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**CONJUGATIONS ON STABLY ALMOST COMPLEX
MANIFOLDS**

ALLAN L. EDELSON

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A stably almost complex structure on a smooth manifold M is an automorphism $J: \tau_M \oplus \theta^k \rightarrow \tau_M \oplus \theta^k$ for some $k \geq 0$, covering the identity map on M , and satisfying $J^2 = -1$. If $k = 0$, J is an almost complex structure. An involution $T: M \rightarrow M$ is a conjugation of (M, J) if there exists an involution $\alpha: \theta^k \rightarrow \theta^k$ covering T , such that $T_* \oplus \alpha$ is conjugate linear, i.e., $(T_* \oplus \alpha) \circ J = -J \circ (T_* \oplus \alpha)$. The bordism theory of conjugations has been studied by R. Stong. In § 2 of this article it is shown that every closed n -manifold can be realized as the fixed point set of a conjugation on a closed, $2n$ -dimensional stably almost complex manifold. This should be compared to the result of Conner and Floyd that the fixed point set of a conjugation on an almost complex $2n$ -manifold is n -dimensional, which is false for stably almost complex manifolds. The proof will use the following result:

LEMMA 1. *Every closed manifold is cobordant to the fixed point set of a conjugation on a closed, almost complex manifold.*

Let $H_{m,n}(C) \subset P^m(C) \times P^n(C)$ with $m \leq n$, be the hypersurface defined as the locus of $w_0 z_0 + w_1 z_1 + \cdots + w_m z_m = 0$ (in homogeneous coordinates (w_0, \cdots, w_m) and (z_0, \cdots, z_n)). Let $H_{m,n}(R)$ be the corresponding real hypersurface. Then generators for the cobordism ring η_* can be taken to be the manifolds $P^{2n}(R)$ and $H_{m,n}(R)$, which are fixed point sets of conjugations on $P^{2n}(C)$ and $H_{m,n}(C)$ respectively. The preceding lemma follows easily.

In § 3, almost complex conjugations on $S^{2q+1} \times S^{2q+1}$ are given, with fixed point set S^{2q+1} . As a consequence, any manifold obtained from $P^{2n}(R)$ or $H_{m,n}(R)$ by surgeries on odd dimensional spheres, is itself the fixed point set of a conjugation on an almost complex manifold.

We will also need the following definition. If T is a free involution on a compact manifold M , a characteristic submanifold for (M, T) is a submanifold $M' \subset M$ of codimension 1, such that $M = W_+ \cup W_-$ (where W_+ and W_- are compact submanifolds of M), $M' = W_+ \cap W_-$, and $T(W_+) = W_-$. M' can always be obtained as the pullback of P^{N-1} by an equivariant map $(M, T) \rightarrow (P^N, A)$, where A is the antipodal map.

2. Stably almost complex structures.

LEMMA 2. *The tangent sphere bundle of a manifold is stably almost complex and the bundle involution is a conjugation.*

Proof. Let $D(M)$ denote the tangent disc bundle of M , and $S(M)$ the sphere bundle, with projection map π . There is an isomorphism $\tau_{D(M)} \cong \pi^*\tau_M \oplus \pi^*\tau_M$, and an almost complex structure can be defined by $(x, y) \rightarrow (-y, x)$. The bundle involution acts as -1 in the bundle tangent to the fibres, identified with the second summand, and is a conjugation. Restricting to $S(M)$ gives a conjugation on $\tau_{S(M)} \oplus \nu_{S(M)}$, ν being the normal bundle to the boundary which is θ^1 .

This lemma provides an important example of stably almost complex manifolds. We are now ready to state the main result of this section.

THEOREM 1. *Every closed n -dimensional manifold is the fixed point set of a conjugation on a closed $2n$ -dimensional stably almost complex manifold.*

Proof. Choose a cobordism (W^{n+1}, F_1, F_2) with F_2 an arbitrary closed n -manifold. Assume F_1 is the fixed point set of a conjugation on the closed, almost complex manifold M_1 . We will construct a closed, stably almost complex $2n$ -manifold M_2 , with conjugation having fixed point set F_2 . Let B denote the tangent sphere bundle to W . Then bB is the unit sphere bundle in $\tau_{bW} \oplus \nu_{bW}$, and the normal bundle of bB in B is trivial. There is then an induced stable almost complex structure and conjugation on bB . Note that throughout this paper, bM will denote the boundary of the manifold M .

LEMMA 3. *The tangent sphere bundle to bW is a stably almost complex submanifold of bB , invariant under the conjugation.*

Proof. Over bW the bundle τ_W splits as $\tau_{bW} \oplus \nu_{bW}$ and $\pi^*\nu_{bW}$ can be identified with the normal bundle in bB , of the tangent sphere bundle to bW . This normal bundle is trivial, so there is an induced stable almost complex structure. Now let S denote the tangent sphere bundle to bW .

