

Pacific Journal of Mathematics

**LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE.
II**

STEPHEN LAVERN CAMPBELL

LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE (II)

STEPHEN L. CAMPBELL

Let (BN) denote the class of all bounded linear operators on a Hilbert space such that T^*T and TT^* commute. Let $(BN)^+$ be those $T \in (BN)$ which are hyponormal. Embry has observed that if $T \in (BN)$, then $0 \in W(T)$ or T is normal. This is used to show that if $T \in (BN)$, then $(T + \lambda I) \notin (BN)$ unless T is normal. It is also shown that if $T \in (BN)^+$, then T^n is hyponormal for $n \geq 1$. An example of a $T \in (BN)^+$ such that $T^2 \notin (BN)$ is given. Paranormality of operators in (BN) is shown to be equivalent to hyponormality. The relationship between T being in (BN) and T being centered is discussed. Finally, all 3×3 matrices in (BN) are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators T acting on a separable Hilbert space \mathcal{H} such that T^*T and TT^* commute. Such operators are called bi-normal and the class of all such operators is denoted (BN) . This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all 2×2 and 3×3 bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in (BN) .

1. All shifts, weighted and unweighted, bilateral and unilateral, are in (BN) . Further, operators in (BN) , if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

THEOREM 1. *If $T \in (BN)$, then either T is normal or zero is in the interior of the numerical range of T , $W(T)$.*

Proof. Embry has shown that if $T \in (BN)$ and T is not normal, then $0 \in W(T)$ [7, Theorem 1]. She has also shown that if $T \in (BN)$ and $T + T^* \geq 0$, then T is normal [5, Theorem 2]. Thus if 0 were on the boundary of $W(T)$, by a suitable choice of α , $|\alpha| = 1$, we could consider $T_1 = \alpha T$ where $T_1 \in (BN)$ and $T_1 + T_1^* \geq 0$. Then T would be normal.

An interesting consequence of Theorem 1 is that no translate of a bi-normal operator can be bi-normal unless the original operator was normal.

For bounded linear operators X and Y let $[X, Y] = XY - YX$.

THEOREM 2. *Suppose that $T \in (BN)$. Then $T + \lambda I \in (BN)$, some complex $\lambda \neq 0$, if and only if T is normal.*

Proof. Suppose $T \in (BN)$. Let $\lambda \neq 0$ be real. Then

$$[(T + \lambda I)^*(T + \lambda I), (T + \lambda I)(T + \lambda I)^*] = 0$$

is equivalent to $[[T^*, T], T + T^*] = 0$. Thus if $T + \lambda I \in (BN)$ for some real $\lambda \neq 0$, then $T + \lambda I \in (BN)$ for all real λ . But $0 \notin W(T + \lambda I)$ for λ sufficiently large so T would be normal by Theorem 1. The case when λ is complex easily reduces to the one when λ is real.

2. One reason that the class (BN) is of interest is that it includes many of the weighted translated operators of Parrott [10], and nonanalytic composition operators, such as those studied by Ridge [12]. In particular, (BN) includes the Bishop operator [10, p. 2] for which the question of invariant subspaces is still open.

The Bishop operator actually falls into the following class which is more restrictive than (BN) .

DEFINITION 1. A bounded linear operator T is called centered if the set $\{T^n T^{*n}, T^{*n} T^n\}_{n=0}^\infty$ consists of pairwise commuting operators.

Centered operators have been studied by Muhly [9] and Morrell [8]. Muhly has shown that centered operators with zero kernels and dense ranges are the direct sums of weighted translation operators [9]. Parrott has asked (in a private communication) whether the same is true for operators in (BN) . We answer this in the negative by exhibiting a $T \in (BN)$ such that $T^2 \notin (BN)$, and T is invertible.

EXAMPLE 1. Let $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. Then $T \in (BN)$, $T^2 \notin (BN)$, and

T is invertible.

3. Powers of hyponormal or bi-normal operators need not be hyponormal or bi-normal. Operators which are both hyponormal and bi-normal are somewhat "nicer". Let $(BN)^+$ denote the hyponormal bi-normal operators.

