

# Pacific Journal of Mathematics

**COMMUTATIVITY PROPERTIES IN BANACH \*-ALGEBRAS**

BERTRAM YOOD

# COMMUTATIVITY PROPERTIES IN BANACH \*-ALGEBRAS

BERTRAM YOOD

Let  $A$  be a Banach  $*$ -algebra which has a faithful  $*$ -representation as bounded linear operators on a Hilbert space. It follows from Fuglede's theorem concerning normal operators on a Hilbert space that  $x^*y = yx^*$  for all  $x, y$  in  $A$  where  $xx^* = x^*x$  and  $xy = yx$ . Other commutativity properties in suitable Banach  $*$ -algebras  $A$  involving elements not necessarily normal are considered.

1. Introduction. Let  $T_1, T_2$ , and  $U$  be bounded linear operators on a Hilbert space where  $T_1$  and  $T_2$  are normal. The well-known theorem of Fuglede [13] asserts that if  $T_1U = UT_1$  then  $T_1^*U = UT_1^*$ . Putnam's generalization [13] states that if  $T_1U = UT_2$  then  $T_1^*U = UT_2^*$ . With this in view Berberian [3] defined an  $FT$ -ring to be a ring with an involution  $x \rightarrow x^*$  such that  $x^*y = yx^*$  whenever  $x$  is normal and  $xy = yx$ . Likewise a  $PT$ -ring is one which gives  $x_1^*y = yx_2^*$  for all  $x_1, x_2$  normal and  $x_1y = yx_2$ . The usual examples of Banach  $*$ -algebras  $A$  [14] are  $FT$  and  $PT$ -algebras since they have faithful  $*$ -representations as bounded linear operators on a Hilbert space.

For our purposes we must demand somewhat more of  $A$ . We suppose that  $A$  is a semisimple hermitian  $*$ -algebra whose maximal commutative  $*$ -subalgebras are Shilov algebras and where  $xx^* \in W$ ,  $W$  a minimal closed two-sided ideal implies that  $x \in W$ . These requirements may seem special, but are actually satisfied by all  $B^*$ -algebras, all  $H^*$ -algebras and all group algebras of compact groups. Suppose that  $b \neq 0$  in  $A$  and  $ba = ab = 0$  for some  $a \neq 0$  in  $A$ . We show that there exist  $c \neq 0, h \neq 0, h$  self-adjoint, with  $bc = cb = 0$  and  $ch = hc = 0$  provided that either  $A$  has two closed two-sided ideals  $I \neq (0), J \neq (0)$  with  $I \cap J = (0)$  or  $A$  has zero socle. Without such hypotheses the conclusion can fail, as it does for the algebra of all  $2 \times 2$  matrices over the complex field.

2. Notation and preliminaries. As is customary, a Banach  $*$ -algebra  $A$  is called *hermitian* if the spectrum of each self-adjoint element is real. Suppose that  $A$  is hermitian and semisimple. Then so is the algebra obtained by adjoining an identity to  $A$ . Therefore, the theory expounded in [12] for hermitian Banach  $*$ -algebras with an identity applies here to show that  $A$  has a faithful  $*$ -representation as bounded linear operators on a Hilbert space. In particular, if  $x \in A$  and  $xx^* = 0$  then  $x = 0$ . Ptak's development [12] involves a pene-

trating study of the function  $p(x) = r(x^*x)^{1/2}$  (where  $r(y)$  is the spectral radius of  $y$ ). It turns out that  $p^{-1}(0) = 0$  and  $p(x)$  is a  $B^*$ -norm (in general incomplete) for  $A$ . If  $h$  is self-adjoint and  $r(h) = 0$ , then  $p(h) = 0$  and  $h = 0$ . It follows from this and [14, Theorem 4.1.3] that each maximal commutative  $*$ -subalgebra  $E$  of  $A$  is hermitian and semisimple. Also the involution on  $A$  is continuous [14, Theorem 4.1.15].

Next let  $B$  be a semisimple commutative Banach algebra with space  $\mathfrak{M}$  of modular maximal ideals. As is customary we say that  $B$  is a *Shilov algebra* if, given  $M_0 \in \mathfrak{M}$  and a closed set  $\mathfrak{F}$  in  $\mathfrak{M}$  not containing  $M_0$ , there exists  $x \in B$  such that  $\hat{x}(M_0) = 1$  and  $\hat{x}(M) = 0$  for all  $M \in \mathfrak{F}$ . Here  $\hat{x}(M)$  is the Gelfand transform of  $x$ .

Our interest in this paper is confined to the study of noncommutative Banach  $*$ -algebras where Shilov's concept enters in the following way.

**DEFINITION.** A Banach  $*$ -algebra  $A$  is called a *noncommutative Shilov  $*$ -algebra* if its maximal commutative  $*$ -subalgebras are Shilov algebras.

