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**LINEAR OPERATORS FOR WHICH  $T^*T$  AND  $TT^*$  COMMUTE.  
II**

STEPHEN LAVERN CAMPBELL

## LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE (II)

STEPHEN L. CAMPBELL

Let  $(BN)$  denote the class of all bounded linear operators on a Hilbert space such that  $T^*T$  and  $TT^*$  commute. Let  $(BN)^+$  be those  $T \in (BN)$  which are hyponormal. Embry has observed that if  $T \in (BN)$ , then  $0 \in W(T)$  or  $T$  is normal. This is used to show that if  $T \in (BN)$ , then  $(T + \lambda I) \notin (BN)$  unless  $T$  is normal. It is also shown that if  $T \in (BN)^+$ , then  $T^n$  is hyponormal for  $n \geq 1$ . An example of a  $T \in (BN)^+$  such that  $T^2 \notin (BN)$  is given. Paranormality of operators in  $(BN)$  is shown to be equivalent to hyponormality. The relationship between  $T$  being in  $(BN)$  and  $T$  being centered is discussed. Finally, all  $3 \times 3$  matrices in  $(BN)$  are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators  $T$  acting on a separable Hilbert space  $\mathcal{H}$  such that  $T^*T$  and  $TT^*$  commute. Such operators are called bi-normal and the class of all such operators is denoted  $(BN)$ . This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all  $2 \times 2$  and  $3 \times 3$  bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in  $(BN)$ .

1. All shifts, weighted and unweighted, bilateral and unilateral, are in  $(BN)$ . Further, operators in  $(BN)$ , if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

**THEOREM 1.** *If  $T \in (BN)$ , then either  $T$  is normal or zero is in the interior of the numerical range of  $T$ ,  $W(T)$ .*

*Proof.* Embry has shown that if  $T \in (BN)$  and  $T$  is not normal, then  $0 \in W(T)$  [7, Theorem 1]. She has also shown that if  $T \in (BN)$  and  $T + T^* \geq 0$ , then  $T$  is normal [5, Theorem 2]. Thus if 0 were on the boundary of  $W(T)$ , by a suitable choice of  $\alpha$ ,  $|\alpha| = 1$ , we could consider  $T_1 = \alpha T$  where  $T_1 \in (BN)$  and  $T_1 + T_1^* \geq 0$ . Then  $T$  would be normal.

An interesting consequence of Theorem 1 is that no translate of a bi-normal operator can be bi-normal unless the original operator was normal.

For bounded linear operators  $X$  and  $Y$  let  $[X, Y] = XY - YX$ .

**THEOREM 2.** *Suppose that  $T \in (BN)$ . Then  $T + \lambda I \in (BN)$ , some complex  $\lambda \neq 0$ , if and only if  $T$  is normal.*

*Proof.* Suppose  $T \in (BN)$ . Let  $\lambda \neq 0$  be real. Then

$$[(T + \lambda I)^*(T + \lambda I), (T + \lambda I)(T + \lambda I)^*] = 0$$

is equivalent to  $[[T^*, T], T + T^*] = 0$ . Thus if  $T + \lambda I \in (BN)$  for some real  $\lambda \neq 0$ , then  $T + \lambda I \in (BN)$  for all real  $\lambda$ . But  $0 \notin W(T + \lambda I)$  for  $\lambda$  sufficiently large so  $T$  would be normal by Theorem 1. The case when  $\lambda$  is complex easily reduces to the one when  $\lambda$  is real.

2. One reason that the class  $(BN)$  is of interest is that it includes many of the weighted translated operators of Parrott [10], and nonanalytic composition operators, such as those studied by Ridge [12]. In particular,  $(BN)$  includes the Bishop operator [10, p. 2] for which the question of invariant subspaces is still open.

The Bishop operator actually falls into the following class which is more restrictive than  $(BN)$ .

**DEFINITION 1.** A bounded linear operator  $T$  is called centered if the set  $\{T^n T^{*n}, T^{*n} T^n\}_{n=0}^{\infty}$  consists of pairwise commuting operators.

Centered operators have been studied by Muhly [9] and Morrell [8]. Muhly has shown that centered operators with zero kernels and dense ranges are the direct sums of weighted translation operators [9]. Parrott has asked (in a private communication) whether the same is true for operators in  $(BN)$ . We answer this in the negative by exhibiting a  $T \in (BN)$  such that  $T^2 \notin (BN)$ , and  $T$  is invertible.

