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Let X be a compact connected metric space and $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of X. In this paper the hyperspaces are investigated with respect to point-wise connectivity properties. Let $M \in C(X)$. Then 2^X is locally connected (connected im kleinen) at M if and only if for each open set U containing M there is a connected open set V such that $M \subset V \subset U$ (there is a component of U which contains M in its interior). This theorem is used to prove the following main result. Let $A \in 2^X$. Then 2^X is locally connected (connected im kleinen) at A if and only if 2^X is locally connected (connected im kleinen) at each component of A. Several related results about $C(X)$ are also obtained.

A continuum X will be a compact connected metric space. $2^X(C(X))$ denotes the hyperspace of closed subsets (subcontinua) of X, each with the finite (Vietoris) topology, and since X is a continuum, each of 2^X and $C(X)$ is also a continuum (see [5]).

One of the earliest results about hyperspaces of continua, due to Wojdyslawski [7], was that each of 2^X and $C(X)$ is locally connected if and only if X is locally connected. As a point-wise property, local connectedness is stronger than connectedness im kleinen, which in turn is stronger than aposyndesis. The author [1] has shown that if X is any continuum, then each of 2^X and $C(X)$ is aposyndetic. It is the purpose of this paper to investigate the internal structure of 2^X and $C(X)$ with respect to these properties. In particular, we determine necessary and sufficient conditions (in terms of the neighborhood structure in X) that 2^X be locally connected at a point and that 2^X be connected im kleinen at a point. We also determine that $C(X)$ has, in general, stronger point-wise connectivity properties that either 2^X or X.

For notational purposes, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $A \subset X$, then A^* (int A) (bd A) will denote the closure (interior) (boundary) of A in X.

Let $x \in X$. Then X is locally connected (l.c.) at x if for each open set U containing x there is a connected open set V such that $x \in V \subset U$. X is connected im kleinen (c.i.k.) at x if for each open set U containing x there is a component of U which contains x in its interior. X is aposyndetic at x if for each $y \in X - x$ there is a continuum M such that $x \in \operatorname{int} M$ and $y \in X - M$.

If $A_1, \cdots, A_n$ are subsets of X, then $N(A_1, \cdots, A_n) = \{B \in 2^X \mid$ for each $i = 1, \cdots, n$, $B \cap A_i \neq \emptyset$, and $B \subset \bigcup_{i=1}^n A_i\}$. The collection of all sets of the form $N(U_1, \cdots, U_n)$, with $U_1, \cdots, U_n$ open in X, is a base for the finite topology. It is easy to establish that

$$N(U_1, \cdots, U_n)^* = N(U_1^*, \cdots, U_n^*)$$

and that $N(V_1, \cdots, V_m) \subset N(U_1, \cdots, U_n)$ if and only if $\bigcup_{j=1}^m V_j \subset \bigcup_{i=1}^n U_i$ and for each U_i there exists a V_j such that $V_j \subset U_i$ (see [5]). We remark also that the finite topology is equivalent to the Hausdorff metric topology on 2^X whenever X is a compact metric space (theorem on page 47 of [4]).

If $\mathscr{A} \subset 2^X$, then $\bigcup \{A \mid A \in \mathscr{A}\}$ is open (closed) in X whenever $\mathscr{A}$ is open (closed) in 2^X (see [5]). Furthermore, if $\mathscr{A} \cap C(X) \neq \emptyset$ and $\mathscr{A}$ is connected, then $\bigcup \{A \mid A \in \mathscr{A}\}$ is connected (Lemma 1.2 of [3]).

If n is a positive integer, then $F_n(X) = \{A \in 2^X \mid A$ has at most n elements$\}$ and $F(X) = \bigcup_{n=1}^\infty F_n(X)$.

An order arc in $2^X(C(X))$ is an arc which is also a chain with respect to the partial order on $2^X(C(X))$ induced by set inclusion. If $A, B \in 2^X$, then there exists an order arc from A to B if and only if $A \subset B$ and each component of B meets A (Lemma 2.3 of [3]). It follows (Lemma 2.6 of [3]) that every order arc whose initial point is an element of $C(X)$ is entirely contained within $C(X)$.