THEOREM 3. *Suppose that $T \in (BN)^+$. Then T^n is hyponormal for $n \geq 1$.*

Proof. If C, D are positive operators such that $C \geq D \geq 0$, then $TCT^* \geq TDT^* \geq 0$ and $T^*CT \geq T^*DT \geq 0$ for any bounded operator T . Suppose now that $T \in (BN)^+$. Since $T^*T \geq TT^*$, we have $T^{*2}T^2 \geq (T^*T)^2$ and $(TT^*)^2 \geq T^2T^{*2}$. But $T^*T \geq TT^*$ and $[T^*T, TT^*] = 0$ implies that $(T^*T)^2 \geq (TT^*)^2$. Hence $T^{*2}T^2 \geq (T^*T)^2 \geq T^2T^{*2}$ and T^2 is hyponormal. Suppose then that $T^{*n}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n}$ for some integer $n \geq 2$. Then $T^{*n}T^n \geq (TT^*)^n$ implies that $T^{*n+1}T^{n+1} \geq (T^*T)^{n+1}$ and $(T^*T)^n \geq T^nT^{*n}$ implies that $(TT^*)^{n+1} \geq T^{n+1}T^{*n+1}$. But $(T^*T)^{n+1} \geq (TT^*)^{n+1}$. The theorem now follows by induction.

4. The assumption that $T \in (BN)$ is hyponormal can be weakened to $T \in (BN)$ is paranormal but no added generality is achieved as the next result shows. Recall that T is paranormal if $\|T^2\phi\| \cdot \|\phi\| \geq \|T\phi\|^2$ for all $\phi \in \mathcal{H}$. See for example [1]. Hyponormal operators are paranormal.

THEOREM 4. *Suppose that $T \in (BN)$. If T is also paranormal, then it is hyponormal.*

Proof. Suppose that T is paranormal. Then $AB^2A - 2\lambda A^2 + \lambda^2 I \geq 0$ for every $\lambda > 0$ where $A = (TT^*)^{1/2}$ and $B = (T^*T)^{1/2}$ [1]. Suppose that $T \in (BN)$. The condition for paranormality becomes

$$(*) \quad A^2B^2 - 2\lambda A^2 + \lambda^2 I \geq 0 \text{ for every } \lambda > 0.$$

Since $[A^2, B^2] = 0$, there exists a spectral measure $E(\cdot)$ such that

$$A^2 = \int f(t)dE(t) \quad \text{and} \quad B^2 = \int g(t)dE(t).$$

Substituting these integrals into (*) gives

$$\int (f(t)g(t) - 2\lambda f(t) + \lambda^2)dE(t) \geq 0.$$

Let $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } xy - 2\lambda x + \lambda^2 \geq 0 \text{ for all } \lambda > 0\}$. Then $(f(t), g(t)) \in \theta$ almost everywhere dE . We will show now that actually $\theta = \{(x, y): x \geq 0, y \geq 0, \text{ and } y \geq x\}$. Then $g(t) \geq f(t)$ almost everywhere dE and $T^*T \geq TT^*$ as desired. To see that $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } y \geq x\}$, observe that $xy - 2\lambda x + \lambda^2 = 0, \lambda > 0$, defines the curve $y = h_\lambda(x) = 2\lambda - \lambda^2/x$ in the first quadrant. The line $y = x$ is tangent to $h_\lambda(x)$ at $x = \lambda$. Since $h_\lambda(x)$ is everywhere

concave down we have that it lies entirely on or below $y = x$. But θ consists of those points in the first quadrant lying above the graph of h_λ for every $\lambda > 0$, that is, above the line $y = x$.

An immediate corollary to Theorem 4 which might save time in the construction of examples is the following.

COROLLARY 1. *There are no weighted shifts which are paranormal and not hyponormal.*

5. Under certain conditions T being in (BN) does imply T is centered. We give two.

THEOREM 5. *Suppose that $\|T\| \leq 1$. If $T^*T = f(TT^*)$ and $TT^* = g(T^*T)$ where f and g are continuous functions from $[0, 1]$ into $[0, 1]$, then T is centered.*

Proof. If $T^*T = f(TT^*)$, then

$$(*) \quad T^{*2}T^2 = T^*f(TT^*)T = f(T^*T)T^*T = f(f(TT^*))f(TT^*) = f_2(TT^*)$$

where f_2 is a continuous function from $[0, 1]$ into $[0, 1]$. The second equality of $(*)$ is trivially valid if f is a polynomial. By taking uniform limits of polynomials it can be seen that it is true for all continuous functions f . From $(*)$ and an induction argument, we get that $T^{*n}T^n = f_n(TT^*)$ and $T^nT^{*n} = g_n(T^*T)$ for continuous functions f_n, g_n mapping $[0, 1]$ into $[0, 1]$, $n \geq 1$. Hence $[T^{*j}T^j, T^i T^{*i}] = 0$ for all integers $i, j \geq 0$.