**EXAMPLE 1.** Let  $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ . Then  $T \in (BN)$ ,  $T^2 \notin (BN)$ , and  $T$  is invertible.

3. Powers of hyponormal or bi-normal operators need not be hyponormal or bi-normal. Operators which are both hyponormal and bi-normal are somewhat "nicer". Let  $(BN)^+$  denote the hyponormal bi-normal operators.

**THEOREM 3.** *Suppose that  $T \in (BN)^+$ . Then  $T^n$  is hyponormal for  $n \geq 1$ .*

*Proof.* If  $C, D$  are positive operators such that  $C \geq D \geq 0$ , then  $TCT^* \geq TDT^* \geq 0$  and  $T^*CT \geq T^*DT \geq 0$  for any bounded operator  $T$ . Suppose now that  $T \in (BN)^+$ . Since  $T^*T \geq TT^*$ , we have  $T^{*2}T^2 \geq (T^*T)^2$  and  $(TT^*)^2 \geq T^2T^{*2}$ . But  $T^*T \geq TT^*$  and  $[T^*T, TT^*] = 0$  implies that  $(T^*T)^2 \geq (TT^*)^2$ . Hence  $T^{*2}T^2 \geq (T^*T)^2 \geq T^2T^{*2}$  and  $T^2$  is hyponormal. Suppose then that  $T^{*n}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n}$  for some integer  $n \geq 2$ . Then  $T^{*n}T^n \geq (TT^*)^n$  implies that  $T^{*(n+1)}T^{n+1} \geq (T^*T)^{n+1}$  and  $(T^*T)^n \geq T^nT^{*n}$  implies that  $(TT^*)^{n+1} \geq T^{n+1}T^{*(n+1)}$ . But  $(T^*T)^{n+1} \geq (TT^*)^{n+1}$ . The theorem now follows by induction.

4. The assumption that  $T \in (BN)$  is hyponormal can be weakened to  $T \in (BN)$  is paranormal but no added generality is achieved as the next result shows. Recall that  $T$  is paranormal if  $\|T^2\phi\| \cdot \|\phi\| \geq \|T\phi\|^2$  for all  $\phi \in \mathcal{L}$ . See for example [1]. Hyponormal operators are paranormal.

**THEOREM 4.** *Suppose that  $T \in (BN)$ . If  $T$  is also paranormal, then it is hyponormal.*

*Proof.* Suppose that  $T$  is paranormal. Then  $AB^2A - 2\lambda A^2 + \lambda^2 I \geq 0$  for every  $\lambda > 0$  where  $A = (TT^*)^{1/2}$  and  $B = (T^*T)^{1/2}$  [1]. Suppose that  $T \in (BN)$ . The condition for paranormality becomes

$$(*) \quad A^2B^2 - 2\lambda A^2 + \lambda^2 I \geq 0 \text{ for every } \lambda > 0.$$

Since  $[A^2, B^2] = 0$ , there exists a spectral measure  $E(\cdot)$  such that

$$A^2 = \int f(t)dE(t) \quad \text{and} \quad B^2 = \int g(t)dE(t).$$

Substituting these integrals into (\*) gives

$$\int (f(t)g(t) - 2\lambda f(t) + \lambda^2)dE(t) \geq 0.$$

Let  $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } xy - 2\lambda x + \lambda^2 \geq 0 \text{ for all } \lambda > 0\}$ . Then  $(f(t), g(t)) \in \theta$  almost everywhere  $dE$ . We will show now that actually  $\theta = \{(x, y): x \geq 0, y \geq 0, \text{ and } y \geq x\}$ . Then  $g(t) \geq f(t)$  almost everywhere  $dE$  and  $T^*T \geq TT^*$  as desired. To see that  $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } y \geq x\}$ , observe that  $xy - 2\lambda x + \lambda^2 = 0, \lambda > 0$ , defines the curve  $y = h_\lambda(x) = 2\lambda - \lambda^2/x$  in the first quadrant. The line  $y = x$  is tangent to  $h_\lambda(x)$  at  $x = \lambda$ . Since  $h_\lambda(x)$  is everywhere

concave down we have that it lies entirely on or below  $y = x$ . But  $\theta$  consists of those points in the first quadrant lying above the graph of  $h_\lambda$  for every  $\lambda > 0$ , that is, above the line  $y = x$ .

An immediate corollary to Theorem 4 which might save time in the construction of examples is the following.

