

Pacific Journal of Mathematics

AN ELEMENTARY PROOF OF THE LIFTING THEOREM

TIM EDEN TRAYNOR

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An elementary proof is given of the lifting theorem for a complete totally finite measure space, which does not use the martingale theorem or Vitali differentiation.

Introduction. In this paper we give a proof of the lifting theorem for a complete totally finite measure space, which involves only elementary properties of measure. The complicated isomorphism theorem of Maharam's original proof [4] is avoided. On the other hand, we do not use the concepts of martingale or of Vitali differentiation ([1] [2] [3] [5]). In fact, the entire construction takes place in the σ -field of measurable sets, without passing to the algebra of essentially bounded measurable functions. We feel this makes it easier to see what is involved.

Throughout what follows:

(S, \mathcal{M}, μ) is a complete measure space with $\mu(S) < \infty$;

$\mathcal{N} = \{A \in \mathcal{M} : \mu(A) = 0\}$;

N is the set of nonnegative integers;

For subsets A, B of S ,

$$\begin{aligned} AB &= A \cap B; \\ A \setminus B &= \{s \in A : s \notin B\}; \\ A^c &= S \setminus A; \\ A \Delta B &= AB^c \cup BA^c; \\ A \doteq B &\text{ iff } A, B \in \mathcal{M} \text{ and } \mu(A \Delta B) = 0. \end{aligned}$$

For a family \mathcal{H} of subsets of S ,

$$\bigcup \mathcal{H} = \bigcup_{E \in \mathcal{H}} E.$$

1. DEFINITIONS. For any field $\mathcal{A} \subset \mathcal{M}$,

(1) d is a (lower) density on \mathcal{A} iff d is a mapping on \mathcal{A} to \mathcal{N} such that, for A, B in \mathcal{A} ,

- (i) $d(A) \doteq A$;
- (ii) $A \doteq B$ implies $d(A) = d(B)$;
- (iii) $d(\emptyset) = \emptyset$, $d(S) = S$;
- (iv) $d(AB) = d(A)d(B)$.

(2) l is a lifting on \mathcal{A} iff l is a density on \mathcal{A} such that

- (v) $l(A^c) = l(A)^c$, for A in \mathcal{A} .

For a detailed study of liftings and their applications, we refer

to A. and C. Ionescu Tulcea [3].

2. REMARKS. Let l be a lifting on the σ -field $\mathcal{A} \subset \mathcal{M}$ and $\mathcal{F} = l|\mathcal{A}$. Then:

- (1) \mathcal{F} is a field in S .
- (2) $\mathcal{F} \subset \{E \in \mathcal{A} : 0 < \mu(E) < \mu(S)\} \cup \{\emptyset, S\}$.
- (3) If, for each n in N , $E_n \in \mathcal{F}$, and $A = \bigcup_n E_n$, then $l(A) \supset A$. (Indeed, for each n , $E_n \setminus l(A) \subset A \setminus l(A) \doteq \emptyset$, so $E_n \setminus l(A) = \emptyset$, by (2).)

3. THEOREM. If d is a density on a field \mathcal{A} with $\mathcal{N} \subset \mathcal{A} \subset \mathcal{M}$, then there exists a lifting l on \mathcal{A} , with

$$(*) \quad d(A) \subset l(A) \subset d(A^c), \quad \text{for } A \text{ in } \mathcal{A}.$$

Proof. For each filterbase $\mathcal{B} \subset \mathcal{A}$, let $\hat{\mathcal{B}}$ denote an ultrafilter containing \mathcal{B} . We recall that for subsets A, B of S ,

- (a) $A \in \hat{\mathcal{B}}$ iff $A^c \notin \hat{\mathcal{B}}$, and
- (b) $A \cap B \in \hat{\mathcal{B}}$ iff $A \in \hat{\mathcal{B}}$ and $B \in \hat{\mathcal{B}}$.

For each s in S , let

$$\mathcal{F}(s) = \{A \in \mathcal{A} : s \in d(A)\}.$$

Since d is a density, $\mathcal{F}(s)$ is a filterbase. Put

$$l(A) = \{s \in S : A \in \hat{\mathcal{F}}(s)\}, \quad \text{for } A \text{ in } \mathcal{A}.$$

By the properties (a), (b) of an ultrafilter, for A, B in \mathcal{A} , we have (v) $l(A^c) = l(A)^c$ and (iv) $l(AB) = l(A)l(B)$. Moreover, if $s \in d(A)$, then $A \in \mathcal{F}(s) \subset \hat{\mathcal{F}}(s)$, so that $s \in l(A)$. Hence, $d(A) \subset l(A)$. Similarly $d(A^c) \subset l(A^c)$. Using (v) we find that (*) holds. Since $d(A) \doteq A \doteq d(A^c)^c$, we have (i) $l(A) \doteq A$. If $N \doteq \emptyset$, then $d(N) = d(\emptyset) = \emptyset$ and $d(N^c) = d(S) = S$, so that, by (*), $l(N) = \emptyset$. Hence, (iii) $l(\emptyset) = \emptyset$, $l(S) = S$ and (ii) if $A \doteq B$, then $l(A)l(B) = l(A \Delta B) = \emptyset$, so that $l(A) = l(B)$. This completes the proof.

