LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE. II

STEPHEN LAVERN CAMPBELL
LINEAR OPERATORS FOR WHICH $T^*T$ AND $TT^*$ COMMUTE (II)

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Let $(BN)$ denote the class of all bounded linear operators on a Hilbert space such that $T^*T$ and $TT^*$ commute. Let $(BN)^+$ be those $T \in (BN)$ which are hyponormal. Embry has observed that if $T \in (BN)$, then $0 \in W(T)$ or $T$ is normal. This is used to show that if $T \in (BN)$, then $(T + \lambda I) \notin (BN)$ unless $T$ is normal. It is also shown that if $T \in (BN)^+$, then $T^*$ is hyponormal for $n \geq 1$. An example of a $T \in (BN)^+$ such that $T^* \notin (BN)$ is given. Paranormality of operators in $(BN)$ is shown to be equivalent to hyponormality. The relationship between $T$ being in $(BN)$ and $T$ being centered is discussed. Finally, all $3 \times 3$ matrices in $(BN)$ are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators $T$ acting on a separable Hilbert space such that $T^*T$ and $TT^*$ commute. Such operators are called bi-normal and the class of all such operators is denoted $(BN)$. This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all $2 \times 2$ and $3 \times 3$ bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in $(BN)$.

1. All shifts, weighted and unweighted, bilateral and unilateral, are in $(BN)$. Further, operators in $(BN)$, if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

**Theorem 1.** If $T \in (BN)$, then either $T$ is normal or zero is in the interior of the numerical range of $T$, $W(T)$.

**Proof.** Embry has shown that if $T \in (BN)$ and $T$ is not normal, then $0 \in W(T)$ [7, Theorem 1]. She has also shown that if $T \in (BN)$ and $T + T^* \geq 0$, then $T$ is normal [5, Theorem 2]. Thus if $0$ were on the boundary of $W(T)$, by a suitable choice of $\alpha$, $|\alpha| = 1$, we could consider $T_1 = \alpha T$ where $T_1 \in (BN)$ and $T_1 + T_1^* \geq 0$. Then $T$ would be normal.
An interesting consequence of Theorem 1 is that no translate of a bi-normal operator can be bi-normal unless the original operator was normal.

For bounded linear operators $X$ and $Y$ let $[X, Y] = XY - YX$.

**THEOREM 2.** Suppose that $T \in (BN)$. Then $T + \lambda I \in (BN)$, some complex $\lambda \neq 0$, if and only if $T$ is normal.

*Proof.* Suppose $T \in (BN)$. Let $\lambda \neq 0$ be real. Then

$$[(T + \lambda I)^*(T + \lambda I), (T + \lambda I)(T + \lambda I)^*] = 0$$

is equivalent to $[[T^*, T], T + T^*] = 0$. Thus if $T + \lambda I \in (BN)$ for some real $\lambda \neq 0$, then $T + \lambda I \in (BN)$ for all real $\lambda$. But $0 \notin W(T + \lambda I)$ for $\lambda$ sufficiently large so $T$ would be normal by Theorem 1. The case when $\lambda$ is complex easily reduces to the one when $\lambda$ is real.

2. One reason that the class $(BN)$ is of interest is that it includes many of the weighted translated operators of Parrott [10], and nonanalytic composition operators, such as those studied by Ridge [12]. In particular, $(BN)$ includes the Bishop operator [10, p. 2] for which the question of invariant subspaces is still open.

The Bishop operator actually falls into the following class which is more restrictive than $(BN)$.

**DEFINITION 1.** A bounded linear operator $T$ is called centered if the set $\{T^n T^*, T^* T^n\}_{n=0}$ consists of pairwise commuting operators.

Centered operators have been studied by Muhly [9] and Morrell [8]. Muhly has shown that centered operators with zero kernels and dense ranges are the direct sums of weighted translation operators [9]. Parrott has asked (in a private communication) whether the same is true for operators in $(BN)$. We answer this in the negative by exhibiting a $T \in (BN)$ such that $T^2 \notin (BN)$, and $T$ is invertible.

**EXAMPLE 1.** Let $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. Then $T \in (BN)$, $T^2 \in (BN)$, and $T$ is invertible.