The assumption that f, g are continuous can be considerably weakened. If h, k are bounded measurable functions from $[0, 1]$ into $[0, 1]$, then let $(h \odot k)(x) = h(k(x))k(x)$. Set $h_1 = h$ and define $h_n = (h_{n-1} \odot h)$ for $n \geq 2$. Then the theorem is true if f_n and g_n are well-defined dE measurable functions for every integer $n \geq 1$. dE is the spectral measure of the $*$ -algebra generated by I, T^*T and TT^* . Clearly the assumption $\|T\| \leq 1$ is not restrictive.

S. K. Parrott has proven the following result (private communication).

THEOREM 6. *If $T \in (BN)$ and T^*T has a cyclic vector, then T is unitarily equivalent to a weighted translation operator.*

6. The operator $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ acting on C^2 shows that Theorem 6 is not valid for an arbitrary $T \in (BN)$. Our next example shows it is also not true for $T \in (BN)^+$.

EXAMPLE 2. Let

$$T_n = \begin{bmatrix} 0 & 0 & \sqrt{2}g(n+1) \\ g(n) & g(n) & 0 \\ g(n) & -g(n) & 0 \end{bmatrix}, n \geq 1,$$

where $g(n)$ is a strictly increasing sequence of positive numbers converging to 1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ T_1 & 0 & 0 & \cdot \\ 0 & T_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

acting on \mathcal{L} where \mathcal{L} is a countable number of copies of C^3 . Then $A \in (BN)^+$, but $A^2 \notin (BN)$. $A \in (BN)$ since A^*A and AA^* are diagonal. $A \in (BN)^+$ since $T_{n+1}^*T_{n+1} \geq T_n T_n^*$, $n \geq 1$. So show $A^2 \notin (BN)$, one need only show that $[(T_{n+1}T_n)(T_{n+1}T_n)^*, (T_{n+3}T_{n+2})^*(T_{n+3}T_{n+2})] \neq 0$ for some $n \geq 1$. Picking $n = 1$ and $g(1) = 0$ makes the calculation easier.

It is easy to modify Example 2 to get an invertible A such that $A \in (BN)^+$ and $A^2 \notin (BN)$. This is done by picking a sequence $\{g(n)\}_{n=-\infty}^{\infty}$ such that $g(n) < g(n+1)$, $\lim_{n \rightarrow -\infty} g(n) = 1$, and $\lim_{n \rightarrow \infty} g(n) = c > 0$. Define A to be a matrix weighted bilateral shift with weights T_n, T_n as in Example 2.

There remains then the problem of determining what types of operators are in $(BN)^+$.

In the process of proving Theorem 1 of [3] we proved the following result which could be helpful.

If C is self-adjoint, let $E_C(\cdot)$ be the spectral measure of C .

PROPOSITION 1. *If $T \in (BN)^+$, then $E_{T^*T}([b, \|T\|])\mathcal{L}$ is an invariant subspace of T for every $b > 0$. Furthermore, $E_{T^*T}([0, b]) \leq E_{TT^*}([0, b])$ for every $b > 0$.*

By considering weighted shifts in $(BN)^+$ it is easy to see that the subspaces need never be reducing and $[b, \|T\|]$ cannot be replaced by a noninterval or by an interval without $\|T\|$ as an end point.

7. The presence of a large number of examples is useful both in making conjectures and in finding counterexamples. There has also been some interest in the condition (BN) when $\dim \mathcal{L} < \infty$ [4]. For these reasons we will now characterize all operators in (BN) when $\dim \mathcal{L} = 2$ and $\dim \mathcal{L} = 3$.

DEFINITION 2. If $\{\phi_i\}$ is an orthonormal basis, D is a diagonal matrix with respect to this basis, and U is a permutation of the basis, then $T = UD$ is called a weighted permutation.

We say that a matrix A is a form for T if T is unitarily equivalent to a scalar multiple of either A or A^* .

THEOREM 7. *If $T \in (BN)$ and $\dim \mathcal{L} = 2$, then the possible forms are:*

$$(I1) \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \text{ } a \text{ an arbitrary complex number.}$$

$$(I2) \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \text{ } b > 0.$$

$$(I3) \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ } a \text{ arbitrary.}$$

THEOREM 8. *If $T \in (BN)$ and $\dim \mathcal{L} = 3$, then the possible forms are:*

$$(II1) \begin{bmatrix} c & 0 & 0 \\ 0 & X & \\ 0 & & \end{bmatrix} \text{ where } X \text{ is (I2), } c \text{ an arbitrary complex number.}$$

(II2) *A weighted permutation.*

$$(II3) \begin{bmatrix} 0 & b & -1 \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, \text{ } b > 0.$$

$$(II4) \begin{bmatrix} 0 & 0 & a \\ u_{21} & u_{22} & 0 \\ u_{31} & u_{32} & 0 \end{bmatrix} \text{ where } a > 0 \text{ and } \begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \text{ is unitary.}$$

Proof. Theorem 7 is easy. Form (II3) is best developed from the form developed in [4] for matrices T such that $[T^+T, TT^+] = 0$ where T^+ is the generalized inverse of T . If $T \in (BN)$, then $[T^+T, TT^+] = 0$. Form (II4) is best developed by looking at the polar form and determining possible unitary parts of T .

