AN ELEMENTARY PROOF OF THE LIFTING THEOREM

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An elementary proof is given of the lifting theorem for
a complete totally finite measure space, which does not use
the martingale theorem or Vitali differentiation.

Introduction. In this paper we give a proof of the lifting
theorem for a complete totally finite measure space, which involves
only elementary properties of measure. The complicated isomorphism
theorem of Maharam's original proof [4] is avoided. On the other
hand, we do not use the concepts of martingale or of Vitali differ-
entiation ([1][2][3][5]). In fact, the entire construction takes place
in the σ-field of measurable sets, without passing to the algebra of
essentially bounded measurable functions. We feel this makes it
easier to see what is involved.

Throughout what follows:

(S, ℳ, μ) is a complete measure space with μ(S) < ∞;
ℳ = {A ∈ ℳ: μ(A) = 0};
N is the set of nonnegative integers;
For subsets A, B of S,

AB = A ∩ B;
A\B = {s ∈ A: s ∈ B};
A^c = S\A;
AΔB = AB^c ∪ BA^c;
A ⊥ B iff A, B ∈ ℳ and μ(AΔB) = 0.

For a family ℬ of subsets of S,

∪ ℬ = ∪ E.

1. DEFINITIONS. For any field ℳ ⊆ ℳ,

(1) d is a (lower) density on ℳ iff d is a mapping on ℳ to
ℳ such that, for A, B in ℳ,

(i) d(A) ⊨ A;
(ii) A ⊨ B implies d(A) = d(B);
(iii) d(∅) = ∅, d(S) = S;
(iv) d(AB) = d(A)d(B).
(2) l is a lifting on ℳ iff l is a density on ℳ such that

(v) l(A^c) = l(A)^c, for A in ℳ.

For a detailed study of liftings and their applications, we refer
2. Remarks. Let \( l \) be a lifting on the \( \sigma \)-field \( \mathcal{A} \subset \mathcal{M} \) and \( \mathcal{F} = \{l(A)\} \). Then:

1. \( \mathcal{F} \) is a field in \( S \).
2. \( \mathcal{F} \subset \{E \in \mathcal{A} : 0 < \mu(E) < \mu(S)\} \cup \{\emptyset, S\}. \)
3. If, for each \( n \) in \( \mathbb{N} \), \( E_n \in \mathcal{F} \), and \( A = \bigcup_n E_n \), then \( l(A) \supset A \).

(Indeed, for each \( n \), \( E_n \setminus l(A) \subset A \setminus l(A) = \emptyset \), so \( E_n \setminus l(A) = \emptyset \), by (2).)

3. Theorem. If \( d \) is a density on a field \( \mathcal{A} \) with \( \mathcal{N} \subset \mathcal{A} \subset \mathcal{M} \), then there exists a lifting \( l \) on \( \mathcal{A} \), with

\[
(*) \quad d(A) \subset \mu(A) \subset d(A^c), \quad \text{for } A \text{ in } \mathcal{A}.
\]

Proof. For each filterbase \( \mathcal{B} \subset \mathcal{A} \), let \( \mathcal{U} \) denote an ultrafilter containing \( \mathcal{B} \). We recall that for subsets \( A, B \) of \( S \),

(a) \( A \in \mathcal{U} \) iff \( A^c \in \mathcal{U} \), and
(b) \( A \cap B \in \mathcal{U} \) iff \( A \in \mathcal{U} \) and \( B \in \mathcal{U} \).

For each \( s \) in \( S \), let

\[
\mathcal{F}(s) = \{A \in \mathcal{A} : s \in d(A)\}.
\]

Since \( d \) is a density, \( \mathcal{F}(s) \) is a filterbase. Put

\[
l(A) = \{s \in S : A \in \mathcal{F}(s)\}, \quad \text{for } A \text{ in } \mathcal{A}.
\]

By the properties (a), (b) of an ultrafilter, for \( A, B \) in \( \mathcal{A} \), we have (v) \( l(A^c) = l(A^c) \) and (iv) \( l(AB) = l(A)l(B) \). Moreover, if \( s \in d(A) \), then \( A \in \mathcal{F}(s) \subset \mathcal{F}(s) \), so that \( s \in l(A) \). Hence, \( d(A) \subset l(A) \). Similarly \( d(A^c) \subset l(A^c) \). Using (v) we find that (*) holds. Since \( d(A) = \hat{A} = \hat{d}(A^c) \), we have (i) \( l(A) = A \). If \( N = \emptyset \), then \( d(N) = d(\emptyset) = \emptyset \) and \( d(N^c) = d(S) = S \), so that, by (*), \( l(N) = \emptyset \). Hence, (iii) \( l(\emptyset) = \emptyset \), \( l(S) = S \) and (ii) if \( A = B \), then \( l(A)l(B) = l(A \cap B) = \emptyset \), so that \( l(A) = l(B) \). This completes the proof.