3. Powers of hyponormal or bi-normal operators need not be hyponormal or bi-normal. Operators which are both hyponormal and bi-normal are somewhat "nicer". Let $(BN)^+$ denote the hyponormal bi-normal operators.
THEOREM 3. Suppose that $T \in (BN)^+$. Then $T^n$ is hyponormal for $n \geq 1$.

**Proof.** If $C, D$ are positive operators such that $C \geq D \geq 0$, then $TCT^* \geq TDT^* \geq 0$ and $T^*CT \geq T^*DT \geq 0$ for any bounded operator $T$. Suppose now that $T \in (BN)^+$. Since $T^*T \geq TT^*$, we have $T^{*2}T^2 \geq (T^*T)^2$ and $(TT^*)^2 \geq T^2T^2$. But $T^*T \geq TT^*$ and $[T^*T, TT^*] = 0$ implies that $(T^*T)^2 \geq (TT^*)^2$. Hence $T^{*2}T^2 \geq (T^*T)^2 \geq T^2T^2$ and $T^2$ is hyponormal. Suppose then that $T^*nT^* \geq (T^*T)^n \geq (TT^*)^n \geq T^*T^n$ for some integer $n \geq 2$. Then $T^*nT^* \geq (TT^*)^n$ implies that $T^*n+1T^n+1 \geq (T^*T)^n+1$ and $(TT^*)^n \geq T^nT^n$ implies that $(TT^*)^n+1 \geq T^{n+1}T^{*n+1}$. But $(TT^*)^n+1 \geq (TT^*)^n+1$. The theorem now follows by induction.

4. The assumption that $T \in (BN)$ is hyponormal can be weakened to $T \in (BN)$ is paranormal but no added generality is achieved as the next result shows. Recall that $T$ is paranormal if $\|\lambda T^2 - T\| \geq 0$ for every $\lambda > 0$. See for example [1]. Hyponormal operators are paranormal.

THEOREM 4. Suppose that $T \in (BN)$. If $T$ is also paranormal, then it is hyponormal.

**Proof.** Suppose that $T$ is paranormal. Then $AB^2A - 2\lambda A^2 + \lambda^2 I \geq 0$ for every $\lambda > 0$ where $A = (TT^*)^{1/2}$ and $B = (T^*T)^{1/2}$ [1]. Suppose that $T \in (BN)$. The condition for paranormality becomes

$$(*) \quad A^2B^2 - 2\lambda A^2 + \lambda^2 I \geq 0 \text{ for every } \lambda > 0.$$  

Since $[A^2, B^2] = 0$, there exists a spectral measure $E(\cdot)$ such that

$$A^2 = \int f(t)dE(t) \quad \text{and} \quad B^2 = \int g(t)dE(t).$$

Substituting these integrals into $(*)$ gives

$$\int (f(t)g(t) - 2\lambda f(t) + \lambda^2) dE(t) \geq 0.$$  

Let $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } xy - 2\lambda \lambda^2 \geq 0 \text{ for all } \lambda > 0\}$. Then $(f(t), g(t)) \in \theta$ almost everywhere $dE$. We will show now that actually $\theta = \{(x, y): x \geq 0, y \geq 0, \text{ and } y \geq x\}$. Then $g(t) \geq f(t)$ almost everywhere $dE$ and $T^*T \geq TT^*$ as desired. To see that $\theta = \{(x, y): x \geq 0, y \geq 0 \text{ and } y \geq x\}$, observe that $xy - 2\lambda x + \lambda^2 = 0$, $\lambda > 0$, defines the curve $y = h_\lambda(x) = 2\lambda - \lambda^2/x$ in the first quadrant. The line $y = x$ is tangent to $h_\lambda(x)$ at $x = \lambda$. Since $h_\lambda(x)$ is everywhere
concave down we have that it lies entirely on or below $y = x$. But $	heta$ consists of those points in the first quadrant lying above the graph of $h_\lambda$ for every $\lambda > 0$, that is, above the line $y = x$.

An immediate corollary to Theorem 4 which might save time in the construction of examples is the following.

**Corollary 1.** There are no weighted shifts which are paranormal and not hyponormal.

5. Under certain conditions $T$ being in $(BN)$ does imply $T$ is centered. We give two.

**Theorem 5.** Suppose that $||T|| \leq 1$. If $T^*T = f(TT^*)$ and $TT^* = g(T^*T)$ where $f$ and $g$ are continuous functions from $[0, 1]$ into $[0, 1]$, then $T$ is centered.

