CONJUGATIONS ON STABLY ALMOST COMPLEX MANIFOLDS

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A stably almost complex structure on a smooth manifold $M$ is an automorphism $J: \tau^k M \oplus \theta^k \to \tau^k M \oplus \theta^k$ for some $k \geq 0$, covering the identity map on $M$, and satisfying $J^2 = -1$. If $k = 0$, $J$ is an almost complex structure. An involution $T: M \to M$ is a conjugation of $(M, J)$ if there exists an involution $\alpha: \theta^k \to \theta^k$ covering $T$, such that $T \circ \alpha$ is conjugate linear, i.e., $(T \circ \alpha) \circ J = -J \circ (T \circ \alpha)$. The bordism theory of conjugations has been studied by R. Stong. In §2 of this article it is shown that every closed $n$-manifold can be realized as the fixed point set of a conjugation on a closed, $2n$-dimensional stably almost complex manifold. This should be compared to the result of Conner and Floyd that the fixed point set of a conjugation on an almost complex $2n$-manifold is $n$-dimensional, which is false for stably almost complex manifolds. The proof will use the following result:

**Lemma 1.** Every closed manifold is cobordant to the fixed point set of a conjugation on a closed, almost complex manifold.

Let $H_{m,n}(C) \subset P^m(C) \times P^n(C)$ with $m \leq n$, be the hypersurface defined as the locus of $w_0z_0 + w_1z_1 + \cdots + w_mz_m = 0$ (in homogeneous coordinates $(w_0, \ldots, w_m)$ and $(z_0, \ldots, z_n)$). Let $H_{m,n}(R)$ be the corresponding real hypersurface. Then generators for the cobordism ring $\eta^*$ can be taken to be the manifolds $P^{2n}(R)$ and $H_{m,n}(R)$, which are fixed point sets of conjugations on $P^{2n}(C)$ and $H_{m,n}(C)$ respectively. The preceding lemma follows easily.

In §3, almost complex conjugations on $S^{2n+1} \times S^{2n+1}$ are given, with fixed point set $S^{2n+1}$. As a consequence, any manifold obtained from $P^{2n}(R)$ or $H_{m,n}(R)$ by surgeries on odd dimensional spheres, is itself the fixed point set of a conjugation on an almost complex manifold.

We will also need the following definition. If $T$ is a free involution on a compact manifold $M$, a characteristic submanifold for $(M, T)$ is a submanifold $M' \subset M$ of codimension 1, such that $M = W_+ \cup W_-$ (where $W_+$ and $W_-$ are compact submanifolds of $M$), $M' = W_+ \cap W_-$, and $T(W_+) = W_-$. $M'$ can always be obtained as the pullback of $P^{2n-1}$ by an equivariant map $(M, T) \to (P^n, A)$, where $A$ is the antipodal map.

2. Stably almost complex structures.
Lemma 2. The tangent sphere bundle of a manifold is stably almost complex and the bundle involution is a conjugation.

Proof. Let $D(M)$ denote the tangent disc bundle of $M$, and $S(M)$ the sphere bundle, with projection map $\pi$. There is an isomorphism $\tau_{D(M)} \cong \pi^*\tau_M \oplus \pi^*\tau_M$, and an almost complex structure can be defined by $(x, y) \mapsto (-y, x)$. The bundle involution acts as $-1$ in the bundle tangent to the fibres, identified with the second summand, and is a conjugation. Restricting to $S(M)$ gives a conjugation on $\tau_{S(M)} \oplus \nu_{S(M)}$, $\nu$ being the normal bundle to the boundary which is $\theta$.

This lemma provides an important example of stably almost complex manifolds. We are now ready to state the main result of this section.

Theorem 1. Every closed $n$-dimensional manifold is the fixed point set of a conjugation on a closed $2n$-dimensional stably almost complex manifold.

Proof. Choose a cobordism $(W^{n+1}, F_1, F_0)$ with $F_1$ an arbitrary closed $n$-manifold. Assume $F_1$ is the fixed point set of a conjugation on the closed, almost complex manifold $M$. We will construct a closed, stably almost complex $2n$-manifold $M$, with conjugation having fixed point set $F_1$. Let $B$ denote the tangent sphere bundle to $W$. Then $bB$ is the unit sphere bundle in $\tau_B \oplus \nu_B$, and the normal bundle of $bB$ in $B$ is trivial. There is then an induced stable almost complex structure and conjugation on $bB$. Note that throughout this paper, $bM$ will denote the boundary of the manifold $M$.

Lemma 3. The tangent sphere bundle to $bW$ is a stably almost complex submanifold of $bB$, invariant under the conjugation.

