c = 0$, T is a nonexpansive mapping, while if $b = 1/2$, T is of a class of mappings investigated by Kannan [10]. A general fixed point theorem in uniformly convex spaces for mappings satisfying condition (1) has recently been proved by Goebel, Kirk, and Shimi in [8]. We now obtain the following application of Theorem 2 to mappings of this type:

THEOREM 4. Let D be a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space X and let $T: D \rightarrow D$ be a continuous mapping satisfying condition (1) above with $b \neq 0$.

Then T has a unique fixed point u , and $F^n x \rightarrow u$, for every $x \in D$.

Proof. By the fixed point theorem of [9] T has at least one fixed point. If $Tu = u$ and $Tv = v$ and $u \neq v$, then by (1) $\|u - v\| \leq (a + 2c)\|u - v\|$, which implies that $b = 0$, a contradiction. Thus T has a unique fixed point which we denote u .

If $x \in D$, then since $Tu = u$

$$(2) \quad \begin{aligned} \|Tx - u\| &\leq a\|x - u\| + b\|x - Tx\| + c[\|x - u\| + \|u - Tx\|] \\ &\leq (a + b + c)\|x - u\| + (b + c)\|u - Tx\|. \end{aligned}$$

By combining terms we obtain for every $x \in D$

$$\|Tx - u\| \leq \|x - u\|.$$

If $x \in D$ we have by inequality (2) above that

$$(1 - c)\|Tx - u\| \leq (a + c)\|x - u\| + b\|x - Tx\|.$$

Thus

$$\begin{aligned} (1 - c)[\|x - u\| - \|x - Tx\|] &\leq (1 - c)\|Tx - u\| \\ &\leq (a + c)\|x - u\| + b\|x - Tx\|. \end{aligned}$$

Collecting terms we obtain

$$(1 + b - c)\|x - Tx\| \geq (1 - a - 2c)\|x - u\|.$$

Since $1 + b - c > 0$ and $1 - a - 2c > 0$ we have for every $x \in D$

$$\|x - Tx\| \geq \frac{1 - a - 2c}{1 + b - c} \|x - u\|.$$

The conditions of Theorem 2 are now satisfied (for $s = 1$ and for every $\rho > 0$), and thus in view of Remarks 1 and 2 above $F^n x \rightarrow u$ for every $x \in D$.

As another example we consider strongly pseudo-contractive mappings. If D is a convex subset of a Banach space X and $C \subset D$, a mapping $T: D \rightarrow D$ is said to be *strongly pseudo-contractive* relative to C [7] if for each $x \in X$ and $r > 0$ there exists a number $\alpha_r(x) < 1$ such that $\|x - y\| \leq \alpha_r(x)\|(1 + r)(x - y) - r(Tx - Ty)\|$, for every $y \in C$. It is easily seen that if T has a fixed point $u \in C$, then u is the only fixed point of T . Conditions for the existence of fixed points for such mappings are given in [7]. The following theorem gives conditions under which strongly pseudo-contractive mappings satisfy condition (i) of Theorem 2.

THEOREM 5. *Let D be a convex subset of a Banach space X*

and let $T: D \rightarrow D$ be strongly pseudo-contractive relative to C . If T has a fixed point $u \in C$, and if for some $r > 0$ $\limsup_{x \rightarrow u} \alpha_r(x) < 1$, then there exists $c > 0$ and an open ball $S(u, \varepsilon)$ of radius ε about u such that if $x \in D \cap S(u, \varepsilon)$ then $\|x - Tx\| \geq c \|x - u\|$.

Proof. Since $\limsup_{x \rightarrow u} \alpha_r(x) < 1$, there exists an open ball $S(u, \varepsilon)$ of radius ε about u and a constant $k \in (0, 1)$ such that if $x \in D \cap S(u, \varepsilon)$ then $\alpha_r(x) \leq k$. Let $c = (1 - k)/(kr)$. Then $(1 - \alpha_r(x))/(\alpha_r(x)r) \geq c$ for each $x \in D \cap S(u, \varepsilon)$. Since $Tu = u$, for each $x \in D \cap S(u, \varepsilon)$

$$\begin{aligned} \|x - u\| &\leq \alpha_r(x) \|(1 + r)(x - u) - r(Tx - u)\| \\ &= \alpha_r(x) \|r(x - Tx) + (x - u)\| \\ &\leq \alpha_r(x)r \|x - Tx\| + \alpha_r(x) \|x - u\|, \end{aligned}$$

yielding

$$\frac{1 - \alpha_r(x)}{\alpha_r(x)r} \|x - u\| \leq \|x - Tx\|.$$

Thus

$$c \|x - u\| \leq \|x - Tx\|$$

for every $x \in D \cap S(u, \varepsilon)$.

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