

# Pacific Journal of Mathematics

**CONJUGATIONS ON STABLY ALMOST COMPLEX  
MANIFOLDS**

ALLAN L. EDELSON

## CONJUGATIONS ON STABLY ALMOST COMPLEX MANIFOLDS

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A stably almost complex structure on a smooth manifold  $M$  is an automorphism  $J: \tau_M \oplus \theta^k \rightarrow \tau_M \oplus \theta^k$  for some  $k \geq 0$ , covering the identity map on  $M$ , and satisfying  $J^2 = -1$ . If  $k = 0$ ,  $J$  is an almost complex structure. An involution  $T: M \rightarrow M$  is a conjugation of  $(M, J)$  if there exists an involution  $\alpha: \theta^k \rightarrow \theta^k$  covering  $T$ , such that  $T_* \oplus \alpha$  is conjugate linear, i.e.,  $(T_* \oplus \alpha) \circ J = -J \circ (T_* \oplus \alpha)$ . The bordism theory of conjugations has been studied by R. Stong. In §2 of this article it is shown that every closed  $n$ -manifold can be realized as the fixed point set of a conjugation on a closed,  $2n$ -dimensional stably almost complex manifold. This should be compared to the result of Conner and Floyd that the fixed point set of a conjugation on an almost complex  $2n$ -manifold is  $n$ -dimensional, which is false for stably almost complex manifolds. The proof will use the following result:

LEMMA 1. *Every closed manifold is cobordant to the fixed point set of a conjugation on a closed, almost complex manifold.*

Let  $H_{m,n}(C) \subset P^m(C) \times P^n(C)$  with  $m \leq n$ , be the hypersurface defined as the locus of  $w_0 z_0 + w_1 z_1 + \cdots + w_m z_m = 0$  (in homogeneous coordinates  $(w_0, \cdots, w_m)$  and  $(z_0, \cdots, z_n)$ ). Let  $H_{m,n}(R)$  be the corresponding real hypersurface. Then generators for the cobordism ring  $\eta_*$  can be taken to be the manifolds  $P^{2n}(R)$  and  $H_{m,n}(R)$ , which are fixed point sets of conjugations on  $P^{2n}(C)$  and  $H_{m,n}(C)$  respectively. The preceding lemma follows easily.

In §3, almost complex conjugations on  $S^{2q+1} \times S^{2q+1}$  are given, with fixed point set  $S^{2q+1}$ . As a consequence, any manifold obtained from  $P^{2n}(R)$  or  $H_{m,n}(R)$  by surgeries on odd dimensional spheres, is itself the fixed point set of a conjugation on an almost complex manifold.

We will also need the following definition. If  $T$  is a free involution on a compact manifold  $M$ , a characteristic submanifold for  $(M, T)$  is a submanifold  $M' \subset M$  of codimension 1, such that  $M = W_+ \cup W_-$  (where  $W_+$  and  $W_-$  are compact submanifolds of  $M$ ),  $M' = W_+ \cap W_-$ , and  $T(W_+) = W_-$ .  $M'$  can always be obtained as the pullback of  $P^{N-1}$  by an equivariant map  $(M, T) \rightarrow (P^N, A)$ , where  $A$  is the antipodal map.

### 2. Stably almost complex structures.

LEMMA 2. *The tangent sphere bundle of a manifold is stably almost complex and the bundle involution is a conjugation.*

*Proof.* Let  $D(M)$  denote the tangent disc bundle of  $M$ , and  $S(M)$  the sphere bundle, with projection map  $\pi$ . There is an isomorphism  $\tau_{D(M)} \cong \pi^*\tau_M \oplus \pi^*\nu_M$ , and an almost complex structure can be defined by  $(x, y) \rightarrow (-y, x)$ . The bundle involution acts as  $-1$  in the bundle tangent to the fibres, identified with the second summand, and is a conjugation. Restricting to  $S(M)$  gives a conjugation on  $\tau_{S(M)} \oplus \nu_{S(M)}$ ,  $\nu$  being the normal bundle to the boundary which is  $\theta^1$ .

