LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE.

II

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LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE (II)

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Let $(BN)$ denote the class of all bounded linear operators on a Hilbert space such that $T^*T$ and $TT^*$ commute. Let $(BN)^+$ be those $T \in (BN)$ which are hyponormal. Embry has observed that if $T \in (BN)$, then $0 \in W(T)$ or $T$ is normal. This is used to show that if $T \in (BN)$, then $(T + \lambda I) \notin (BN)$ unless $T$ is normal. It is also shown that if $T \in (BN)^+$, then $T^n$ is hyponormal for $n \geq 1$. An example of an $T \in (BN)^+$ such that $T^2 \in (BN)$ is given. Paranormality of operators in $(BN)$ is shown to be equivalent to hyponormality. The relationship between $T$ being in $(BN)$ and $T$ being centered is discussed. Finally, all $3 \times 3$ matrices in $(BN)$ are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators $T$ acting on a separable Hilbert space $H$ such that $T^*T$ and $TT^*$ commute. Such operators are called bi-normal and the class of all such operators is denoted $(BN)$. This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all $2 \times 2$ and $3 \times 3$ bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in $(BN)$.

1. All shifts, weighted and unweighted, bilateral and unilateral, are in $(BN)$. Further, operators in $(BN)$, if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

**Theorem 1.** If $T \in (BN)$, then either $T$ is normal or zero is in the interior of the numerical range of $T$, $W(T)$.

**Proof.** Embry has shown that if $T \in (BN)$ and $T$ is not normal, then $0 \in W(T)$ [7, Theorem 1]. She has also shown that if $T \in (BN)$ and $T + T^* \geq 0$, then $T$ is normal [5, Theorem 2]. Thus if $0$ were on the boundary of $W(T)$, by a suitable choice of $\alpha$, $|\alpha| = 1$, we could consider $T_1 = \alpha T$ where $T_1 \in (BN)$ and $T_1 + T_1^* \geq 0$. Then $T$ would be normal.
An interesting consequence of Theorem 1 is that no translate of a bi-normal operator can be bi-normal unless the original operator was normal.

For bounded linear operators $X$ and $Y$ let $[X, Y] = XY - YX$.

**Theorem 2.** Suppose that $T \in (BN)$. Then $T + \lambda I \in (BN)$, some complex $\lambda \neq 0$, if and only if $T$ is normal.

**Proof.** Suppose $T \in (BN)$. Let $\lambda \neq 0$ be real. Then

$$[(T + \lambda I)^*(T + \lambda I), (T + \lambda I)(T + \lambda I)^*] = 0$$

is equivalent to $[[T^*, T], T + T^*] = 0$. Thus if $T + \lambda I \in (BN)$ for some real $\lambda \neq 0$, then $T + \lambda I \in (BN)$ for all real $\lambda$. But $0 \in W(T + \lambda I)$ for $\lambda$ sufficiently large so $T$ would be normal by Theorem 1. The case when $\lambda$ is complex easily reduces to the one when $\lambda$ is real.

2. One reason that the class $(BN)$ is of interest is that it includes many of the weighted translated operators of Parrott [10], and nonanalytic composition operators, such as those studied by Ridge [12]. In particular, $(BN)$ includes the Bishop operator [10, p. 2] for which the question of invariant subspaces is still open.

The Bishop operator actually falls into the following class which is more restrictive than $(BN)$.

**Definition 1.** A bounded linear operator $T$ is called centered if the set $\{T^n T^*, T^* T^n\}_{n=0}^\infty$ consists of pairwise commuting operators.

Centered operators have been studied by Muhly [9] and Morrell [8]. Muhly has shown that centered operators with zero kernels and dense ranges are the direct sums of weighted translation operators [9]. Parrott has asked (in a private communication) whether the same is true for operators in $(BN)$. We answer this in the negative by exhibiting a $T \in (BN)$ such that $T^2 \notin (BN)$, and $T$ is invertible.

**Example 1.** Let $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. Then $T \in (BN)$, $T^2 \notin (BN)$, and $T$ is invertible.

3. Powers of hyponormal or bi-normal operators need not be hyponormal or bi-normal. Operators which are both hyponormal and bi-normal are somewhat "nicer". Let $(BN)^+$ denote the hyponormal bi-normal operators.
THEOREM 3. Suppose that \( T \in (BN)^* \). Then \( T^* \) is hyponormal for \( n \geq 1 \).