**Proof.** If $T^*T = f(TT^*)$, then

$$(*) \quad T^*T^2 = T^*f(TT^*)T = f(T^*T)T^*T = f(f(TT^*))f(TT^*) = f_2(TT^*)$$

where $f_2$ is a continuous function from $[0, 1]$ into $[0, 1]$. The second equality of $(*)$ is trivially valid if $f$ is a polynomial. By taking uniform limits of polynomials it can be seen that it is true for all continuous functions $f$. From $(*)$ and an induction argument, we get that $T^*T^n = f_n(TT^*)$ and $T^*T^{*n} = g_n(T^*T)$ for continuous functions $f_n, g_n$ mapping $[0, 1]$ into $[0, 1]$, $n \geq 1$. Hence $[T^*jT^i, T^*jT^i] = 0$ for all integers $i, j \geq 0$.

The assumption that $f, g$ are continuous can be considerably weakened. If $h, k$ are bounded measurable functions from $[0, 1]$ into $[0, 1]$, then let $(h \bullet k)(x) = h(k(x))k(x)$. Set $h_1 = h$ and define $h_n = (h_{n-1} \bullet h)$ for $n \geq 2$. Then the theorem is true if $f_n$ and $g_n$ are well-defined $dE$ measurable functions for every integer $n \geq 1$. $dE$ is the spectral measure of the $^*$-algebra generated by $I, T^*T$ and $TT^*$. Clearly the assumption $||T|| \leq 1$ is not restrictive.

S. K. Parrott has proven the following result (private communication).

**Theorem 6.** If $T \in (BN)$ and $T^*T$ has a cyclic vector, then $T$ is unitarily equivalent to a weighted translation operator.

6. The operator $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ acting on $C^2$ shows that Theorem 6 is not valid for an arbitrary $T \in (BN)$. Our next example shows it is also not true for $T \in (BN)^+$. 
EXAMPLE 2. Let
\[
T_n = \begin{bmatrix} 0 & 0 & \sqrt{2} g(n+1) \\ g(n) & g(n) & 0 \\ g(n) & -g(n) & 0 \end{bmatrix}, \quad n \geq 1,
\]

where \(g(n)\) is a strictly increasing sequence of positive numbers converging to 1. Let
\[
A = \begin{bmatrix} 0 & 0 & 0 \\ T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ \vdots & & \ddots 
\end{bmatrix}
\]
acting on \(\mathcal{K}\) where \(\mathcal{K}\) is a countable number of copies of \(C^n\). Then \(A \in (BN)^+\), but \(A^2 \notin (BN)\). \(A \in (BN)\) since \(A^*A\) and \(AA^*\) are diagonal. \(A \in (BN)^+\) since \(T_{n+1}^* T_{n+1} \geq T_n T_n^*, \ n \geq 1\). So show \(A^2 \notin (BN)\), one need only show that \([(T_{n+1}^* T_n^*)(T_{n+1} T_n), (T_{n+2}^* T_{n+2})^*(T_{n+1} T_{n+2})] \neq 0\) for some \(n \geq 1\). Picking \(n = 1\) and \(g(1) = 0\) makes the calculation easier.

It is easy to modify Example 2 to get an invertible \(A\) such that \(A \in (BN)^+\) and \(A^2 \notin (BN)\). This is done by picking a sequence \(\{g(n)\}_{n=-\infty}^\infty\) such that \(g(n) < g(n+1), \lim_{n \to \infty} g(n) = 1, \text{and} \lim_{n \to -\infty} g(n) = c > 0\). Define \(A\) to be a matrix weighted bilateral shift with weights \(T_n, \ T_n\) as in Example 2.

There remains then the problem of determining what types of operators are in \((BN)^+\).

In the process of proving Theorem 1 of [3] we proved the following result which could be helpful.

If \(C\) is self-adjoint, let \(E_C(\cdot)\) be the spectral measure of \(C\).

**PROPOSITION 1.** If \(T \in (BN)^+\), then \(E_{T\cdot T}([b, \ ||T||]) \mathcal{K}\) is an invariant subspace of \(T\) for every \(b > 0\). Furthermore, \(E_{T\cdot T}([0, b]) \subseteq E_{T\cdot T}([0, b])\) for every \(b > 0\).