Example 1 was found by considering an operator of form (II4). The blocks in Example 2 are also (II4) forms.

In looking for (BN) matrices the following matrix version of Theorem 6 is useful.

THEOREM 9. *Suppose that $T \in (BN)$ and that $\dim \mathcal{L} = n < \infty$. If T^*T has n different eigenvalues, then T is a weighted permutation.*

Theorem 9 can be given a simple matrix proof by observing that if $T = U(T^*T)^{1/2}$ and $T \in (BN)$, then $U(T^*T) = (TT^*)U$ and T^*T and TT^* may be simultaneously diagonalized. Furthermore, T^*T and TT^* have the same spectrum. It is then easy to see that the only

possible U are permutations of the basis that diagonalizes T^*T and TT^* .

It is easy to verify that in all of the forms in Theorem 7 and Theorem 8, except possibly (II4), that zero is in the convex hull of $\sigma(T)$. Is this always true when $n = \dim \mathcal{L} < \infty$? Is it true when $\dim \mathcal{L}$ is infinite? If it is not always true, for what dimensions is it true?

8. All of the two-dimensional bi-normal operators have a square which is normal. Such operators are automatically bi-normal (though never nontrivially hyponormal). This result was proved in [4] and observed independently by Embry in a private communication.

Operators such that T^2 is normal have been studied by Embry [6] and completely characterized by Radjavi and Rosenthal [11].

The author would like to thank Mary Embry and S. K. Parrott for their helpful comments.

REFERENCES

1. T. Ando, *Operators with a norm condition*, Acta Sci. Math., (Szeged), **33** (1972), 169-178.
2. Arlen Brown, *The unitary equivalence of bi-normal operators*, Amer. J. Math., **76** (1954), 414-434.
3. Stephen L. Campbell, *Linear operators for which T^*T and TT^* commute*, Proc. Amer. Math. Soc., **34** (1972), 177-180.
4. Stephen L. Campbell and Carl D. Meyer, *EP operators and generalized inverses*, Canad. Math. Bull., (to appear).
5. Mary R. Embry, *Conditions implying normality in Hilbert space*, Pacific J. Math., **18** (1966), 457-460.
6. ———, *N^{th} roots of Operators*, Proc. Amer. Math. Soc., **19** (1968), 63-68.
7. ———, *Similarities involving normal operators on Hilbert space*, Pacific J. Math., **35** (1970), 331-336.
8. Bernard B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J., **23** (1973), 497-511.
9. Paul S. Muhly, *Imprimitive operators*, unpublished preprint, 1972.
10. Stephen K. Parrott, *Weighted Translation Operators*, Ph. D. Dissertation, Univ. of Michigan, 1965.
11. Heydar Radjavi and Peter Rosenthal, *On roots of normal operators*, J. Math. Anal. Appl., **34** (1971), 653-664.
12. W. C. Ridge, *Spectrum of a composition operator*, Proc. Amer. Math. Soc., **37** (1973), 121-127.

Received April 11, 1973.

NORTH CAROLINA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1973 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i>	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i>	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i>	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i>	347
Stephen LaVern Campbell, <i>Linear operators for which T^*T and TT^* commute.</i> <i>II</i>	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i>	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i>	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i>	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in 2^X and $C(X)$</i>	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i>	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i>	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i>	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i>	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i>	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i>	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ..	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i>	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i>	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i>	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot</i>	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i>	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i>	539
Dorte Olesen, <i>Derivations of AW^*-algebras are inner</i>	555
Dorte Olesen and Gert Kjærsgaard Pedersen, <i>Derivations of C^*-algebras have semi-continuous generators</i>	563
Duane O'Neill, <i>On conjugation cobordism</i>	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i>	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of $L^1(\mu; E)$</i>	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i>	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i>	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i>	627
Carl E. Swenson, <i>Direct sum subset decompositions of Z</i>	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i>	635
Robert S. Wilson, <i>Representations of finite rings</i>	643