**COROLLARY 1.** *There are no weighted shifts which are par-normal and not hyponormal.*

5. Under certain conditions  $T$  being in  $(BN)$  does imply  $T$  is centered. We give two.

**THEOREM 5.** *Suppose that  $\|T\| \leq 1$ . If  $T^*T = f(TT^*)$  and  $TT^* = g(T^*T)$  where  $f$  and  $g$  are continuous functions from  $[0, 1]$  into  $[0, 1]$ , then  $T$  is centered.*

*Proof.* If  $T^*T = f(TT^*)$ , then

$$(*) \quad T^{*2}T^2 = T^*f(TT^*)T = f(T^*T)T^*T = f(f(TT^*))f(TT^*) = f_2(TT^*)$$

where  $f_2$  is a continuous function from  $[0, 1]$  into  $[0, 1]$ . The second equality of  $(*)$  is trivially valid if  $f$  is a polynomial. By taking uniform limits of polynomials it can be seen that it is true for all continuous functions  $f$ . From  $(*)$  and an induction argument, we get that  $T^{*n}T^n = f_n(TT^*)$  and  $T^nT^{*n} = g_n(T^*T)$  for continuous functions  $f_n, g_n$  mapping  $[0, 1]$  into  $[0, 1]$ ,  $n \geq 1$ . Hence  $[T^{*i}T^j, T^i T^{*j}] = 0$  for all integers  $i, j \geq 0$ .

The assumption that  $f, g$  are continuous can be considerably weakened. If  $h, k$  are bounded measurable functions from  $[0, 1]$  into  $[0, 1]$ , then let  $(h \odot k)(x) = h(k(x))k(x)$ . Set  $h_1 = h$  and define  $h_n = (h_{n-1} \odot h)$  for  $n \geq 2$ . Then the theorem is true if  $f_n$  and  $g_n$  are well-defined  $dE$  measurable functions for every integer  $n \geq 1$ .  $dE$  is the spectral measure of the  $*$ -algebra generated by  $I, T^*T$  and  $TT^*$ . Clearly the assumption  $\|T\| \leq 1$  is not restrictive.

S. K. Parrott has proven the following result (private communication).

**THEOREM 6.** *If  $T \in (BN)$  and  $T^*T$  has a cyclic vector, then  $T$  is unitarily equivalent to a weighted translation operator.*

6. The operator  $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$  acting on  $C^2$  shows that Theorem 6 is not valid for an arbitrary  $T \in (BN)$ . Our next example shows it is also not true for  $T \in (BN)^+$ .

EXAMPLE 2. Let

$$T_n = \begin{bmatrix} 0 & 0 & \sqrt{2}g(n+1) \\ g(n) & g(n) & 0 \\ g(n) & -g(n) & 0 \end{bmatrix}, n \geq 1,$$

where  $g(n)$  is a strictly increasing sequence of positive numbers converging to 1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ T_1 & 0 & 0 & \cdot \\ 0 & T_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

acting on  $\mathcal{L}$  where  $\mathcal{L}$  is a countable number of copies of  $C^3$ . Then  $A \in (BN)^+$ , but  $A^2 \notin (BN)$ .  $A \in (BN)$  since  $A^*A$  and  $AA^*$  are diagonal.  $A \in (BN)^+$  since  $T_{n+1}^*T_{n+1} \geq T_nT_n^*$ ,  $n \geq 1$ . So show  $A^2 \notin (BN)$ , one need only show that  $[(T_{n+1}T_n)(T_{n+1}T_n)^*, (T_{n+3}T_{n+2})^*(T_{n+3}T_{n+2})] \neq 0$  for some  $n \geq 1$ . Picking  $n = 1$  and  $g(1) = 0$  makes the calculation easier.

It is easy to modify Example 2 to get an invertible  $A$  such that  $A \in (BN)^+$  and  $A^2 \notin (BN)$ . This is done by picking a sequence  $\{g(n)\}_{n=-\infty}^\infty$  such that  $g(n) < g(n+1)$ ,  $\lim_{n \rightarrow \infty} g(n) = 1$ , and  $\lim_{n \rightarrow -\infty} g(n) = c > 0$ . Define  $A$  to be a matrix weighted bilateral shift with weights  $T_n, T_n$  as in Example 2.

There remains then the problem of determining what types of operators are in  $(BN)^+$ .

In the process of proving Theorem 1 of [3] we proved the following result which could be helpful.