Note that any such  $A$  must be semisimple. For let  $J$  be the radical of  $A$ .  $J = J^*$ . If  $h$  is self-adjoint and  $h \in J$ , then  $r(h) = 0$ . Since  $h$  lies in a commutative Shilov algebras,  $h = 0$ . Therefore  $J = (0)$ .

As in [7] we say that  $A$  is a *CC algebra* if the mappings  $x \rightarrow ax$  and  $x \rightarrow xa$  are completely continuous on  $A$ .

**PROPOSITION 2.1.** *Let  $A$  be a semisimple CC Banach  $*$ -algebra where  $x = 0$  if  $xx^* = 0$ . Then  $A$  is a hermitian noncommutative Shilov  $*$ -algebra. If  $W$  is a minimal closed two-sided ideal in  $A$  containing  $xx^*$  then  $x \in W$ .*

*Proof.* A result of Barnes [2, Theorem 7.2] asserts that  $A$  is a modular annihilator algebra. It follows from the arguments of [5, Theorem 3.8] that the involution is hermitian.

Let  $E$  be a maximal commutative  $*$ -subalgebra with  $\mathfrak{M}$  as its space of modular maximal ideals. In this situation, as noted above,  $E$  is semisimple. Again using [2, Theorem 7.2] we see that if  $M_0 \in \mathfrak{M}$  there exists  $x \neq 0$ ,  $x \in E$ , such that  $xM_0 = (0)$ . Then  $\hat{x}(M_0) \neq 0$  while  $\hat{x}(M) = 0$ ,  $M \neq M_0$ . Therefore  $B$  is a Shilov algebra.

Take a minimal closed two-sided ideal  $W$  in  $A$ . By [7, Theorem 14],  $W$  is finite-dimensional. Let  $x \rightarrow \alpha(x)$  be a faithful  $*$ -representation of  $A$  as a subalgebra of  $B(H)$ , all the bounded linear operators on a Hilbert space  $H$ . Since  $W$  is finite-dimensional,  $\alpha(W)$  is a closed two-sided ideal in  $K$ , the closure of  $\alpha(A)$  in  $B(H)$ . If  $xx^* \in W$  then

$\alpha(x)(\alpha(x))^* \in \alpha(W)$ . From this we see that  $\alpha(x) \in \alpha(W)$  via [14, Corollary 4.9.3] so that  $x \in W$ .

Examples of algebras satisfying the hypotheses of Proposition 2.1 include the group algebra of a compact group  $G$  and, in addition,  $C(G)$  with convolution multiplication, the sup norm and the involution  $f^*(t) = \overline{f(t^{-1})}$ . See [7]. These algebras have the following more specific property (P) than that given for the minimal closed two-sided ideals by Proposition 2.1.

(P) Let  $x \in A$  and  $I$  be a closed two-sided ideal in  $A$ . If  $xx^* \in I$  then  $x \in I$ .

To show this for  $C(G)$  we use the natural inner product for  $C(G)$  given by

$$(f, g) = \int_G f(t) \overline{g(t)} dt$$

where the integration is taken with respect to normalized Haar measure. We call on the following properties of  $A = C(G)$ : (a)  $A^2 \subset I \oplus I^\perp$ , (b)  $x \in \overline{xA} \cap \overline{Ax}$  and (c)  $I = I^{\perp\perp}$ . Suppose that  $xx^* \in I$  and  $z \in A$ . Then  $(zx)(zx)^* \in I$ . Let  $zx = u + v$ ,  $u \in I$ ,  $v \in I^\perp$  and let  $w \in I^\perp$ . Then

$$0 = (zx x^* z^*, w) = (v v^*, w).$$

Therefore,  $v v^* \in I \cap I^\perp = (0)$  so that  $v = 0$  and  $zx \in I$ . By (b) we see that  $x \in I$ .

That  $L(G)$ ,  $G$  compact, has property (P) follows from the theory of closed two-sided ideals in  $L(G)$  developed in [6, Chapter IX]. We refer, in particular, to [6, Theorem 3.8.7] (see also [6, Theorem 28.40]) but do not give details here.

**PROPOSITION 2.2.**  *$B^*$ -algebras and  $H^*$ -algebras are hermitian noncommutative Shilov\*-algebras with property (P).*

*Proof.* For  $B^*$ -algebras see [14, Chapter IV]. That a maximal commutative\*-subalgebra  $E$  of an  $H^*$ -algebra  $A$  is a Shilov algebra follows from the fact that  $E$  is a commutative  $H^*$ -algebra and [1, Corollary 4.1]. That  $A$  has property (P) follows from the same analysis used for  $C(G)$  above.