The proof of the following theorem usually uses martingales or Vitali differentiation. We use neither. However, the reader familiar with Sion [5] will recognize the connection with his method. (See Remark 7 below.)

4. THEOREM. Suppose that, for each n in N , \mathcal{A}_n is a σ -field with $\mathcal{N} \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \mathcal{M}$ and l_n is a lifting on \mathcal{A}_n with $l_n = l_{n+1}|_{\mathcal{A}_n}$. Put $\mathcal{A} = \sigma\text{-field}(\bigcup_n \mathcal{A}_n)$. Then there is a lifting l on \mathcal{A} with $l_n = l|_{\mathcal{A}_n}$, for each n in N .

Proof. The result will follow immediately from Theorem 3 if we can construct a density d on \mathcal{A} with $d(A) = l_n(A)$ for A in \mathcal{A}_n . To this end, for each k in N , let \mathcal{F}_k denote $l_k[\mathcal{A}_k]$. For each A in \mathcal{A} , k in N , and $r < 1$, put

$$\begin{aligned} \mathcal{D}(A; k, r) &= \{E \in \mathcal{F}_k: \mu(AF) \geq r\mu(F), \text{ whenever } E \supset F \in \mathcal{F}_k\}, \\ d(A; k, r) &= \bigcup \mathcal{D}(A; k, r), \text{ and} \\ d(A) &= \bigcap_{r < 1} \bigcup_{n \in N} \bigcap_{k \geq n} d(A; k, r). \end{aligned}$$

We will show that d is a suitable density function on \mathcal{A} .

For fixed A, r , and k , let \mathcal{H} be a maximal disjoint subfamily of $\mathcal{D}(A; k, r)$. Then \mathcal{H} is countable. Put $B = l_k(\cup \mathcal{H})$. Clearly, $B \in \mathcal{D}(A; k, r)$. Moreover, if $E \in \mathcal{D}(A; k, r)$, $E \setminus B = \emptyset$, by Remark 2(3) and the maximality of \mathcal{H} . This shows that $d(A; k, r) = B$ is the largest element of $\mathcal{D}(A; k, r)$. In particular, $d(A; k, r) \in \mathcal{F}_k \subset \mathcal{A}$. If $r < s < 1$, we have $d(A; k, r) \supset d(A; k, s)$, so we need only consider rational r . Since \mathcal{A} is a σ -field, we conclude that $d(A) \in \mathcal{A}$.

There is no difficulty in showing that $A \doteq B \in \mathcal{A}$ implies $d(A) = d(B)$, or that $d(A) = l_n(A)$, for A in \mathcal{A}_n . In particular, $d(\emptyset) = \emptyset$ and $d(S) = S$. We have left to check conditions (i) and (iv) of the definition of a density.

To check condition (iv), let $A, B \in \mathcal{A}$, $k \in N$, $r < 1$. For each F in \mathcal{F}_k contained in $d(A; k, (r + 1)/2) \cap d(B; k, (r + 1)/2)$, we have

$$\begin{aligned} \mu(ABF) &= \mu(AF) + \mu(BF) - \mu((A \cup B)F) \\ &\geq ((r + 1)/2)\mu(F) + ((r + 1)/2)\mu(F) - \mu(F) \\ &= r\mu(F). \end{aligned}$$

Hence, $d(A; k, (r + 1)/2) \cap d(B; k, (r + 1)/2) \subset d(AB; k, r)$. By direct computation, this yields $d(A)d(B) \subset d(AB)$. On the other hand, for each k and r , $d(AB; k, r) \subset d(A; k, r) \cap d(B; k, r)$, so that $d(AB) \subset d(A)d(B)$, establishing (iv).

To verify condition (i), let $A \in \mathcal{A}$ and put

$$d'(A) = \bigcup_{0 < r < 1} \bigcap_{n \in N} \bigcup_{k \geq n} d(A; k, r).$$

We will show that

- (a) $d'(A)A^c \doteq \emptyset$,
- (b) $Ad'(A^c) \doteq \emptyset$, and
- (c) $d'(A^c) \supset d(A)^c$, $d'(A) \supset d(A)$,

from which we get

$$d(A)\Delta A = d(A)A^c \cup Ad'(A^c) \subset d'(A)A^c \cup Ad'(A^c) \doteq \emptyset,$$

as required.