Proof. If \( C, D \) are positive operators such that \( C \geq D \geq 0 \), then \( T^*CT \geq T^*DT \geq 0 \) for any bounded operator \( T \). Suppose now that \( T \in (BN)^* \). Since \( T^*T \geq TT^* \), we have \( T^*T^2 \geq (TT^*)^2 \) and \( (TT^*)^2 \geq T^2T^* \). But \( T^*T \geq TT^* \) and \( [T^*T, TT^*] = 0 \) implies that \( (TT^*)^2 \geq (TT^*)^2 \). Hence \( T^*T^2 \geq (TT^*)^2 \) and \( T^2T^2 \) is hyponormal. Suppose then that \( T^*T^n \geq (TT^*)^n \) for some integer \( n \geq 2 \). Then \( T^*T^* \geq (TT^*)^n \) implies that \( T^*T^{n+1} \geq (TT^*)^{n+1} \) and \( (TT^*)^n \geq T^*T^n \). But \((TT^*)^n \geq (TT^*)^{n+1} \). The theorem now follows by induction.

4. The assumption that \( T \in (BN) \) is hyponormal can be weakened to \( T \in (BN) \) is paranormal but no added generality is achieved as the next result shows. Recall that \( T \) is paranormal if \( \| T^* \| \leq \| T \| \) for all \( \phi \in \mathcal{A} \). See for example [1]. Hyponormal operators are paranormal.

THEOREM 4. Suppose that \( T \in (BN) \). If \( T \) is also paranormal, then it is hyponormal.

Proof. Suppose that \( T \) is paranormal. Then \( AB^2A - 2\lambda A^2 + \lambda^2 I \geq 0 \) for every \( \lambda > 0 \) where \( A = (TT^*)^{1/2} \) and \( B = (T^*T)^{1/2} \) [1]. Suppose that \( T \in (BN) \). The condition for paranormality becomes

\[
A^2B^2 - 2\lambda A^2 + \lambda^2 I \geq 0 \text{ for every } \lambda > 0.
\]

Since \([A^2, B^2] = 0\), there exists a spectral measure \( E(\cdot) \) such that

\[
A^2 = \int f(t)dE(t) \text{ and } B^2 = \int g(t)dE(t).
\]

Substituting these integrals into (*) gives

\[
\int (f(t)g(t) - 2\lambda f(t) + \lambda^2)dE(t) \geq 0.
\]

Let \( \theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } xy - 2\lambda x + \lambda^2 \geq 0 \} \text{ for all } \lambda > 0 \}. \) Then \((f(t), g(t)) \in \theta \) almost everywhere \( dE \). We will show now that actually \( \theta = \{(x, y): x \geq 0, y \geq 0 \} \). Then \( g(t) \geq f(t) \) almost everywhere \( dE \) and \( T^*T \geq TT^* \) as desired. To see that \( \theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } y \geq x \} \), observe that \( xy - 2\lambda x + \lambda^2 = 0, \lambda > 0, \) defines the curve \( y = h_\lambda(x) = 2\lambda - \lambda^2/x \) in the first quadrant. The line \( y = x \) is tangent to \( h_\lambda(x) \) at \( x = \lambda \). Since \( h_\lambda(x) \) is everywhere
concave down we have that it lies entirely on or below \( y = x \). But \( \theta \) consists of those points in the first quadrant lying above the graph of \( h_2 \) for every \( \lambda > 0 \), that is, above the line \( y = x \).

An immediate corollary to Theorem 4 which might save time in the construction of examples is the following.

**Corollary 1.** There are no weighted shifts which are paranormal and not hyponormal.

5. Under certain conditions \( T \) being in \((BN)\) does imply \( T \) is centered. We give two.

**Theorem 5.** Suppose that \( \| T \| \leq 1 \). If \( T^*T = f(TT^*) \) and \( TT^* = g(T^*T) \) where \( f \) and \( g \) are continuous functions from \([0, 1]\) into \([0, 1]\), then \( T \) is centered.