Proof. Over $bW$ the bundle $\tau_w$ splits as $\tau_{bW} \oplus \nu_{bW}$ and $\pi^*\nu_{bW}$ can be identified with the normal bundle in $bB$, of the tangent sphere bundle to $bW$. This normal bundle is trivial, so there is an induced stable almost complex structure. Now let $S$ denote the tangent sphere bundle to $bW$.

Lemma 4. There is a stably almost complex submanifold $V \subset B$, invariant under the conjugation, with $bV = V \cap bB = S$.

Proof. The involution $T$ on $B$ is free, and $S$ is a characteristic submanifold for the restriction $T|_{bB}$. There is a map $f: bB/T \to P^\infty$,
for $N$ sufficiently large, that is transverse regular on $P^{N-1}$ and with $S/T = f^{-1}(P^{N-1})$. $f$ extends to a map $F: B/T \to P^N$, transverse regular on $P^{N-1}$. Pulling back $P^{N-1}$ under $F$ and lifting to the two-fold covering gives the desired submanifold $V$. Notice that $S$ is the disjoint union of the tangent sphere bundles to $F_1$ and $F_2$.

There is a submanifold, $V'$, of the tangent disc bundle to $W$ consisting of $V$ and the tangent disc bundle of $bW$. This has trivial normal bundle and is invariant under $T$. There are corners along $S$, which can be rounded off preserving the triviality of the normal bundle, and we obtain a smooth, stably almost complex manifold with conjugation. The fixed point set of the conjugation is $F_1 \cup F_2$.

Choose a neighborhood $N'$ of $F_1$ in $V'$, equivariantly diffeomorphic to the tangent bundle of $F_1$. Similarly choose a neighborhood $N$ of $F_1$ in $M_i$. Define a diffeomorphism from $N \setminus F_1 \to N' \setminus F_1$ by sending $(x, v) \mapsto (x, -v/\|v\|^2)$, where $v$ is a tangent vector at $x$. This is smooth, and preserves the almost complex structure along the unit sphere bundle. Form a smooth manifold $M_i$ from $V \setminus F_1 \cup M \setminus F_1$ by identifying the above submanifolds. There are almost complex structures on $V \setminus N'_1$ and $M_i \setminus N_i$, where $N'_1$ and $N_i$ are the vectors of length $\leq 1$. These agree on sphere bundles, and hence $M_i$ has a stable almost complex structure, provided we add to $\tau_p$ a trivial complex line bundle. The involution on $M_i \setminus F_1$ is free and hence the fixed point set is $F_1$. This completes the proof of Theorem 1.

3. Conjugations on $S^{2q+1} \times S^{2q+1}$. In [1], Calabi and Eckmann have described almost complex structures on $S^{2q+1} \times S^{2q+1}$. In this section we will describe a conjugation having fixed point set $S^{2q+1}$. We begin with a description of the principal bundles involved.

Let $\{U_{ij}\}_{0 \leq i \leq q}$ be the standard open covering of $P^q(C)$ by coordinate neighborhoods. Then $\{U_i \times U_j\}_{0 \leq i, j \leq q}$ is an open covering of $P^q(C) \times P^q(C)$ by coordinate neighborhoods. Let $U_{ia} = U_i \times U_a$. As in [4, Ch. 9], define a principal bundle $B$ over $P^q(C) \times P^q(C)$ with group $G = S^1 \times S^1$ and transition functions $\Psi_{ia,jb}: U_{ia} \cap U_{j\beta} \to G$ given by

$$\Psi_{ia,jb}([Z], [W]) = \left( \frac{z_i}{z_j}, \frac{z_j}{z_i}, \frac{w_a}{w_\beta}, \frac{w_\beta}{w_a} \right).$$

Note that $Z = (z_0, \ldots, z_q), W = (w_0, \ldots, w_q)$, and $Z \in S^{2q+1}, W \in S^{2q+1}$. Then $\Psi_{ia,jb}^{f_{ia,jb}} = \Psi_{ia,kb}$. Now let $B_{ia} = U_{ia} \times G$ and define $\tilde{T}_{ia}: B_{ia} \to B_{ia}$ by $\tilde{T}_{ia}([Z], [W], \nu, \mu) = ([\tilde{W}], [\tilde{Z}], \tilde{\nu}, \tilde{\mu})$.

**Lemma 5.** The map $\tilde{T}: B \to B$ defined by $\tilde{T}|_{B_{ia}} = \tilde{T}_{ia}$ is a well-defined involution covering $T([Z], [W]) = ([\tilde{W}], [\tilde{Z}])$. 