**PROPOSITION 2.3.** *If  $A$  is a noncommutative Shilov \*-algebra so is every closed two-sided \*-ideal  $I$  in  $A$ .*

*Proof.* Let  $B$  be a maximal commutative \*-subalgebra of  $I$ . Certainly  $B$  is contained in a maximal commutative \*-subalgebra  $E$  of  $A$ . We show that  $B$  is an ideal in  $E$ . For  $y \in B$  and  $z \in E$  we have  $yz$  normal and permuting with each  $x \in B$ . Moreover  $yz \in I$ .

Then by the maximality of  $B$ ,  $yz \in B$ . Since  $E$  is a Shilov algebra, so is  $B$  by [10, Proposition 9.2].

For a self-adjoint element  $h$ , we write  $h \geq 0$  in case its spectrum is contained in the set of nonnegative real numbers. By a *minimal idempotent* we mean an idempotent generator of a minimal one-sided ideal.

**3. Two-sided annihilation in Banach  $*$ -algebras.** We write  $x \# y$  if  $xy = yx = 0$ . The involution  $x \rightarrow x^*$  in a ring is called *proper* if  $x^*x = 0$  implies that  $x = 0$ . The *FT*-property gives information on annihilation properties of normal elements which we put in the following form to point up what must be faced in the discussion below for nonnormal elements.

**PROPOSITION 3.1.** *Let  $A$  be an FT-ring with proper involution. Suppose  $b \in A$  is normal,  $b \neq 0$  and there exists  $a \neq 0$  in  $A$  where  $b \# a$ . Then*

- (1) *there exists a self-adjoint element  $h \neq 0$  such that  $b \# h$  and*
- (2) *there exist  $c \neq 0$  and  $h \neq 0$ ,  $h$  self-adjoint, where  $b \# c$  and  $c \# h$ .*

*Proof.* Since  $ab = ba$  we get, from the *FT*-property that  $ba^* = a^*b$ . Therefore  $0 = baa^* = aa^*b$ . Then  $b \# aa^*$  and  $aa^* \neq 0$ .

Note also that  $b^*a = ab^*$ . Then  $ab^*b = b^*ba = 0$  and (2) is also verified.

Now we start to examine what can happen when  $b$  is not normal but otherwise satisfies all the hypotheses of Proposition 3.1. Let  $A$  be the algebra of all  $2 \times 2$  matrices over the complex field and set

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One readily verified that  $b \# b$  and that, for  $a \in A$ ,  $b \# a$  if and only if  $a$  is a scalar multiple of  $b$ . Therefore, both of the conclusions (1) and (2) of Proposition 3.1 fail to hold for the element  $b$ .

Next consider the case of the algebra  $A$  of all  $3 \times 3$  matrices over the complex field. Consider

$$b = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}.$$

One verifies that  $b \# a$ , for  $a \in A$ , if and only if  $a$  is a scalar multiple of

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,  $b$  fails to satisfy the conclusion (1) of Proposition 3.1. Easy computations show that (2) is satisfied by  $b$ . Thus we can have (2) without (1).

Our treatment below of these questions makes essential use of ideas and techniques from Ono's interesting paper [11]. The next lemma is a modification to fit our needs of work in [11, pp. 155–156].

**LEMMA 3.2.** *Let  $A$  be a hermitian noncommutative Shilov \*-algebra. Suppose that  $h \neq 0$  is self-adjoint in  $A$ ,  $h \geq 0$  and  $h$  not a scalar multiple of a minimal idempotent. Then there exist non-zero self-adjoint elements  $u \geq 0$ ,  $v \geq 0$  in  $A$  such that  $h$ ,  $u$ ,  $v$  commute pairwise,  $hu \neq 0$ ,  $hv \neq 0$ , and  $uv = 0$ .*

*Proof.* Suppose first that  $h$  has at least two nonzero numbers in its spectrum. Let  $E$  be a maximal commutative \*-subalgebra of  $A$  containing  $h$  with space  $\mathfrak{M}$  of modular maximal ideals. There exist  $M_0$  and  $M_1$  in  $\mathfrak{M}$  such that  $0 < \hat{h}(M_1) < \hat{h}(M_0) = r(h)$ . We choose positive numbers  $r_1$ ,  $r_2$ , and  $r_3$  such that

$$0 < r_1 < \hat{h}(M_0) < r_2 < r_3 < \hat{h}(M_0).$$

Next we consider the open sets in  $\mathfrak{M}$  defined by

$$U(M_0) = \{M \in \mathfrak{M} : \hat{h}(M) > r_3\} \text{ and } V(M_1) = \{M \in \mathfrak{M} : r_1 < \hat{h}(M) < r_2\}.$$

Since  $E$  is a Shilov algebra there exist  $u_1, v_1 \in E$  where  $\hat{u}_1(M_0) = 1$ ,  $\hat{u}_1(M) = 0$ ,  $M \notin U(M_0)$ ,  $\hat{v}_1(M_1) = 1$  and  $\hat{v}_1(M) = 0$ ,  $M \notin V(M_0)$ . Since  $E$  is a hermitian \*-algebra,  $\hat{x}^*(M) = \overline{\hat{x}(M)}$ ,  $x \in E$ ,  $M \in \mathfrak{M}$ . Then  $u = u_1 u_1^*$ ,  $v = v_1 v_1^*$  have the desired properties.