**Proof.** If \( T^*T = f(TT^*) \), then

\[
(*) \quad T^{*2}T^2 = T^*f(TT^*)T = f(T^*T)T^*T = f(f(TT^*)g(TT^*)) = f_2(TT^*)
\]

where \( f_2 \) is a continuous function from \([0, 1]\) into \([0, 1]\). The second equality of \((*)\) is trivially valid if \( f \) is a polynomial. By taking uniform limits of polynomials it can be seen that it is true for all continuous functions \( f \). From \((*)\) and an induction argument, we get that \( T^{*n}T^* = f_2(TT^*) \) and \( T^*T^{*n} = g_2(T^*T) \) for continuous functions \( f_2, g_2 \) mapping \([0, 1]\) into \([0, 1]\), \( n \geq 1 \). Hence \( [T^{*i}T^j, T^{*j}T^i] = 0 \) for all integers \( i, j \geq 0 \).

The assumption that \( f, g \) are continuous can be considerably weakened. If \( h, k \) are bounded measurable functions from \([0, 1]\) into \([0, 1]\), then let \( (h \boxtimes k)(x) = h(k(x))k(x) \). Set \( h_1 = h \) and define \( h_n = (h_{n-1} \boxtimes h) \) for \( n \geq 2 \). Then the theorem is true if \( f_n \) and \( g_n \) are well-defined \( dE \) measurable functions for every integer \( n \geq 1 \). \( dE \) is the spectral measure of the \(*\)-algebra generated by \( I, T^*T \) and \( TT^* \). Clearly the assumption \( \| T \| \leq 1 \) is not restrictive.

S. K. Parrott has proven the following result (private communication).

**Theorem 6.** If \( T \in (BN) \) and \( T^*T \) has a cyclic vector, then \( T \) is unitarily equivalent to a weighted translation operator.

6. The operator \( T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \) acting on \( C^2 \) shows that Theorem 6 is not valid for an arbitrary \( T \in (BN) \). Our next example shows it is also not true for \( T \in (BN)^{+} \).
Example 2. Let

\[ T_n = \begin{bmatrix} 0 & 0 & \sqrt{2} g(n + 1) \\ g(n) & g(n) & 0 \\ g(n) & -g(n) & 0 \end{bmatrix}, \quad n \geq 1, \]

where \( g(n) \) is a strictly increasing sequence of positive numbers converging to 1. Let

\[ A = \begin{bmatrix} 0 & 0 & 0 \\ T_1 & 0 & 0 \\ 0 & T_2 & 0 \end{bmatrix} \]

acting on \( \mathcal{A} \) where \( \mathcal{A} \) is a countable number of copies of \( C^* \). Then \( A \in (BN)^+ \), but \( A^* \notin (BN) \). \( A \in (BN) \) since \( A^*A \) and \( AA^* \) are diagonal. \( A \in (BN)^+ \) since \( T_{n+1}^* T_{n+1} \geq T_n T_n^* \), \( n \geq 1 \). So show \( A^* \in (BN) \), one need only show that \( [(T_{n+1}^* T_n)(T_{n+1}^* T_n)]^* \), \( (T_{n+2}^* T_{n+2})^* (T_{n+2}^* T_{n+2}) \) \( \neq 0 \) for some \( n \geq 1 \). Picking \( n = 1 \) and \( g(1) = 0 \) makes the calculation easier.

It is easy to modify Example 2 to get an invertible \( A \) such that \( A \in (BN)^+ \) and \( A^* \notin (BN) \). This is done by picking a sequence \( \{g(n)\}_{n=\infty}^\infty \) such that \( g(n) < g(n + 1), \lim_{n \to \infty} g(n) = 1, \) and \( \lim_{n \to \infty} g(n) = c > 0 \). Define \( A \) to be a matrix weighted bilateral shift with weights \( T_n, T_n \) as in Example 2.

There remains then the problem of determining what types of operators are in \( (BN)^+ \).

In the process of proving Theorem 1 of [3] we proved the following result which could be helpful.

If \( C \) is self-adjoint, let \( E_C(\cdot) \) be the spectral measure of \( C \).

Proposition 1. If \( T \in (BN)^+ \), then \( E_{T^* T}([b, ||T||]) < \) is an invariant subspace of \( T \) for every \( b > 0 \). Furthermore, \( E_{T^* T}([0, b]) \subseteq E_{T^* T}([0, b]) \) for every \( b > 0 \).