Proof. We need to show that the diagram

\[
\begin{array}{ccc}
B_{i\alpha} & \xrightarrow{T_{i\alpha}} & B_{n\alpha} \\
\downarrow & & \downarrow \\
B_{j\beta} & \xrightarrow{T_{j\beta}} & B_{n\beta}
\end{array}
\]

in which the vertical maps are the identifications defined on the appropriate intersections, is commutative. We have

\[
\varphi_{i\alpha,j\beta}T_{i\alpha}([Z], [W], \lambda, \mu) = \left( [W], [Z], \frac{\overline{w}_\alpha}{\overline{w}_\beta}, \frac{\overline{z}_i}{\overline{z}_j}, \frac{\overline{z}_j}{\overline{z}_i}, \lambda \right)
\]

and so the diagram commutes. The remainder of the lemma is clear. Note the use of the symbol $\varphi_{i\alpha,j\beta}$ to denote the map $B_{i\alpha} \to B_{j\beta}$ defined on the appropriate intersection.

Define a map $h_{i\alpha}: B_{i\alpha} \to S^{2q+1} \times S^{2q+1}$ by

\[
h_{i\alpha}([Z], [W], \lambda, \mu) = \left( \lambda \frac{z_i}{|z_i|} Z, \frac{\mu}{|w_a|} W \right).
\]

Then $h_{j\beta} \varphi_{i\alpha,j\beta} = h_{i\alpha}$ so that there is a well-defined diffeomorphism $h: B \to S^{2q+1} \times S^{2q+1}$.

Lemma 6. The involution

\[ hT_i^{-1}: S^{2q+1} \times S^{2q+1} \longrightarrow S^{2q+1} \times S^{2q+1} \]

is given by $(Z, W) \mapsto (\overline{W}, \overline{Z})$.

Proof. We have

\[
h_{i\alpha}T_{i\alpha}([Z], [W], \lambda, \mu) = \left( \frac{\overline{w}_a}{|w_a|}, \overline{W}, \frac{\overline{z}_i}{|z_i|} \right),
\]

and the lemma follows.

Again following [4], consider the principal bundle $B'$ over $P^s(C) \times P^s(C)$ with group $G' = C/D$ where $D$ is the subgroup of $C$ generated by the complex numbers $\{1, i\}$. Define transition functions $\varphi'_{i\alpha,j\beta}: U_{i\alpha} \cap U_{j\beta} \to G'$ by

\[
\varphi'_{i\alpha,j\beta}([Z], [W]) = -\frac{1}{2\pi i}(\log |z_i| + i \log |w_a|) + \frac{1}{2\pi i}(\log \frac{z_i}{z_j} + i \log \frac{w_a}{w_j})
\]

\[
+ \frac{1}{2\pi i}(\log |z_j| + i \log |w_j|).
\]
We wish to define a bundle equivalence \( f: B \to B' \). First define an isomorphism \( g: G \to G' \) by
\[
g(\lambda, \mu) = \left( \frac{1}{2\pi i} \log \lambda \right) + i \left( \frac{1}{2\pi i} \log \mu \right).
\]
It follows that \( g\mathcal{F}_{ia, j\beta} = \mathcal{F}'_{ia, j\beta}: U_{ia} \cap U_{j\beta} \to G' \), and hence that \( f \) can be defined by defining \( f_{ia} = 1 \times g: B_{ia} \to B'_{ia} \). There is an induced involution \( \bar{T}'_{ia} = (1 \times g)\bar{T}_{ia}(1 \times g^{-1}): B'_{ia} \to B'_{ai} \) given by
\[
\bar{T}'_{ia}(\{Z, [W], [v]\}) = (\{\bar{W}, [Z], [iv]\}),
\]
and an involution \( \bar{T}': B' \to B' \). Here \([v]\) denotes the class in \( G' \) of the complex number \( v \).

**Lemma 6.** \( \bar{T}' \) is a conjugation of the complex manifold \( B' \).

**Proof.** In local coordinates, \( \bar{T}' \) is given by \( \bar{T}'([Z], [W], [v]) = ([\bar{W}], [Z], [iv]) \). We need only verify that the map \([v] \to [iv] \) is a conjugation of the complex manifold \( G' = \mathbb{C}/D \). Since this map sends \([iv] \) to \([(-i)v]\), the lemma follows.

**Theorem 2.** \( S^{2q+1} \) is the fixed point set of a conjugation \( S^{2q+1} \times S^{2q+1} \).

**Proof.** The diffeomorphisms \( f: (B, \bar{T}) \to (B', \bar{T}') \) and \( h: (B, \bar{T}) \to (S^{2q+1} \times S^{2q+1}, \bar{T}) \) are equivariant with respect to the given involutions, and commute with the projections onto \( P^q(C) \times P^q(C) \). Note that \( \bar{T} \) is defined by \( \bar{T}(Z, W) = (\bar{W}, \bar{Z}) \). Then theorem follows since the fixed point set of \( \bar{T} \) is diffeomorphic to \( S^{2q+1} \).

**References**


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