The proof of the following theorem usually uses martingales or Vitali differentiation. We use neither. However, the reader familiar with Sion [5] will recognize the connection with his method. (See Remark 7 below.)

4. Theorem. Suppose that, for each \( n \) in \( \mathbb{N} \), \( \mathcal{A}_n \) is a \( \sigma \)-field with \( \mathcal{N} \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \mathcal{M} \) and \( l_n \) is a lifting on \( \mathcal{A}_n \) with \( l_n = l_{n+1} | \mathcal{A}_n \). Put \( \mathcal{A} = \sigma \)-field \( (\bigcup_n \mathcal{A}_n) \). Then there is a lifting \( l \) on \( \mathcal{A} \) with \( l_n = l | \mathcal{A}_n \), for each \( n \) in \( \mathbb{N} \).
Proof. The result will follow immediately from Theorem 3 if we can construct a density $d$ on $\mathcal{S}$ with $d(A) = \ell_n(A)$ for $A$ in $\mathcal{S}_n$. To this end, for each $k$ in $N$, let $\mathcal{F}_k$ denote $\ell_k[\mathcal{S}_n]$. For each $A$ in $\mathcal{S}$, $k$ in $N$, and $r < 1$, put

$$
D(A; k, r) = \{E \in \mathcal{F}_k : \mu(AF) \geq r\mu(F)\}, \quad d(A; k, r) = \bigcup \{D(A; k, r)\}, \quad d(A) = \bigcap_{r < 1} \bigcup_{k \in N} \bigcap_{n \in \mathbb{N}} d(A; k, r).
$$

We will show that $d$ is a suitable density function on $\mathcal{S}$.

For fixed $A$, $r$, and $k$, let $\mathcal{K}$ be a maximal disjoint subfamily of $D(A; k, r)$. Then $\mathcal{K}$ is countable. Put $B = \ell_k(\cup \mathcal{K})$. Clearly, $B \in D(A; k, r)$. Moreover, if $E \in D(A; k, r)$, $E \setminus B = \emptyset$, by Remark 2(b) and the maximality of $\mathcal{K}$. This shows that $d(A; k, r) = B$ is the largest element of $D(A; k, r)$. In particular, $d(A; k, r) \in \mathcal{F}_k \subset \mathcal{S}_n$. If $r < s < 1$, we have $d(A; k, r) \supset d(A; k, s)$, so we need only consider rational $r$. Since $\mathcal{S}$ is a $\sigma$-field, we conclude that $d(A) \in \mathcal{S}_n$.

There is no difficulty in showing that $A \sqsupset B \in \mathcal{S}_n$ implies $d(A) = d(B)$, or that $d(S) = S$, for $A$ in $\mathcal{S}_n$. In particular, $d(\emptyset) = \emptyset$ and $d(S) = S$. We have left to check conditions (i) and (iv) of the definition of a density.

To check condition (iv), let $A, B \in \mathcal{S}_n$, $k \in N$, $r < 1$. For each $F$ in $\mathcal{F}_k$ contained in $d(A; k, (r + 1)/2) \cap d(B; k, (r + 1)/2)$, we have

$$
\mu(ABF) = \mu(AF) + \mu(BF) - \mu((A \cup B)F) \\
\geq ((r + 1)/2)\mu(F) + ((r + 1)/2)\mu(F) - \mu(F) \\
= r\mu(F).
$$

Hence, $d(A; k, (r + 1)/2) \cap d(B; k, (r + 1)/2) \subseteq d(AB; k, r)$. By direct computation, this yields $d(A)d(B) \subseteq d(AB)$. On the other hand, for each $k$ and $r$, $d(AB; k, r) \supset d(A; k, r) \cap d(B; k, r)$, so that $d(AB) \subseteq d(A)d(B)$, establishing (iv).