Fix r in $(0, 1)$ and write $D_k = d(A; k, r)$, for k in N . Since $D_k \in \mathcal{D}(A; k, r)$, we have for each B in \mathcal{A}_k ,

$$\mu(ABD_k) = \mu(AI_k(B)D_k) \geq r\mu(I_k(B)D_k) = r\mu(BD_k).$$

Suppose $B \in \bigcup_n \mathcal{A}_n$. Then there exists an n in N such that $B \in \mathcal{A}_n$. For $m \geq n$, $\mathcal{A}_m \supset \mathcal{A}_n$, and putting $C_m = BD_m \setminus \bigcup_{n \leq k < m} D_k$, we have

$$\begin{aligned} \mu(AB \bigcup_{k \geq n} D_k) &= \sum_{m \geq n} \mu(AC_m) \\ &\geq \sum_{m \geq n} r\mu(C_m) \\ &= r\mu(B \bigcup_{k \geq n} D_k). \end{aligned}$$

Taking intersections over n we have

$$\mu(AB \bigcap_n \bigcup_{k \geq n} D_k) \geq r\mu(B \bigcap_n \bigcup_{k \geq n} D_k).$$

By considering monotone sequences of such B we see that this holds for all B in \mathcal{A} , the σ -field generated by the field $\bigcup_n \mathcal{A}_n$. In particular, putting $B = A^c$ we have $0 \geq r\mu(A^c \bigcap_n \bigcup_{k \geq n} D_k)$. But $r > 0$, so $\mu(A^c \bigcap_n \bigcup_{k \geq n} D_k) = 0$. Taking the union over rational r in $(0, 1)$ we have $A^c d'(A) \doteq \emptyset$. This proves (a). Replacing A by A^c we have (b).

To prove (c) we let $k \in N$, $0 < r < 1$ and show

$$d(A; k, r)^c \subset d(A^c; k, 1 - r).$$

To this end suppose $\emptyset \neq E \in \mathcal{F}_k$ and $E \subset d(A; k, r)^c$. Then $E \notin \mathcal{D}(A; k, r)$, so there exists F in \mathcal{F}_k contained in E with $\mu(AF) < r\mu(F)$. Let \mathcal{H} be a maximal disjoint collection of such F . By Remark 2(3) and maximality of \mathcal{H} we have $E \setminus I_k(\bigcup \mathcal{H}) = \emptyset$, so $E = I_k(\bigcup \mathcal{H})$. Moreover, $\mu(AE) = \sum_{F \in \mathcal{H}} \mu(AF) \leq \sum_{F \in \mathcal{H}} r\mu(F) = r\mu(E)$. In other words, $\mu(A^c E) \geq (1 - r)\mu(E)$. This shows that $d(A; k, r)^c \in \mathcal{D}(A^c; k, 1 - r)$, so $d(A; k, r)^c \subset d(A^c; k, 1 - r)$. Hence,

$$\begin{aligned} d(A)^c &= \bigcup_{r \in (0,1)} \bigcap_n \bigcup_{k \geq n} d(A; k, r)^c \\ &\subset \bigcup_{r \in (0,1)} \bigcap_n \bigcup_{k \geq n} d(A^c; k, 1 - r) \\ &= d'(A^c). \end{aligned}$$

Since it is clear that $d(A) \subset d'(A)$, this proves (c) and completes the proof of the theorem.

To prove the lifting theorem, we need one more lemma, due to A. and C. Ionescu Tulcea [2]. For completeness, we include a proof here.

5. LEMMA. *Let \mathcal{A} be a σ -field with $\mathcal{N} \subset \mathcal{A} \subset \mathcal{M}$, l a lifting*

on \mathcal{A} . If $A \in \mathcal{M} \setminus \mathcal{A}$ and $\mathcal{A}' = \text{field}(\mathcal{A} \cup \{A\})$, then there exists a lifting on \mathcal{A}' extending l .

Proof. Let $\mathcal{F} = \mathcal{U}[\mathcal{A}]$, $\mathcal{E} = \{E \in \mathcal{F} : \mu(EA^c) = 0\}$. Let \mathcal{K} be a maximal disjoint subfamily of \mathcal{E} and let $A_1 = \mathcal{U}(\mathcal{K})$. Then $A_1 \in \mathcal{E}$ and, by maximality of \mathcal{K} and Remark 2(3), $E \setminus A_1 = \emptyset$, for all E in \mathcal{E} , so that A_1 is the largest element of \mathcal{E} . Similarly, let A_2 be the largest E in \mathcal{F} with $\mu(EA) = 0$. Put $\bar{A} = (A \cup A_1) \setminus A_2$. Then $\bar{A} \doteq A$. (Indeed, $\bar{A} \Delta A \subset A_1 A^c \cup A_2 A \doteq \emptyset$.) Thus, $\mathcal{A}' = \text{field}(\mathcal{A} \cup \{\bar{A}\}) (= \{(C\bar{A} \cup D\bar{A}^c : C, D \in \mathcal{A})\})$. For E, F in \mathcal{F} ,

- (a) $E\bar{A} \doteq F\bar{A}$ implies $E\bar{A} = F\bar{A}$, and
- (b) $E\bar{A}^c \doteq F\bar{A}^c$ implies $E\bar{A}^c = F\bar{A}^c$.