Next suppose that the spectrum of  $h$  contains exactly one non-zero element (which we may take to be the number one without loss of generality). Then  $\hat{h}(M)$  is either 0 or 1 for each  $M \in \mathfrak{M}$ . As  $E$  is semisimple,  $h$  is a self-adjoint idempotent. By hypothesis,  $h$  is not a minimal idempotent so that there exist  $w \in A$  where  $hwh$  is not a scalar multiple of  $h$ . We can certainly, and so do, select  $w$  to be self-adjoint.

For  $x \in hAh$ , the nonzero spectrum of  $x$  is the same whether computed in  $A$  or in  $hAh$  by [9, Lemma 3]. Then  $hAh$  is a hermitian Banach algebra with identity  $h$ . Moreover, as zero is the only self-adjoint element in  $hAh$  with spectrum solely zero, we see that  $hAh$  is semisimple. Next we can select  $\lambda > 0$  so large that  $\text{sp}(z | hAh)$ ,

the spectrum of  $z = \lambda h + hwh$  computed in  $hAh$ , is contained in the open set  $(0, \infty)$ . Therefore,  $z^{-1}$  exists in  $hAh$ . We rule out the possibility that  $\text{sp}(z|_{hAh})$  consists of just one number  $\alpha$ . For in that case,  $\alpha^{-1}z$  would be a invertible idempotent in  $hAh$  and therefore  $\alpha^{-1}z = h$ . From this we see that  $hwh = (\alpha - \lambda)h$ , contrary to the choice of  $w$ .

Therefore  $\text{sp}(z)$ , computed in  $A$ , contains at least two nonzero numbers. By the first part of the proof of this lemma, working in a maximal commutative  $*$ -subalgebra  $E$  of  $A$  containing  $h$  and  $z$ , we see that there exist self-adjoint  $u \geq 0, v \geq 0, u \neq 0, v \neq 0$  in  $E$  such that  $zu \neq 0, zv \neq 0$ , and  $uv = 0$ . Then  $(\lambda h + hwh)u \neq 0$  so that  $hu \neq 0$ . Likewise  $hv \neq 0$ . Therefore,  $u$  and  $v$  have the desired properties.

The next lemma is also a modification and extension of work in [11].

As a preliminary we show that, in the algebra  $A$  under consideration,  $xx^*$  is a nonzero scalar multiple of a minimal idempotent if and only if  $x^*x$  also enjoys this property. For let  $xx^* = \lambda e$  where  $\lambda \neq 0, e = e^2 \neq 0$ . Now  $xx^* \geq 0$  by the Shirali-Ford theorem [12, Theorem 5.9] and  $\text{sp}(e)$  consists of the numbers 0 and 1. Therefore  $\lambda > 0$ . Then setting  $z = \lambda^{-1/2}x$ , we get  $zz^* = e$ . By [16, Proposition 3] we see that  $z^*z$  is also a minimal idempotent. Note that  $e$  must be self-adjoint.

**LEMMA 3.3.** *Let  $A$  be a hermitian noncommutative Shilov  $*$ -algebra. Let  $b \neq 0$  in  $A$  where  $b \# a$  for some  $a \neq 0$ . Then either*

( $\alpha$ )  *$b^*b(c^*c)^2b^*b$  is a nonzero scalar multiple of a minimal idempotent, for each  $c \neq 0$  in  $A$  where  $b \# c$*   
or

( $\beta$ ) *there exist  $c \neq 0, h \neq 0, h = h^*$  in  $A$  such that  $b \# c$  and  $c \# h$ .*

*Proof.* Suppose  $c \neq 0, b \# c$  and  $b^*b(c^*c)^2b^*b = 0$ . Then  $b^*bc^*c = 0 = c^*cb^*(c^*cb^*)^*$ . As the involution is proper,  $c^*cb^* = 0$ . But this gives  $b(c^*c) = 0 = (c^*c)b$ . We then have  $b \# c^*c$  and  $c^*c \# b^*b$ .