To verify condition (i), let $A \in \mathcal{S}_n$ and put

$$
d'(A) = \bigcup_{0 < r < 1} \bigcap_{n \in \mathbb{N}} d(A; k, r).
$$

We will show that

(a) $d'(A)A^c = \emptyset$,

(b) $Ad'(A^c) = \emptyset$, and

(c) $d'(A^c) \supset d(A)^c$, $d'(A) \supset d(A)$,

from which we get

$$
d(A)A = d(A)A^c \cup Ad'(A)^c \subset d'(A)A^c \cup Ad'(A^c) = \emptyset,
$$

as required.
Fix \( r \) in \((0,1)\) and write \( D_k = d(A; k, r)\), for \( k \) in \( \mathbb{N} \). Since \( D_k \in \mathcal{D}(A; k, r)\), we have for each \( B \) in \( \mathcal{J} \),
\[
\mu(ABD_k) = \mu(Al_k(B)D_k) \geq r\mu(l_k(B)D_k) = r\mu(BD_k).
\]
Suppose \( B \in \bigcup \mathcal{J} \). Then there exists an \( n \) in \( \mathbb{N} \) such that \( B \in \mathcal{J}_n \). For \( m \geq n \), \( \mathcal{J}_m \supset \mathcal{J}_n \), and putting \( C_m = BD_m \setminus \bigcup_{n\geq k<m} D_k \), we have
\[
\mu(AB \bigcup_{k\geq n} D_k) = \sum_{m \geq n} \mu(AC_m) \geq \sum_{m \geq n} r\mu(C_m) = r\mu(B \bigcup_{k\geq n} D_k).
\]
Taking intersections over \( n \) we have
\[
\mu(AB \bigcap_{n \geq n} \bigcup_{k\geq n} D_k) \geq r\mu(B \bigcap_{n \geq n} \bigcup_{k\geq n} D_k).
\]
By considering monotone sequences of such \( B \) we see that this holds for all \( B \) in \( \mathcal{J} \), the \( \sigma \)-field generated by the field \( \bigcup \mathcal{J} \). In particular, putting \( B = A^c \) we have \( 0 \geq r\mu(A^c \bigcup_{k\geq n} D_k) \). But \( r > 0 \), so \( \mu(A^c \bigcap_{k\geq n} D_k) = 0 \). Taking the union over rational \( r \) in \((0,1)\) we have \( A^c d'(A) = \emptyset \). This proves (a). Replacing \( A \) by \( A^c \) we have (b).

To prove (c) we let \( k \in \mathbb{N} \), \( 0 < r < 1 \) and show
\[
d(A; k, r)^c \subset d(A; k, 1 - r).
\]
To this end suppose \( \emptyset \neq E \in \mathcal{F}_k \) and \( E \subset d(A; k, r)^c \). Then \( E \in \mathcal{D}(A; k, r) \), so there exists \( F \in \mathcal{F}_k \) contained in \( E \) with \( \mu(AF) < r\mu(F) \). Let \( \mathcal{K} \) be a maximal disjoint collection of such \( F \). By Remark 2(3) and maximality of \( \mathcal{K} \) we have \( E \setminus \bigcup \mathcal{K} = \emptyset \), so \( E = l_k(\bigcup \mathcal{K}) \). Moreover, \( \mu(AE) = \sum_{F \in \mathcal{K}} \mu(AF) \leq \sum_{F \in \mathcal{K}} r\mu(F) = r\mu(E) \). In other words, \( \mu(A^c E) \geq (1 - r)\mu(E) \). This shows that \( d(A; k, r)^c \in \mathcal{D}(A^c; k, 1 - r) \), so \( d(A; k, r)^c \subset d(A^c; k, 1 - r) \). Hence,
\[
d(A)^e = \bigcup_{r \in (0,1)} \bigcap_{k \geq n} d(A; k, r)^e \subset \bigcup_{r \in (0,1)} \bigcap_{k \geq n} d(A^c; k, 1 - r) = d'(A^c).
\]
Since it is clear that \( d(A) \subset d'(A) \), this proves (c) and completes the proof of the theorem.

To prove the lifting theorem, we need one more lemma, due to A. and C. Ionescu Tulcea [2]. For completeness, we include a proof here.