By considering weighted shifts in \( (BN)^+ \) it is easy to see that the subspaces need never be reducing and \( [b, ||T||] \) cannot be replaced by a noninterval or by an interval without \( ||T|| \) as an end point.

7. The presence of a large number of examples is useful both in making conjectures and in finding counterexamples. There has also been some interest in the condition \( (BN) \) when \( \dim \mathcal{A} < \infty \) [4]. For these reasons we will now characterize all operators in \( (BN) \) when \( \dim \mathcal{A} = 2 \) and \( \dim \mathcal{A} = 3 \).
DEFINITION 2. If \( \{ \phi_i \} \) is an orthonormal basis, \( D \) is a diagonal matrix with respect to this basis, and \( U \) is a permutation of the basis, then \( T = UD \) is called a weighted permutation.

We say that a matrix \( A \) is a form for \( T \) if \( T \) is unitarily equivalent to a scalar multiple of either \( A \) or \( A^* \).

**Theorem 7.** If \( T \in (BN) \) and \( \dim \mathcal{A} = 2 \), then the possible forms are:

1. \[
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}, \text{ } a \text{ an arbitrary complex number.}
\]
2. \[
\begin{bmatrix}
1 & b \\
0 & -1
\end{bmatrix}, \text{ } b > 0.
\]
3. \[
\begin{bmatrix}
0 & 1 \\
a & 0
\end{bmatrix}, \text{ an arbitrary.}
\]

**Theorem 8.** If \( T \in (BN) \) and \( \dim \mathcal{A} = 3 \), then the possible forms are:

1. \[
\begin{bmatrix}
c & 0 & 0 \\
0 & X & 0 \\
0 & 0 & 0
\end{bmatrix}, \text{ where } X \text{ is (I2), } c \text{ an arbitrary complex number.}
\]
2. A weighted permutation.
3. \[
\begin{bmatrix}
0 & b & -1 \\
0 & 1 & b \\
0 & 0 & 0
\end{bmatrix}, \text{ } b > 0.
\]
4. \[
\begin{bmatrix}
0 & 0 & a \\
u_{21} & u_{22} & 0 \\
u_{21} & u_{22} & 0
\end{bmatrix}, \text{ } a > 0 \text{ and } \begin{bmatrix} u_{21} & u_{22} \end{bmatrix} \text{ is unitary.}
\]

**Proof.** Theorem 7 is easy. Form (I3) is best developed from the form developed in [4] for matrices \( T \) such that \( [T'T, TT'] = 0 \) where \( T' \) is the generalized inverse of \( T \). If \( T \in (BN) \), then \( [T'T, TT'] = 0 \). Form (I4) is best developed by looking at the polar form and determining possible unitary parts of \( T \).

Example 1 was found by considering an operator of form (I4). The blocks in Example 2 are also (I4) forms.

In looking for \( (BN) \) matrices the following matrix version of Theorem 6 is useful.

**Theorem 9.** Suppose that \( T \in (BN) \) and that \( \dim \mathcal{A} = n < \infty \). If \( T^*T \) has \( n \) different eigenvalues, then \( T \) is a weighted permutation.

Theorem 9 can be given a simple matrix proof by observing that if \( T = U(T^*T)^{1/2} \) and \( T \in (BN) \), then \( U(T^*T) = (TT^*)U \) and \( T^*T \) and \( TT^* \) may be simultaneously diagonalized. Furthermore, \( T^*T \) and \( TT^* \) have the same spectrum. It is then easy to see that the only
possible $U$ are permutations of the basis that diagonalizes $T^*T$ and $TT^*$.

It is easy to verify that in all of the forms in Theorem 7 and Theorem 8, except possibly (II4), that zero is in the convex hull of $\sigma(T)$. Is this always true when $n = \dim \mathcal{A} < \infty$? Is it true when $\dim \mathcal{A}$ is infinite? If it is not always true, for what dimensions is it true?

8. All of the two-dimensional bi-normal operators have a square which is normal. Such operators are automatically bi-normal (though never nontrivially hyponormal). This result was proved in [4] and observed independently by Embry in a private communication.

Operators such that $T^2$ is normal have been studied by Embry [6] and completely characterized by Radjavi and Rosenthal [11].

The author would like to thank Mary Embry and S. K. Parrott for their helpful comments.

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Received April 11, 1973.

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