We therefore may suppose that we have  $a \neq 0, b \# a$  with  $b^*b(a^*a)^2b^*b \neq 0$  and not a scalar multiple of a minimal idempotent. This entails the fact that  $y = a^*a(b^*b)^2a^*a$  is not a nonzero scalar multiple of a minimal idempotent and  $y \neq 0$ . The Shirali-Ford theorem [12, Theorem 5.9] tells us that  $y \geq 0$ . Lemma 3.2 provides  $u \geq 0, v \geq 0$ , different from zero, permuting with each other and  $y$  such that  $uv = 0$  but  $yu \neq 0, yv \neq 0$ . In particular

$$(1) \quad a^*au \neq 0, \quad (b^*b)^2a^*av \neq 0.$$

On the basis of (1), we shall show

$$(2) \quad aua^*a \neq 0.$$

Consider the faithful \*-representation  $x \rightarrow \tau(x)$  of  $A$  as bounded linear operators on the Hilbert space  $H$ . In the algebra  $B(H)$  of all bounded linear operators on  $H$  the element  $\tau(u)$  has a positive self-adjoint square root  $W$ . If (2) is not valid then  $a^*aua^*a = 0$  and

$$[\tau(a)]^*\tau(a)W^2[\tau(a)]^*\tau(a) = 0,$$

from which we derive  $a^*au = 0$  contrary to (1).

We set

$$z = (b^*b)^2a^*ava^*ab^*b.$$

Clearly  $zb^*b$  is self-adjoint. Moreover  $zb^*b \neq 0$ . For suppose  $zb^*b = 0$ . Now  $\tau(v)$  has the form  $W^2$ ,  $W$  self-adjoint, in  $B(H)$ . For convenience, set  $x = (b^*b)^2a^*a$ . Then  $\tau(x)W^2\tau(x^*) = 0$ . This makes  $\tau(x)W^2 = 0$  and therefore  $xv = 0$  contrary to (1).

Next observe that  $b \# aua^*a$  and the latter is nonzero by (2). We complete the proof by showing that  $aua^*a \# zb^*b$ . Clearly  $(zb^*b)(aua^*a) = 0$ . On the other hand,

$$(aua^*a)zb^*b = auyva^*a(b^*b)^2 = 0$$

since  $u$  permutes with  $y$  and  $uv = 0$ .

In view of Lemma 3.3 we find it convenient to introduce the following notation. We set, for the algebra  $A$ ,

$$Q = \{b \neq 0 \text{ in } A : b \# a \text{ for some } a \neq 0 \text{ in } A\}$$

$$R = \{b \neq 0 \text{ in } A : b \# c, c \# h \text{ where } c \neq 0, h \neq 0, h = h^* \text{ in } A\}.$$

Clearly  $Q^* = Q$  and  $R^* = R$ .

**THEOREM 3.4.** *Let  $A$  be a hermitian noncommutative Shilov \*-algebra with zero socle. Let  $I$  be any two-sided closed \*-ideal in  $A$ . Then, in the algebra  $I$ ,  $Q \subset R$ .*

*Proof.* By [14, Theorem 4.1.9]  $I$  is a hermitian \*-algebra. That  $I$  has zero socle follows from [15, Lemma 3.10]. Proposition 2.3 and Lemma 3.3 give the desired result.

Consider a semisimple topological ring algebra  $A$ . Let  $e$  be a minimal idempotent in  $A$ . It is readily shown that  $I = AeA$  is a minimal two-sided ideal in  $A$  and that the closure of  $I$  is a minimal closed two-sided ideal in  $A$ . We say that  $I$  is *generated by the minimal idempotent  $e$* . In our situation where  $A$  has an involution we need to consider the following property. The example of § 2 give ample motivation here.



DEFINITION.  $A$  has *property M* if, given a minimal closed two-sided ideal  $I$  generated by a minimal idempotent,  $x \in I$  whenever  $xx^* \in I$ .

As usual  $A$  is considered to have property  $M$  if there are no such ideals  $I$ . Also note that, under property  $M$ ,  $I^* = I$ .

THEOREM 3.5. *Let  $A$  be a hermitian noncommutative Shilov  $*$ -algebra with property  $M$ . If there exist closed two-sided ideals  $I_1 \neq (0)$ ,  $I_2 \neq (0)$  in  $A$  with  $I_1 \cap I_2 = (0)$  then  $Q \subset R$ .*

*Proof.* Let  $b \in Q$  where  $b \# a$  for  $a \neq 0$ . We show  $b \in R$ . By Lemma 3.3 we may suppose that  $b^*b(a^*a)^2b^*b \neq 0$  is a positive scalar multiple of a minimal idempotent  $e$ , for otherwise  $b \in R$ . Let  $W$  denote the closure of  $AeA$ . Then  $W$  is a minimal closed ideal. Therefore,  $W \cap I_j = I_j$  or  $W \cap I_j = (0)$ ,  $j = 1, 2$ . For at least one of  $j = 1, 2$  we must have  $W \cap I_j = (0)$ . Then, for that  $j$ ,  $WI_j = I_jW = (0)$ . By [4, Theorem 7] the left and right annihilators of  $W$  in  $A$  coincide. Call this two-sided ideal  $K$ . Moreover, since  $W = W^*$ , it follows that  $K = K^* \neq (0)$ . Thus there exists a self-adjoint  $h \neq 0$  so that  $hW = Wh = (0)$ .