5. **Lemma.** Let \( \mathcal{A} \) be a \( \sigma \)-field with \( \mathcal{N} \subset \mathcal{A} \subset \mathcal{M} \), \( l \) a lifting
on $\mathcal{A}$. If $A \in \mathcal{M} \setminus \mathcal{A}$ and $\mathcal{A}' = \text{field } (\mathcal{A} \cup \{A\})$, then there exists a lifting on $\mathcal{A}'$ extending $l$.

Proof. Let $\mathcal{F} = \mathcal{F}(\mathcal{A})$, $\mathcal{C} = \{E \in \mathcal{F} : \mu(EA') = 0\}$. Let $\mathcal{H}$ be a maximal disjoint subfamily of $\mathcal{C}$ and let $A_i = l(\bigcup \mathcal{H})$. Then $A_i \in \mathcal{C}$ and, by maximality of $\mathcal{H}$ and Remark 2(3), $E \setminus A_i = \emptyset$, for all $E$ in $\mathcal{C}$, so that $A_i$ is the largest element of $\mathcal{C}$. Similarly, let $A_j$ be the largest $E$ in $\mathcal{F}$ with $\mu(EA) = 0$. Put $\bar{A} = (A \cup A_i) \setminus A_j$. Then $\bar{A} \subseteq A$. (Indeed, $\bar{A} \supseteq A_i \subseteq A_j \subseteq \bar{A}$.) Thus, $\mathcal{A}' = \text{field } (\mathcal{A} \cup \{\bar{A}\}) = \{ (C \cup D \bar{A} : C, D \in \mathcal{A}) \}$. For $E, F$ in $\mathcal{F}$,

\begin{align*}
(a) & \quad EA = FA \implies EA = FA, \\
(b) & \quad E\bar{A} = FA \implies E\bar{A} = FA^c.
\end{align*}

Indeed, $EA = FA$ implies $\mu((E \setminus FA)A) = 0$, so that, by definition of $A_2$, $E\bar{A} \subseteq A_2 \subseteq \bar{A}^c$. Thus, $(E\bar{A})\bar{A} = \emptyset$, so $E\bar{A} = FA$. The proof of (b) is similar.

Now define $l'$ on $\mathcal{A}'$ by

$$l'(C \cup D \bar{A}) = l(C)\bar{A} \cup l(D)\bar{A}^c,$$

for $C, D$ in $\mathcal{A}$. Using (a) and (b) we see that $l'$ is well-defined and that for $M, N$ in $\mathcal{A}'$, $M \subseteq N$ implies $l'(M) \subseteq l'(N)$. The other properties of a lifting are easily verified. Moreover, for $C$ in $\mathcal{A}$, $l'(C) = l(C)\bar{A} \cup l(C)\bar{A}^c = l(C)$, so $l'$ extends $l$.

We can now prove the lifting theorem:

6. Theorem. Let $(S, \mathcal{M}, \mu)$ be a complete measure space with $\mu(S) < \infty$. Then, there exists a lifting on $\mathcal{M}$.

Proof. Let $\mathcal{H}$ be the set of pairs $(\mathcal{A}, l)$ where $\mathcal{A}$ is a $\sigma$-field with $\mathcal{M} \subset \mathcal{A} \subset \mathcal{M}$ and $l$ is a lifting on $\mathcal{A}$, with the ordering: $(\mathcal{A}', l') \leq (\mathcal{A}, l)$ iff $\mathcal{T} \subset \mathcal{A}'$ and $l = l' | \mathcal{T}$. We show that $\mathcal{H}$ has a maximal element. Indeed, suppose $\mathcal{H}' = \{ (\mathcal{A}_i, l_i) : i \in I \}$ is a totally ordered subfamily of $\mathcal{H}$. We distinguish two cases:

(a) If $\mathcal{H}'$ has no countable cofinal subfamily, put $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ and $l(A) = l_i(A)$, for $A$ in $\mathcal{A}_i$, $i$ in $I$. Then $(\mathcal{A}, l)$ is an upper bound for $\mathcal{H}'$ in $\mathcal{H}$.

(b) If $\mathcal{H}'$ has a countable cofinal subfamily $\mathcal{H}'' = \{ (\mathcal{A}_n, l_n) : n \in \mathcal{N} \}$, then by Theorem 4, $\mathcal{H}'''$ (and hence $\mathcal{H}'$) has an upper bound in $\mathcal{H}$. By Zorn's lemma, we conclude that $\mathcal{H}$ has a maximal element, $(\mathcal{A}, l)$.