By considering weighted shifts in \((BN)^+\) it is easy to see that the subspaces need never be reducing and \([b, ||T||]\) cannot be replaced by a noninterval or by an interval without \(||T||\) as an end point.

7. The presence of a large number of examples is useful both in making conjectures and in finding counterexamples. There has also been some interest in the condition \((BN)\) when \(\dim \mathcal{K} < \infty\) [4]. For these reasons we will now characterize all operators in \((BN)\) when \(\dim \mathcal{K} = 2\) and \(\dim \mathcal{K} = 3\).
DEFINITION 2. If \( \{ \phi_i \} \) is an orthonormal basis, \( D \) is a diagonal matrix with respect to this basis, and \( U \) is a permutation of the basis, then \( T = UD \) is called a weighted permutation.

We say that a matrix \( A \) is a form for \( T \) if \( T \) is unitarily equivalent to a scalar multiple of either \( A \) or \( A^* \).

**THEOREM 7.** If \( T \in (BN) \) and \( \dim \mathcal{L} = 2 \), then the possible forms are:

1. \[
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}, \ a \text{ an arbitrary complex number.}
\]
2. \[
\begin{bmatrix}
1 & b \\
0 & -1
\end{bmatrix}, \ b > 0.
\]
3. \[
\begin{bmatrix}
0 & 1 \\
a & 0
\end{bmatrix}, \text{ an arbitrary.}
\]

**THEOREM 8.** If \( T \in (BN) \) and \( \dim \mathcal{L} = 3 \), then the possible forms are:

1. \[(I1) \begin{bmatrix} c & 0 & 0 \\ 0 & X \\ 0 & 0 & 0 \end{bmatrix} \text{ where } X \text{ is (I2), } c \text{ an arbitrary complex number.}
\]
2. \[(I2) \text{ A weighted permutation.}
\]
3. \[
\begin{bmatrix}
0 & b & -1 \\
0 & 1 & b \\
0 & 0 & 0
\end{bmatrix}, \ b > 0.
\]
4. \[
\begin{bmatrix}
0 & 0 & a \\
u_{21} & u_{22} & 0 \\
u_{31} & u_{32} & 0
\end{bmatrix} \text{ where } a > 0 \text{ and } \begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \text{ is unitary.}
\]

**Proof.** Theorem 7 is easy. Form (II3) is best developed from the form developed in [4] for matrices \( T \) such that \([T' T, TT'] = 0\) where \( T' \) is the generalized inverse of \( T \). If \( T \in (BN) \), then \([T' T, TT'] = 0\). Form (II4) is best developed by looking at the polar form and determining possible unitary parts of \( T \).

Example 1 was found by considering an operator of form (II4). The blocks in Example 2 are also (II4) forms.

In looking for \( (BN) \) matrices the following matrix version of Theorem 6 is useful.

**THEOREM 9.** Suppose that \( T \in (BN) \) and that \( \dim \mathcal{L} = n < \infty \). If \( T^* T \) has \( n \) different eigenvalues, then \( T \) is a weighted permutation.

Theorem 9 can be given a simple matrix proof by observing that if \( T = U(T^* T)^{1/2} \) and \( T \in (BN) \), then \( U(T^* T) = (TT^*)U \) and \( T^* T \) and \( TT^* \) may be simultaneously diagonalized. Furthermore, \( T^* T \) and \( TT^* \) have the same spectrum. It is then easy to see that the only
possible $U$ are permutations of the basis that diagonalizes $T^*T$ and $TT^*$.

It is easy to verify that in all of the forms in Theorem 7 and Theorem 8, except possibly (II4), that zero is in the convex hull of $\sigma(T)$. Is this always true when $n = \dim \mathcal{H} < \infty$? Is it true when $\dim \mathcal{H}$ is infinite? If it is not always true, for what dimensions is it true?

8. All of the two-dimensional bi-normal operators have a square which is normal. Such operators are automatically bi-normal (though never nontrivially hyponormal). This result was proved in [4] and observed independently by Embry in a private communication.

Operators such that $T^2$ is normal have been studied by Embry [6] and completely characterized by Radjavi and Rosenthal [11].

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