This lemma provides an important example of stably almost complex manifolds. We are now ready to state the main result of this section.

THEOREM 1. *Every closed  $n$ -dimensional manifold is the fixed point set of a conjugation on a closed  $2n$ -dimensional stably almost complex manifold.*

*Proof.* Choose a cobordism  $(W^{n+1}, F_1, F_2)$  with  $F_2$  an arbitrary closed  $n$ -manifold. Assume  $F_1$  is the fixed point set of a conjugation on the closed, almost complex manifold  $M_1$ . We will construct a closed, stably almost complex  $2n$ -manifold  $M_2$ , with conjugation having fixed point set  $F_2$ . Let  $B$  denote the tangent sphere bundle to  $W$ . Then  $bB$  is the unit sphere bundle in  $\tau_{bW} \oplus \nu_{bW}$ , and the normal bundle of  $bB$  in  $B$  is trivial. There is then an induced stable almost complex structure and conjugation on  $bB$ . Note that throughout this paper,  $bM$  will denote the boundary of the manifold  $M$ .

LEMMA 3. *The tangent sphere bundle to  $bW$  is a stably almost complex submanifold of  $bB$ , invariant under the conjugation.*

*Proof.* Over  $bW$  the bundle  $\tau_W$  splits as  $\tau_{bW} \oplus \nu_{bW}$  and  $\pi^*\nu_{bW}$  can be identified with the normal bundle in  $bB$ , of the tangent sphere bundle to  $bW$ . This normal bundle is trivial, so there is an induced stable almost complex structure. Now let  $S$  denote the tangent sphere bundle to  $bW$ .

LEMMA 4. *There is a stably almost complex submanifold  $V \subset B$ , invariant under the conjugation, with  $bV = V \cap bB = S$ .*

*Proof.* The involution  $T$  on  $B$  is free, and  $S$  is a characteristic submanifold for the restriction  $T|_{bB}$ . There is a map  $f: bB/T \rightarrow P^N$ ,

for  $N$  sufficiently large, that is transverse regular on  $P^{N-1}$  and with  $S/T = f^{-1}(P^{N-1})$ .  $f$  extends to a map  $F: B/T \rightarrow P^N$ , transverse regular on  $P^{N-1}$ . Pulling back  $P^{N-1}$  under  $F$  and lifting to the two-fold covering gives the desired submanifold  $V$ . Notice that  $S$  is the disjoint union of the tangent sphere bundles to  $F_1$  and  $F_2$ .

There is a submanifold,  $V'$ , of the tangent disc bundle to  $W$  consisting of  $V$  and the tangent disc bundle of  $bW$ . This has trivial normal bundle and is invariant under  $T$ . There are corners along  $S$ , which can be rounded off preserving the triviality of the normal bundle, and we obtain a smooth, stably almost complex manifold with conjugation. The fixed point set of the conjugation is  $F_1 \cup F_2$ .

Choose a neighborhood  $N'$  of  $F_1$  in  $V'$ , equivariantly diffeomorphic to the tangent bundle of  $F_1$ . Similarly choose a neighborhood  $N$  of  $F_1$  in  $M_1$ . Define a diffeomorphism from  $N \setminus F_1 \rightarrow N' \setminus F_1$  by sending  $(x, v) \rightarrow (x, -v/\|v\|^2)$ , where  $v$  is a tangent vector at  $x$ . This is smooth, and preserves the almost complex structure along the unit sphere bundle. Form a smooth manifold  $M_2$  from  $V \setminus F_1 \cup M_1 \setminus F_1$  by identifying the above submanifolds. There are almost complex structures on  $V \setminus N'_1$  and  $M_1 \setminus N_1$ , where  $N'_1$  and  $N_2$  are the vectors of length  $\leq 1$ . These agree on sphere bundles, and hence  $M_2$  has a stable almost complex structure, provided we add to  $\tau_{V'}$ , a trivial complex line bundle. The involution on  $M_1 \setminus F_1$  is free and hence the fixed point set is  $F_2$ . This completes the proof of Theorem 1.