LEMMA 4. *There is a stably almost complex submanifold $V \subset B$, invariant under the conjugation, with $bV = V \cap bB = S$.*

Proof. The involution T on B is free, and S is a characteristic submanifold for the restriction $T|_{bB}$. There is a map $f: bB/T \rightarrow P^N$,

for N sufficiently large, that is transverse regular on P^{N-1} and with $S/T = f^{-1}(P^{N-1})$. f extends to a map $F: B/T \rightarrow P^N$, transverse regular on P^{N-1} . Pulling back P^{N-1} under F and lifting to the two-fold covering gives the desired submanifold V . Notice that S is the disjoint union of the tangent sphere bundles to F_1 and F_2 .

There is a submanifold, V' , of the tangent disc bundle to W consisting of V and the tangent disc bundle of bW . This has trivial normal bundle and is invariant under T . There are corners along S , which can be rounded off preserving the triviality of the normal bundle, and we obtain a smooth, stably almost complex manifold with conjugation. The fixed point set of the conjugation is $F_1 \cup F_2$.

Choose a neighborhood N' of F_1 in V' , equivariantly diffeomorphic to the tangent bundle of F_1 . Similarly choose a neighborhood N of F_1 in M_1 . Define a diffeomorphism from $N \setminus F_1 \rightarrow N' \setminus F_1$ by sending $(x, v) \rightarrow (x, -v/\|v\|^2)$, where v is a tangent vector at x . This is smooth, and preserves the almost complex structure along the unit sphere bundle. Form a smooth manifold M_2 from $V' \setminus F_1 \cup M_1 \setminus F_1$ by identifying the above submanifolds. There are almost complex structures on $V' \setminus N'_1$ and $M_1 \setminus N_1$, where N'_1 and N_2 are the vectors of length ≤ 1 . These agree on sphere bundles, and hence M_2 has a stable almost complex structure, provided we add to $\tau_{V'}$, a trivial complex line bundle. The involution on $M_1 \setminus F_1$ is free and hence the fixed point set is F_2 . This completes the proof of Theorem 1.

3. **Conjugations on $S^{2q+1} \times S^{2q+1}$.** In [1], Calabi and Eckmann have described almost complex structures on $S^{2q+1} \times S^{2q+1}$. In this section we will describe a conjugation having fixed point set S^{2q+1} . We begin with a description of the principal bundles involved.

Let $\{U_i\}_{0 \leq i \leq q}$ be the standard open covering of $P^q(C)$ by coordinate neighborhoods. Then $\{U_i \times U_\alpha\}_{0 \leq i, \alpha \leq q}$ is an open covering of $P^q(C) \times P^q(C)$ by coordinate neighborhoods. Let $U_{i\alpha} = U_i \times U_\alpha$. As in [4, Ch. 9], define a principal bundle B over $P^q(C) \times P^q(C)$ with group $G = S^1 \times S^1$ and transition functions $\Psi_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G$ given by

$$\Psi_{i\alpha, j\beta}([Z], [W]) = \left(\frac{z_i |z_j|}{z_j |z_i|}, \frac{w_\alpha |w_\beta|}{w_\beta |w_\alpha|} \right).$$

Note that $Z = (z_0, \dots, z_q)$, $W = (w_0, \dots, w_q)$, and $Z \in S^{2q+1}$, $W \in S^{2q+1}$. Then $\Psi_{i\alpha, j\beta} \Psi_{j\beta, k\gamma} = \Psi_{i\alpha, k\gamma}$. Now let $B_{i\alpha} = U_{i\alpha} \times G$ and define $\tilde{T}_{i\alpha}: B_{i\alpha} \rightarrow B_{i\alpha}$ by $\tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = ([\bar{W}], [\bar{Z}], \bar{\mu}, \bar{\lambda})$.

LEMMA 5. *The map $\tilde{T}: B \rightarrow B$ defined by $\tilde{T}|_{B_{i\alpha}} = \tilde{T}_{i\alpha}$ is a well-defined involution covering $T([Z], [W]) = ([\bar{W}], [\bar{Z}])$.*

Proof. We need to show that the diagram

$$\begin{array}{ccc} B_{i\alpha} & \xrightarrow{\tilde{T}_{i\alpha}} & B_{\alpha i} \\ \downarrow & & \downarrow \\ B_{j\beta} & \xrightarrow{\tilde{T}_{j\beta}} & B_{\beta j} \end{array}$$

in which the vertical maps are the identifications defined on the appropriate intersections, is commutative. We have

$$\begin{aligned} \Psi_{\alpha i, \beta j} \tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) &= \left([W], [Z], \frac{\bar{w}_\alpha |\bar{w}_\beta|}{\bar{w}_\beta |\bar{w}_\alpha|} \bar{\mu}, \frac{\bar{z}_i |\bar{z}_j|}{\bar{z}_j |\bar{z}_i|} \bar{\lambda} \right) \\ &= \tilde{T}_{j\beta} \Psi_{i\alpha, j\beta}([Z], [W], \lambda, \mu) \end{aligned}$$

and so the diagram commutes. The remainder of the lemma is clear. Note the use of the symbol $\Psi_{i\alpha, j\beta}$ to denote the map $B_{i\alpha} \rightarrow B_{j\alpha}$ defined on the appropriate intersection.