By Lemma 3, and maximality, $\mathcal{A} = \mathcal{M}$, and the theorem is proved.

7. Remarks.

(1) To see the relationship of our method to that of Sion [5],...
for each $k$ in $N$ and $s$ in $S$, let $\mathcal{F}_k(s) = \{ F \in \mathcal{F} : s \in F \}$, directed downward by inclusion. Then,

$$d(A; k, r) = l_k \left( \left\{ s \in S : \lim_{F \in \mathcal{F}_k(s)} \frac{\mu(AF)}{\mu(F)} \geq r \right\} \right).$$

(One inclusion is obvious, the other follows from Sion's Theorem 2'.)

(2) As several authors have pointed out (see, for example, Sion [5], and for more references, Sion [6]), liftings provide very special Vitali differentiation system, even when no others are available. (If $l$ is a lifting on $\mathcal{M}$, such a system is obtained by assigning to each $s$ in $S$, $\{ F : s \in F \in l[\mathcal{M}] \}$, directed downward by inclusion.) Apart from our desire for an elementary proof, this was our main motivation in looking for a construction of a lifting without using differentiation concepts.

(3) Added in proof. S. Graf [On the existence of strong liftings in second countable topological spaces, (to appear)] has noticed that one may change the word "lifting" to "density" in the statement of Theorem 4. The proof is essentially contained in our proof. Graf has independently obtained a proof of this result (using Radon-Nikodym derivatives).

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Martin Bartelt, Strongly unique best approximates to a function on a set, and a finite subset thereof ................................................................. 1
S. J. Bernau, Theorems of Korovkin type for $L_p$-spaces ........................................ 11
S. J. Bernau and Howard E. Lacey, The range of a contractive projection on an $L_p$-space ................................................................. 21
Marilyn Breen, Decomposition theorems for 3-convex subsets of the plane ........... 43
Ronald Elroy Bruck, Jr., A common fixed point theorem for a commuting family of nonexpansive mappings ........................................... 59
Aiden A. Bruen and J. C. Fisher, Blocking sets and complete k-arcs .................... 73
R. Creighton Buck, Approximation properties of vector valued functions ............. 85
Mary Rodriguez Embry and Marvin Rosenblum, Spectra, tensor products, and linear operator equations ..................................................... 95
Edward William Formanek, Maximal quotient rings of group rings .................. 109
Barry J. Gardner, Some aspects of $T$-nilpotence ............................................. 117
Juan A. Gatica and William A. Kirk, A fixed point theorem for $k$-set-contractions defined in a cone ........................................... 131
Kenneth R. Goodearl, Localization and splitting in hereditary noetherian prime rings ................................................................. 137
James Victor Herod, Generators for evolution systems with quasi continuous trajectories .................................................................................. 153
C. V. Hinkle, The extended centralizer of an $S$-set ................................................. 163
I. Martin (Irving) Isaacs, Lifting Brauer characters of $p$-solvable groups .............. 171
Bruce R. Johnson, Generalized Lerch zeta function ............................................. 189
Erwin Kleinfeld, A generalization of $(-1, 1)$ rings ............................................. 195
Horst Leptin, On symmetry of some Banach algebras ......................................... 203
Paul Weldon Lewis, Strongly bounded operators ................................................. 207
Arthur Larry Lieberman, Spectral distribution of the sum of self-adjoint operators ................................................................. 211
I. J. Maddox and Michael A. L. Willey, Continuous operators on paranormed spaces and matrix transformations ............................................. 217
James Dolan Reid, On rings on groups ............................................................... 229
Richard Miles Schori and James Edward West, Hyperspaces of graphs are Hilbert cubes ................................................................. 239
William H. Specht, A factorization theorem for $p$-constrained groups ................. 253
Robert L. Thele, Iterative techniques for approximation of fixed points of certain nonlinear mappings in Banach spaces ........................................ 259
Tim Eden Traynor, An elementary proof of the lifting theorem ............................ 267
Charles Irvin Vinsonhaler and William Jennings Wickless, Completely decomposable groups which admit only nilpotent multiplications ................. 273
Raymond O’Neil Wells, Jr, Comparison of de Rham and Dolbeault cohomology for proper surjective mappings .............................................. 281
David Lee Wright, The non-minimality of induced central representations ........... 301
Bertram Yood, Commutativity properties in Banach $\ast$-algebras ....................... 307