**3. Conjugations on  $S^{2q+1} \times S^{2q+1}$ .** In [1], Calabi and Eckmann have described almost complex structures on  $S^{2q+1} \times S^{2q+1}$ . In this section we will describe a conjugation having fixed point set  $S^{2q+1}$ . We begin with a description of the principal bundles involved.

Let  $\{U_i\}_{0 \leq i \leq q}$  be the standard open covering of  $P^q(C)$  by coordinate neighborhoods. Then  $\{U_i \times U_\alpha\}_{0 \leq i, \alpha \leq q}$  is an open covering of  $P^q(C) \times P^q(C)$  by coordinate neighborhoods. Let  $U_{i\alpha} = U_i \times U_\alpha$ . As in [4, Ch. 9], define a principal bundle  $B$  over  $P^q(C) \times P^q(C)$  with group  $G = S^1 \times S^1$  and transition functions  $\Psi_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G$  given by

$$\Psi_{i\alpha, j\beta}([Z], [W]) = \left( \frac{z_i | z_j |}{z_j | z_i |}, \frac{w_\alpha | w_\beta |}{w_\beta | w_\alpha |} \right).$$

Note that  $Z = (z_0, \dots, z_q)$ ,  $W = (w_0, \dots, w_q)$ , and  $Z \in S^{2q+1}$ ,  $W \in S^{2q+1}$ . Then  $\Psi_{i\alpha, j\beta} \Psi_{j\beta, k\gamma} = \Psi_{i\alpha, k\gamma}$ . Now let  $B_{i\alpha} = U_{i\alpha} \times G$  and define  $\tilde{T}_{i\alpha}: B_{i\alpha} \rightarrow B_{\alpha i}$  by  $\tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = ([\bar{W}], [\bar{Z}], \bar{\mu}, \bar{\lambda})$ .

**LEMMA 5.** *The map  $\tilde{T}: B \rightarrow B$  defined by  $\tilde{T}|_{B_{i\alpha}} = \tilde{T}_{i\alpha}$  is a well-defined involution covering  $T([Z], [W]) = ([\bar{W}], [\bar{Z}])$ .*

*Proof.* We need to show that the diagram

$$\begin{array}{ccc} B_{i\alpha} & \xrightarrow{\tilde{T}_{i\alpha}} & B_{\alpha i} \\ \downarrow & & \downarrow \\ B_{j\beta} & \xrightarrow{\tilde{T}_{j\beta}} & B_{\beta j} \end{array}$$

in which the vertical maps are the identifications defined on the appropriate intersections, is commutative. We have

$$\begin{aligned} \Psi_{\alpha i, \beta j} \tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) &= \left( [W], [Z], \frac{\bar{w}_\alpha | \bar{w}_\beta | \bar{\mu}}{\bar{w}_\beta | \bar{w}_\alpha | \bar{\mu}}, \frac{\bar{z}_i | \bar{z}_j | \bar{\lambda}}{\bar{z}_j | \bar{z}_i | \bar{\lambda}} \right) \\ &= \tilde{T}_{j\beta} \Psi_{i\alpha, j\beta}([Z], [W], \lambda, \mu) \end{aligned}$$

and so the diagram commutes. The remainder of the lemma is clear. Note the use of the symbol  $\Psi_{i\alpha, j\beta}$  to denote the map  $B_{i\alpha} \rightarrow B_{j\alpha}$  defined on the appropriate intersection.

Define a map  $h_{i\alpha}: B_{i\alpha} \rightarrow S^{2q+1} \times S^{2q+1}$  by

$$h_{i\alpha}([Z], [W], \lambda, \mu) = \left( \lambda \frac{z_i}{|z_i|} Z, \mu \frac{w_\alpha}{|w_\alpha|} W \right).$$

Then  $h_{j\beta} \Psi_{i\alpha, j\beta} = h_{i\alpha}$  so that there is a well-defined diffeomorphism  $h: B \rightarrow S^{2q+1} \times S^{2q+1}$ .