Define a map $h_{i\alpha}: B_{i\alpha} \rightarrow S^{2q+1} \times S^{2q+1}$ by

$$h_{i\alpha}([Z], [W], \lambda, \mu) = \left(\lambda \frac{z_i}{|z_i|} Z, \mu \frac{w_\alpha}{|w_\alpha|} W \right).$$

Then $h_{j\beta} \Psi_{i\alpha, j\beta} = h_{i\alpha}$ so that there is a well-defined diffeomorphism $h: B \rightarrow S^{2q+1} \times S^{2q+1}$.

LEMMA 6. *The involution*

$$h \tilde{T} h^{-1}: S^{2q+1} \times S^{2q+1} \longrightarrow S^{2q+1} \times S^{2q+1}$$

is given by $(Z, W) \rightarrow (\bar{W}, \bar{Z})$.

Proof. We have

$$h_{\alpha i} \tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = \left(\bar{\mu} \frac{\bar{w}_\alpha}{|\bar{w}_\alpha|} \bar{W}, \bar{\lambda} \frac{\bar{z}_i}{|\bar{z}_i|} \bar{Z} \right),$$

and the lemma follows.

Again following [4], consider the principal bundle B' over $P^q(C) \times P^q(C)$ with group $G' = C/D$ where D is the subgroup of C generated by the complex numbers $\{1, i\}$. Define transition functions $\Psi'_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G'$ by

$$\begin{aligned} \Psi'_{i\alpha, j\beta}([Z], [W]) &= -\frac{1}{2\pi i} (\log |z_i| + i \log |w_\alpha|) + \frac{1}{2\pi i} \left(\log \frac{z_i}{z_j} + i \log \frac{w_\alpha}{w_\beta} \right) \\ &\quad + \frac{1}{2\pi i} (\log |z_j| + i \log |w_\beta|). \end{aligned}$$

We wish to define a bundle equivalence $f: B \rightarrow B'$. First define an isomorphism $g: G \rightarrow G'$ by

$$g(\lambda, \mu) = \left(\frac{1}{2\pi i} \log \lambda\right) + i\left(\frac{1}{2\pi i} \log \mu\right).$$

It follows that $g\Psi_{i\alpha j\beta} = \Psi'_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G'$, and hence that f can be defined by defining $f_{i\alpha} = 1 \times g: B_{i\alpha} \rightarrow B'_{i\alpha}$. There is an induced involution $\tilde{T}'_{i\alpha} = (1 \times g)\tilde{T}_{i\alpha}(1 \times g^{-1}): B'_{i\alpha} \rightarrow B'_{i\alpha}$ given by

$$\tilde{T}'_{i\alpha}([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}]),$$

and an involution $\tilde{T}': B' \rightarrow B'$. Here $[v]$ denotes the class in G' of the complex number v .

LEMMA 6. \tilde{T}' is a conjugation of the complex manifold B' .

Proof. In local coordinates, \tilde{T}' is given by $\tilde{T}'([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}])$. We need only verify that the map $[v] \rightarrow [i\bar{v}]$ is a conjugation of the complex manifold $G' = C/D$. Since this map sends $[iv]$ to $[(-i)i\bar{v}]$, the lemma follows.

THEOREM 2. S^{2q+1} is the fixed point set of a conjugation $S^{2q+1} \times S^{2q+1}$.

Proof. The diffeomorphisms $f: (B, \tilde{T}) \rightarrow (B', \tilde{T}')$ and $h: (B, \tilde{T}) \rightarrow (S^{2q+1} \times S^{2q+1}, \bar{T})$ are equivariant with respect to the given involutions, and commute with the projections onto $P^q(C) \times P^q(C)$. Note that \bar{T} is defined by $\bar{T}(Z, W) = (\bar{W}, \bar{Z})$. Then theorem follows since the fixed point set of \bar{T} is diffeomorphic to S^{2q+1} .

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Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i>	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i>	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i>	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i>	347
Stephen LaVern Campbell, <i>Linear operators for which T^*T and TT^* commute. II</i>	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i>	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i>	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i>	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in 2^X and $C(X)$</i>	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i>	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i>	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i>	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i>	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i>	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i>	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ...	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i>	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i>	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i>	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot</i>	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i>	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i>	539
Dorte Olesen, <i>Derivations of AW^*-algebras are inner</i>	555
Dorte Olesen and Gert Kjærgaard Pedersen, <i>Derivations of C^*-algebras have semi-continuous generators</i>	563
Duane O'Neill, <i>On conjugation cobordism</i>	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i>	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of $L^1(\mu; E)$</i>	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i>	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i>	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i>	627
Carl E. Swenson, <i>Direct sum subset decompositions of Z</i>	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i>	635
Robert S. Wilson, <i>Representations of finite rings</i>	643