By the definition of  $e$  we have  $b^*ba^*a \neq 0$ . Moreover

$$b^*ba^*a(b^*ba^*a)^* \in W.$$

This tells us that  $b^*ba^*a \in W$ . It follows that  $ab^*ba^*a \neq 0$ . For otherwise  $a^*ab^*(a^*ab^*)^* = 0$  and  $a^*ab^*b = 0 = b^*ba^*a$ . But clearly  $b \# ab^*ba^*a$ . Inasmuch as  $h \# ab^*ba^*a$  we see that  $b \in R$ .

THEOREM 3.6. *Let  $A$  be a hermitian noncommutative Shilov  $*$ -algebra with property  $M$ . Either  $Q \subset R$  or there exists a unique minimal closed two-sided ideal  $W$  and  $W$  is generated by a minimal idempotent. If  $b \in Q$  and  $b \notin R$  then  $b \# c$  for some  $c \neq 0$  in  $W$ . Also  $ba^*$  and  $a^*b$  are different nonzero elements of  $W$  for all  $a \neq 0$  such that  $b \# a$ .*

*Proof.* Suppose  $Q \not\subset R$ . Lemma 3.3 and Theorem 3.5 show there exists a unique  $W$  as asserted. The arguments of Theorem 3.4 provide us with the desired  $c = ab^*ba^*a$  (where  $b \# a$  and  $a \neq 0$ ).

We continue discussing the setup  $b \# a$ ,  $a \neq 0$ , using all the notation of the proof of Theorem 3.5. As in that proof  $b^*ba^*a \in W$  and, obviously,  $ba^* \neq 0$ . We have  $a^*ab^*(a^*ab^*)^* \in W$  and therefore  $a^*ab^* \in W$ . Since  $W = W^*$  we have  $ba^*a \in W$ . Then  $ba^*(ba^*)^* \in W$  so that  $ba^* \in W$ .

It remains to see that  $a^*b \neq 0$ ,  $a^*b \in W$ , and  $a^*b \neq ba^*$ . Note that  $b^* \in Q$ ,  $b^* \notin R$ , and  $b^* \# a^*$ . Via the uniqueness of the minimal closed

ideal  $W$  we see, from the above proof, that  $b^*a \neq 0$  and  $b^*a \in W$ . If  $ba^* = a^*b$  then  $b \# a^*a$  and  $a^*a \# b^*b$ . This would put  $b$  in  $R$ .

In case  $A$  is an  $AW^*$ -algebra [8] we can obtain more. For if  $c \neq h$  where  $c \neq 0$ ,  $h \neq 0$  and  $h$  is self-adjoint there is a nonzero projection  $p$  with  $c \# p$ . To see this consider a maximal commutative  $*$ -subalgebra  $E$  of  $A$  containing  $h$ . By [8, Lemma 2.1] there exists  $y \in E$  such that  $hy = p$  is a nonzero projection. It is easy to see that  $c \# p$ .

**4. On the  $FT$  and  $PT$ -properties.** Let  $A$  be a Banach  $*$ -algebra. For a maximal commutative subalgebra  $W$  let  $N(W)$  denote the set of normal elements of  $A$  lying in  $W$ . These notions are intimately related to the  $FT$ -property.

**PROPOSITION 4.1.** *A Banach  $*$ -algebra is an  $FT$ -Banach algebra if and only if  $N(W) = [N(W)]^*$  for each maximal commutative subalgebra  $W$  of  $A$ . In that case  $N(W) = W \cap W^*$  and  $N(W)$  is a closed  $*$ -subalgebra of  $A$ .*

*Proof.* Suppose that  $N(W) = [N(W)]^*$  for each maximal commutative subalgebra  $W$  of  $A$ . Let  $x$  be normal in  $A$ ,  $y \in A$  and  $xy = yx$ . Let  $W$  be a maximal commutative subalgebra of  $A$  containing  $x$  and  $y$ . Since  $x^* \in W$ , we get  $x^*y = yx^*$ . Thus  $A$  has the  $FT$ -property.