If  $C$  is self-adjoint, let  $E_C(\cdot)$  be the spectral measure of  $C$ .

PROPOSITION 1. *If  $T \in (BN)^+$ , then  $E_{T^*T}([b, \|T\|])_{\mathcal{L}}$  is an invariant subspace of  $T$  for every  $b > 0$ . Furthermore,  $E_{T^*T}([0, b]) \leq E_{TT^*}([0, b])$  for every  $b > 0$ .*

By considering weighted shifts in  $(BN)^+$  it is easy to see that the subspaces need never be reducing and  $[b, \|T\|]$  cannot be replaced by a noninterval or by an interval without  $\|T\|$  as an end point.

7. The presence of a large number of examples is useful both in making conjectures and in finding counterexamples. There has also been some interest in the condition  $(BN)$  when  $\dim \mathcal{L} < \infty$  [4]. For these reasons we will now characterize all operators in  $(BN)$  when  $\dim \mathcal{L} = 2$  and  $\dim \mathcal{L} = 3$ .

DEFINITION 2. If  $\{\phi_i\}$  is an orthonormal basis,  $D$  is a diagonal matrix with respect to this basis, and  $U$  is a permutation of the basis, then  $T = UD$  is called a weighted permutation.

We say that a matrix  $A$  is a form for  $T$  if  $T$  is unitarily equivalent to a scalar multiple of either  $A$  or  $A^*$ .

THEOREM 7. If  $T \in (BN)$  and  $\dim \mathcal{L} = 2$ , then the possible forms are:

$$(I1) \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \text{ } a \text{ an arbitrary complex number.}$$

$$(I2) \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \text{ } b > 0.$$

$$(I3) \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ } a \text{ an arbitrary.}$$

THEOREM 8. If  $T \in (BN)$  and  $\dim \mathcal{L} = 3$ , then the possible forms are:

$$(II1) \begin{bmatrix} c & 0 & 0 \\ 0 & X & \\ 0 & & \end{bmatrix} \text{ where } X \text{ is (I2), } c \text{ an arbitrary complex number.}$$

(II2) A weighted permutation.

$$(II3) \begin{bmatrix} 0 & b & -1 \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, \text{ } b > 0.$$

$$(II4) \begin{bmatrix} 0 & 0 & a \\ \begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} & 0 & \\ & & 0 \end{bmatrix} \text{ where } a > 0 \text{ and } \begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \text{ is unitary.}$$

*Proof.* Theorem 7 is easy. Form (II3) is best developed from the form developed in [4] for matrices  $T$  such that  $[T^\dagger T, TT^\dagger] = 0$  where  $T^\dagger$  is the generalized inverse of  $T$ . If  $T \in (BN)$ , then  $[T^\dagger T, TT^\dagger] = 0$ . Form (II4) is best developed by looking at the polar form and determining possible unitary parts of  $T$ .

Example 1 was found by considering an operator of form (II4). The blocks in Example 2 are also (II4) forms.

In looking for  $(BN)$  matrices the following matrix version of Theorem 6 is useful.

THEOREM 9. Suppose that  $T \in (BN)$  and that  $\dim \mathcal{L} = n < \infty$ . If  $T^*T$  has  $n$  different eigenvalues, then  $T$  is a weighted permutation.

Theorem 9 can be given a simple matrix proof by observing that if  $T = U(T^*T)^{1/2}$  and  $T \in (BN)$ , then  $U(T^*T) = (TT^*)U$  and  $T^*T$  and  $TT^*$  may be simultaneously diagonalized. Furthermore,  $T^*T$  and  $TT^*$  have the same spectrum. It is then easy to see that the only

possible  $U$  are permutations of the basis that diagonalizes  $T^*T$  and  $TT^*$ .

It is easy to verify that in all of the forms in Theorem 7 and Theorem 8, except possibly (II4), that zero is in the convex hull of  $\sigma(T)$ . Is this always true when  $n = \dim \mathcal{H} < \infty$ ? Is it true when  $\dim \mathcal{H}$  is infinite? If it is not always true, for what dimensions is it true?

8. All of the two-dimensional bi-normal operators have a square which is normal. Such operators are automatically bi-normal (though never nontrivially hyponormal). This result was proved in [4] and observed independently by Embry in a private communication.

Operators such that  $T^2$  is normal have been studied by Embry [6] and completely characterized by Radjavi and Rosenthal [11].

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