Indeed, $E\bar{A} \doteq F\bar{A}$ implies $\mu((E\Delta F)A) = \mu((E\Delta F)\bar{A}) = 0$, so that, by definition of A_2 , $E\Delta F \subset A_2 \subset \bar{A}^c$. Thus, $(E\Delta F)\bar{A} = \emptyset$, so $E\bar{A} = F\bar{A}$. The proof of (b) is similar.

Now define l' on \mathcal{A}' by

$$l'(C\bar{A} \cup D\bar{A}^c) = \mathcal{U}(C)\bar{A} \cup \mathcal{U}(D)\bar{A}^c, \text{ for } C, D \text{ in } \mathcal{A}.$$

Using (a) and (b) we see that l' is well-defined and that for M_1, M_2 in \mathcal{A}' , $M_1 \doteq M_2$ implies $l'(M_1) = l'(M_2)$. The other properties of a lifting are easily verified. Moreover, for C in \mathcal{A} , $l'(C) = \mathcal{U}(C)\bar{A} \cup \mathcal{U}(C)\bar{A}^c = \mathcal{U}(C)$, so l' extends l .

We can now prove the lifting theorem:

6. THEOREM. Let (S, \mathcal{M}, μ) be a complete measure space with $\mu(S) < \infty$. Then, there exists a lifting on \mathcal{M} .

Proof. Let \mathcal{H} be the set of pairs (\mathcal{A}, l) where \mathcal{A} is a σ -field with $\mathcal{N} \subset \mathcal{A} \subset \mathcal{M}$ and l is a lifting on \mathcal{A} , with the ordering: $(\mathcal{A}, l) \leq (\mathcal{A}', l')$ iff $\mathcal{A} \subset \mathcal{A}'$ and $l = l' \upharpoonright \mathcal{A}$. We show that \mathcal{H} has a maximal element. Indeed, suppose $\mathcal{H}' = \{(\mathcal{A}_i, l_i) : i \in I\}$ is a totally ordered subfamily of \mathcal{H} . We distinguish two cases:

(a) If \mathcal{H}' has no countable cofinal subfamily, put $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ and $l(A) = l_i(A)$, for A in \mathcal{A}_i , i in I . Then (\mathcal{A}, l) is an upper bound for \mathcal{H}' in \mathcal{H} .

(b) If \mathcal{H}' has a countable cofinal subfamily $\mathcal{H}'' = \{(\mathcal{A}_{i_n}, l_{i_n}) : n \in \mathbb{N}\}$, then by Theorem 4, \mathcal{H}''' (and hence \mathcal{H}') has an upper bound in \mathcal{H} . By Zorn's lemma, we conclude that \mathcal{H} has a maximal element, (\mathcal{A}, l) .

By Lemma 3, and maximality, $\mathcal{A} = \mathcal{M}$, and the theorem is proved.

7. REMARKS.

(1) To see the relationship of our method to that of Sion [5],

for each k in N and s in S , let $\widehat{\mathcal{F}}_k(s) = \{F \in \mathcal{F}_k : s \in F\}$, directed downward by inclusion. Then,

$$d(A; k, r) = l_k \left(\left\{ s \in S : \lim_{F \in \widehat{\mathcal{F}}_k(s)} \frac{\mu(AF)}{\mu(F)} \geq r \right\} \right).$$

(One inclusion is obvious, the other follows from Sion's Theorem 2'.)

(2) As several authors have pointed out (see, for example, Sion [5], and for more references, Sion [6]), liftings provide very special Vitali differentiation system, even when no others are available. (If l is a lifting on \mathcal{M} , such a system is obtained by assigning to each s in S , $\{F : s \in F \in l[\mathcal{M}]\}$, directed downward by inclusion.) Apart from our desire for an elementary proof, this was our main motivation in looking for a construction of a lifting without using differentiation concepts.

(3) *Added in proof.* S. Graf [On the existence of strong liftings in second countable topological spaces, (to appear)] has noticed that one may change the word "lifting" to "density" in the statement of Theorem 4. The proof is essentially contained in our proof. Graf has independently obtained a proof of this result (using Radon-Nikodým derivatives).

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Received August 1, 1973 and in revised form October 3, 1973. This work was supported by the Summer Research Institute of the Canadian Mathematical Congress.

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Pacific Journal of Mathematics

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