Suppose, conversely, that  $A$  is an  $FT$ -algebra. Let  $W$  be a maximal commutative subalgebra. We have  $x^*y = yx^*$  for all  $x \in N(W)$  and  $y \in W$ . From the maximality of  $W$  we see that  $x^* \in N(W)$ . Suppose also that  $y \in N(W)$ . Then  $(x + y)^*(x + y) = (x + y)(x + y)^*$  and  $xy \in N(W)$ . Clearly  $N(W) \subset W \cap W^*$ . If  $z \in W \cap W^*$ ,  $z$  is normal and so lie in  $N(W)$ . Finally, inasmuch as  $W$  and  $W^*$  are maximal commutative subalgebras, we see that they are closed in  $A$  and, therefore,  $N(W)$  is closed. Note that this does not require the involution to be continuous.

**PROPOSITION 4.2.** *Let  $W$  be a maximal commutative subalgebra of a  $B^*$ -algebra  $A$  with unit. Then  $N(W)$  separates the maximal ideals of  $W$  if and only if  $W = N(W) \oplus R$  where  $R$  is the radical of  $W$ .*

*Proof.* By [14, Theorem 4.8.11],  $A$  is an  $FT$ -algebra. Let  $\mathfrak{M}$  be the space of maximal ideals of  $W$  and suppose that  $N(W)$  separates every  $M_1 \neq M_2$  in  $\mathfrak{M}$ . For  $x \in W$ , the spectrum,  $\text{sp}(x)$  of  $x$  is, by [14, p. 35] the same as the set  $\{\hat{x}(M) : M \in \mathfrak{M}\}$  where  $\tau : x \rightarrow \hat{x}(M)$  denotes the Gelfand transform of  $W$  into  $C(\mathfrak{M})$ . If  $x \in N(W)$ ,  $\|x\| =$

$\sup(|\hat{x}(M)|: M \in \mathfrak{M})$ . Therefore,  $\tau$  is an isometry on  $N(W)$ . Since the spectrum of a self-adjoint element is real we see that  $x^*(M) = \overline{\hat{x}(M)}$  for  $x \in N(W)$  and  $M \in \mathfrak{M}$ . By Proposition 4.1,  $N(W)$  is a  $B^*$ -algebra. The Stone-Weierstrass theorem insures that  $\tau(N(W)) = C(\mathfrak{M})$ . Therefore  $\tau(A) = C(\mathfrak{M})$ . The desired conclusion now follows from the semisimplicity of  $N(W)$ .

Maximal commutative subalgebra of this sort exist. Let  $A$  be the  $B^*$ -direct sum of  $B = C[0, 1]$  and  $M_2$  the  $B^*$ -algebra of all  $2 \times 2$  matrices over the complex field. Let  $I$  be the identity  $2 \times 2$  matrix and let  $T$  be the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One checks that  $W = B \oplus \{\lambda I + \mu T: \lambda, \mu \text{ complex}\}$  is a maximal commutative subalgebra of  $A$ , that  $N(W) = W \cap W^* = B \oplus \{\lambda I: \lambda \text{ complex}\}$  and that  $N(W)$  separates the maximal ideals of  $W$ .

**THEOREM 4.3.** *A noncommutative Shilov  $*$ -algebra is a  $PT$ -algebra if and only if it is hermitian.*

*Proof.* Suppose that  $A$  is an  $PT$ -algebra. Let  $h$  be a self-adjoint element,  $h \neq 0$  and  $B$  be a maximal commutative  $*$ -subalgebra containing  $h$ , with  $\mathfrak{M}$  as its space of modular maximal ideals. By [5, Theorem 2.2] there exists a homeomorphism  $\sigma$  of  $\mathfrak{M}$  onto  $\mathfrak{M}$  of period two such that  $\hat{x}^*(M) = \overline{\hat{x}(\sigma(M))}$  for all  $x \in B$ ,  $M \in \mathfrak{M}$ . Our task is to see that  $\sigma$  is the identity mapping. Suppose otherwise that  $M_0 \neq \sigma(M_0)$  for some  $M_0 \in \mathfrak{M}$ . Let  $U$  and  $V$  be disjoint open sets with  $M_0 \in U$  and  $\sigma(M_0) \in V$ . We select  $x, y \in B$  such that  $\hat{x}(M_0) = 1$ ,  $\hat{x}(M) = 0$ ,  $M \notin U$ ,  $\hat{y}(\sigma(M_0)) = 1$  and  $\hat{y}(M) = 0$ ,  $M \notin V$ . Then  $xy = 0$ . The  $PT$ -property yields  $xy^* = 0$ . Then  $\hat{x}(M_0)\hat{y}(\overline{\sigma(M_0)}) = 0$  which is impossible.