It will be convenient to begin our study by considering points of $C(X)$.

THEOREM 1. *Let $M \in C(X)$. Then 2^X is c.i.k. at M if and only if for each open set U containing M there is a component of U which contains M in its interior.*

Proof. Suppose 2^X is c.i.k. at M. Let U be an open set containing M. Then $M \in N(U)$, so there exists a component $\mathscr{C}$ of $N(U)$ containing M in its interior. It follows that $\bigcup \{A \mid A \in \mathscr{C}\}$ is a connected set containing M in its interior and lying in U.

Conversely, suppose that for each open set U containing M there is a component of U which contains M in its interior. Let $N(U_1, \cdots, U_n)$ be a basic open set containing M and let $N(V_1, \cdots, V_m)$ be a basic open set such that $M \in N(V_1, \cdots, V_m) \subset N(V_1, \cdots, V_m)^* \subset N(U_1, \cdots, U_n)$. Let $V = \bigcup_{i=1}^m V_i$. Then there is a component C of V which contains M in its interior. For each $i = 1, \cdots, m$, let $W_i = V_i \cap \operatorname{int} C$. Then $M \in N(W_1, \cdots, W_m) \subset N(V_1, \cdots, V_m)$. If $A \in N(W_1, \cdots, W_m)$, then $A \subset C^*$, and A, $C^* \in N(V_1^*, \cdots, V_m^*) =$

$N(V_1, \cdots, V_m)^* \subset N(U_1, \cdots, U_n)$. Since C^* is connected there exists an order arc in $N(U_1, \cdots, U_n)$ from A to C^*. It follows that there is a component of $N(U_1, \cdots, U_n)$ which contains M in its interior.

COROLLARY 1. *Let $x \in X$. Then 2^X is c.i.k. at $\{x\}$ if and only if X is c.i.k. at x.*

LEMMA 1. *Let V be a connected open set and $V_1, \cdots, V_n$ be open sets such that $\bigcup_{i=1}^n V_i = V$. Then $N(V_1, \cdots, V_n)$ is connected.*

Proof. Let p be the smallest positive integer such that $F_p(X) \cap N(V_1, \cdots, V_n) \neq \varnothing$. We will show that

$$\mathscr{F} = \bigcup_{i=p}^{\infty} (F_i(X) \cap N(V_1, \cdots, V_n))$$

is connected.

Let $\mathscr{A} = \{\{x_1, \cdots, x_n\} \mid \text{for each } i = 1, \cdots, n, x_i \in V_i, \text{ and } x_i = x_j \text{ if and only if } i = j\}$. We will first establish that $\mathscr{A}$ lies in a connected subset of $\mathscr{F}$. Let $\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\} \in \mathscr{A}$. Define $\mathscr{A}_1 = \{\{x_1, \cdots, x_n, y\} \mid y \in V\}$ and $\mathscr{B}_1 = \{\{y_1, x_2, \cdots, x_n, y\} \mid y \in V\}$. Then each of $\mathscr{A}_1$ and $\mathscr{B}_1$ is the continuous image of the connected set V, so $\mathscr{A}_1$ is a connected subset of $\mathscr{F}$ which contains $\{x_1, \cdots, x_n\}$ and $\{x_1, \cdots, x_n, y_1\}$ and $\mathscr{B}_1$ is a connected subset of $\mathscr{F}$ which contains $\{x_1, \cdots, x_n, y_1\}$ and $\{y_1, x_2, \cdots, x_n\}$. Similarly, for each $i = 2, \cdots, n-1$ define $\mathscr{A}_i = \{\{y_1, \cdots, y_{i-1}, x_i, \cdots, x_n, y\} \mid y \in V\}$ and

$$\mathscr{B}_i = \{\{y_1, \cdots, y_i, x_{i+1}, \cdots, x_n, y\} \mid y \in V\} .$$