LEMMA 6. *The involution*

$$h \tilde{T} h^{-1}: S^{2q+1} \times S^{2q+1} \longrightarrow S^{2q+1} \times S^{2q+1}$$

is given by  $(Z, W) \rightarrow (\bar{W}, \bar{Z})$ .

*Proof.* We have

$$h_{\alpha i} \tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = \left( \bar{\mu} \frac{\bar{w}_\alpha}{|\bar{w}_\alpha|} \bar{W}, \bar{\lambda} \frac{\bar{z}_i}{|\bar{z}_i|} \bar{Z} \right),$$

and the lemma follows.

Again following [4], consider the principal bundle  $B'$  over  $P^q(C) \times P^q(C)$  with group  $G' = C/D$  where  $D$  is the subgroup of  $C$  generated by the complex numbers  $\{1, i\}$ . Define transition functions  $\Psi'_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G'$  by

$$\begin{aligned} \Psi'_{i\alpha, j\beta}([Z], [W]) &= -\frac{1}{2\pi i} (\log |z_i| + i \log |w_\alpha|) + \frac{1}{2\pi i} \left( \log \frac{z_i}{z_j} + i \log \frac{w_\alpha}{w_\beta} \right) \\ &\quad + \frac{1}{2\pi i} (\log |z_j| + i \log |w_\beta|). \end{aligned}$$

We wish to define a bundle equivalence  $f: B \rightarrow B'$ . First define an isomorphism  $g: G \rightarrow G'$  by

$$g(\lambda, \mu) = \left( \frac{1}{2\pi i} \log \lambda \right) + i \left( \frac{1}{2\pi i} \log \mu \right).$$

It follows that  $g\Psi_{i\alpha j\beta} = \Psi'_{i\alpha, j\beta}: U_{i\alpha} \cap U_{j\beta} \rightarrow G'$ , and hence that  $f$  can be defined by defining  $f_{i\alpha} = 1 \times g: B_{i\alpha} \rightarrow B'_{i\alpha}$ . There is an induced involution  $\tilde{T}'_{i\alpha} = (1 \times g)\tilde{T}_{i\alpha}(1 \times g^{-1}): B'_{i\alpha} \rightarrow B'_{i\alpha}$  given by

$$\tilde{T}'_{i\alpha}([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}]),$$

and an involution  $\tilde{T}': B' \rightarrow B'$ . Here  $[v]$  denotes the class in  $G'$  of the complex number  $v$ .

LEMMA 6.  $\tilde{T}'$  is a conjugation of the complex manifold  $B'$ .

*Proof.* In local coordinates,  $\tilde{T}'$  is given by  $\tilde{T}'([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}])$ . We need only verify that the map  $[v] \rightarrow [i\bar{v}]$  is a conjugation of the complex manifold  $G' = C/D$ . Since this map sends  $[iv]$  to  $[(-i)i\bar{v}]$ , the lemma follows.

THEOREM 2.  $S^{2q+1}$  is the fixed point set of a conjugation  $S^{2q+1} \times S^{2q+1}$ .

*Proof.* The diffeomorphisms  $f: (B, \tilde{T}) \rightarrow (B', \tilde{T}')$  and  $h: (B, \tilde{T}) \rightarrow (S^{2q+1} \times S^{2q+1}, \bar{T})$  are equivariant with respect to the given involutions, and commute with the projections onto  $P^q(C) \times P^q(C)$ . Note that  $\bar{T}$  is defined by  $\bar{T}(Z, W) = (\bar{W}, \bar{Z})$ . Then theorem follows since the fixed point set of  $\bar{T}$  is diffeomorphic to  $S^{2q+1}$ .

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