## REFERENCES

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc., **57** (1945), 364-386.
2. B. A. Barnes, *Modular annihilator algebras*, Canad. J. Math., **18** (1966), 566-578.
3. S. K. Berberian, *Note on a theorem of Fuglede and Putnam*, Proc. Amer. Math. Soc., **10** (1959), 175-182.
4. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc., (3) **4** (1954), 154-167.
5. P. Civin and B. Yood, *Involutions on Banach algebras*, Pacific J. Math., **9** (1959), 415-436.
6. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, vol. II, New York, 1970.
7. I. Kaplansky, *Dual rings*, Ann. of Math., **49** (1948), 689-701.
8. ———, *Projections in Banach algebras*, Ann. of Math., **53** (1951), 235-249.
9. ———, *Ring isomorphisms of Banach algebras*, Canad. J. Math., **6** (1954), 374-381.

10. H. Mirkil, *The Work of Silov on Commutative Banach Algebras*, Rio De Janeiro, 1959.
11. T. Ono, *Note on a  $B^*$ -algebra*, J. Math. Soc., Japan, **11** (1959), 146-158.
12. V. Pták, *Banach algebras with involution*, Manuscripta Math., **6** (1972), 245-290.
13. C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, New York, 1967.
14. C. E. Rickart, *General Theory of Banach Algebras*, Princeton, 1960.
15. B. Yood, *Ideals in topological rings*, Canad. J. Math., **16** (1964), 28-45.
16. ———, *On Kadison's condition for extreme points of the unit ball in a  $B^*$ -algebra*, Proc. Edinburgh Math. Soc., **16** (1969), 245-250.

Received November 12, 1973. This research was supported by the National Science Foundation Grant GP 38966.

THE PENNSYLVANIA STATE UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

# Pacific Journal of Mathematics

Vol. 53, No. 1

March, 1974

Martin Bartelt, <i>Strongly unique best approximates to a function on a set, and a finite subset thereof</i> .....	1
S. J. Bernau, <i>Theorems of Korovkin type for <math>L_p</math>-spaces</i> .....	11
S. J. Bernau and Howard E. Lacey, <i>The range of a contractive projection on an <math>L_p</math>-space</i> .....	21
Marilyn Breen, <i>Decomposition theorems for 3-convex subsets of the plane</i> .....	43
Ronald Elroy Bruck, Jr., <i>A common fixed point theorem for a commuting family of nonexpansive mappings</i> .....	59
Aiden A. Bruen and J. C. Fisher, <i>Blocking sets and complete <math>k</math>-arcs</i> .....	73
R. Creighton Buck, <i>Approximation properties of vector valued functions</i> .....	85
Mary Rodriguez Embry and Marvin Rosenblum, <i>Spectra, tensor products, and linear operator equations</i> .....	95
Edward William Formanek, <i>Maximal quotient rings of group rings</i> .....	109
Barry J. Gardner, <i>Some aspects of <math>T</math>-nilpotence</i> .....	117
Juan A. Gatica and William A. Kirk, <i>A fixed point theorem for <math>k</math>-set-contractions defined in a cone</i> .....	131
Kenneth R. Goodearl, <i>Localization and splitting in hereditary noetherian prime rings</i> .....	137
James Victor Herod, <i>Generators for evolution systems with quasi continuous trajectories</i> .....	153
C. V. Hinkle, <i>The extended centralizer of an <math>S</math>-set</i> .....	163
I. Martin (Irving) Isaacs, <i>Lifting Brauer characters of <math>p</math>-solvable groups</i> .....	171
Bruce R. Johnson, <i>Generalized Lerch zeta function</i> .....	189
Erwin Kleinfeld, <i>A generalization of <math>(-1, 1)</math> rings</i> .....	195
Horst Leptin, <i>On symmetry of some Banach algebras</i> .....	203
Paul Weldon Lewis, <i>Strongly bounded operators</i> .....	207
Arthur Larry Lieberman, <i>Spectral distribution of the sum of self-adjoint operators</i> .....	211
I. J. Maddox and Michael A. L. Willey, <i>Continuous operators on paranormed spaces and matrix transformations</i> .....	217
James Dolan Reid, <i>On rings on groups</i> .....	229
Richard Miles Schori and James Edward West, <i>Hyperspaces of graphs are Hilbert cubes</i> .....	239
William H. Specht, <i>A factorization theorem for <math>p</math>-constrained groups</i> .....	253
Robert L Thele, <i>Iterative techniques for approximation of fixed points of certain nonlinear mappings in Banach spaces</i> .....	259
Tim Eden Traynor, <i>An elementary proof of the lifting theorem</i> .....	267
Charles Irvin Vinsonhaler and William Jennings Wickless, <i>Completely decomposable groups which admit only nilpotent multiplications</i> .....	273
Raymond O'Neil Wells, Jr, <i>Comparison of de Rham and Dolbeault cohomology for proper surjective mappings</i> .....	281
David Lee Wright, <i>The non-minimality of induced central representations</i> .....	301
Bertram Yood, <i>Commutativity properties in Banach <math>*</math>-algebras</i> .....	307