Then $\mathscr{A}_i$ is a connected subset of $\mathscr{F}$ which contains $\{y_1, \cdots, y_{i-1}, x_i, \cdots, x_n\}$ and $\{y_1, \cdots, y_i, x_i, \cdots, x_n\}$ and $\mathscr{B}_i$ is a connected subset of $\mathscr{F}$ which contains $\{y_1, \cdots, y_i, x_i, \cdots, x_n\}$ and $\{y_1, \cdots, y_i, x_{i+1}, \cdots, x_n\}$. Define $\mathscr{A}_n = \{\{y_1, \cdots, y_{n-1}, x_n, y\} \mid y \in V\}$ and

$$\mathscr{B}_n = \{\{y_1, \cdots, y_n, y\} \mid y \in V\} .$$

Then $\mathscr{A}_n$ is a connected subset of $\mathscr{F}$ which contains $\{y_1, \cdots, y_{n-1}, x_n\}$ and $\{y_1, \cdots, y_n, x_n\}$ and $\mathscr{B}_n$ is a connected subset of $\mathscr{F}$ which contains $\{y_1, \cdots, y_n, x_n\}$ and $\{y_1, \cdots, y_n\}$. It follows that $\bigcup_{i=1}^n (\mathscr{A}_i \cup \mathscr{B}_i)$ is a connected subset of $\mathscr{F}$ which contains $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$.

Now let $\{x_1, \cdots, x_m\} \in \mathscr{F} - \mathscr{A}$. If $p \leqq m < n$, choose $n - m$ distinct elements $x_{m+1}, \cdots, x_n$ such that $\{x_1, \cdots, x_m, x_{m+1}, \cdots, x_n\} \in \mathscr{A}$. For each $i = 1, \cdots, n - m$ let $\mathscr{C}_i = \{\{x_1, \cdots, x_{m+(i-1)}, y\} \mid y \in V\}$. Then $\mathscr{C}_i$ is a connected subset of $\mathscr{F}$ containing $\{x_1, \cdots, x_{m+(i-1)}\}$ and $\{x_1, \cdots, x_{m+i}\}$. Hence $\bigcup_{i=1}^{n-m} \mathscr{C}_i$ is a connected subset of $\mathscr{F}$ containing $\{x_1, \cdots, x_m\}$ and $\{x_1, \cdots, x_n\}$.

If $m \geqq n$, let $\{y_1, \cdots, y_n\} \in \mathscr{A}$. Let $\mathscr{D}_1 = \{\{x_1, \cdots, x_m, y\} \mid y \in V\}$.

Then $\mathscr{D}_1$ is a connected subset of $\mathscr{F}$ containing $\{x_1, \cdots, x_m\}$ and $\{x_1, \cdots, x_m, y_1\}$. For each $i = 2, \cdots, n$, let $\mathscr{D}_i = \{\{x_1, \cdots, x_m, y_1, \cdots, y_{i-1}, y\} \mid y \in V\}$. Then $\mathscr{D}_i$ is a connnected subset of $\mathscr{F}$ containing $\{x_1, \cdots, x_m, y_1, \cdots, y_{i-1}\}$ and $\{x_1, \cdots, x_m, y_1, \cdots, y_i\}$. Hence $\bigcup_{i=1}^n \mathscr{D}_i$ is a connected subset of $\mathscr{F}$ containing $\{x_1, \cdots, x_m\}$ and $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$. With an analogous construction we can show that there is a connected subset of $\mathscr{F}$ which contains $\{y_1, \cdots, y_n\}$ and $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$. It follows that there is a connected subset of $\mathscr{F}$ which contains $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_n\}$.

We have now established that $\mathscr{A}$ lies in a connected subset of $\mathscr{F}$ and that each member of $\mathscr{F} - \mathscr{A}$ lies in a connected subset of $\mathscr{F}$ which meets $\mathscr{A}$. Hence $\mathscr{F}$ is connected. Since $\mathscr{F}$ is dense in $N(V_1, \cdots, V_n)$, it follows that $N(V_1, \cdots, V_n)$ is connected.

THEOREM 2. *Let $M \in C(X)$. Then 2^X is l.c. at M if and only if for each open set U containing M there exists a connected open set V such that $M \subset V \subset U$.*

Proof. Suppose 2^X is l.c. at M. Let U be an open set containing M. Then $M \in N(U)$, so there exists a connected open set $\mathscr{V}$ such that $M \in \mathscr{V} \subset N(U)$. It follows that $M \subset \bigcup \{A \mid A \in \mathscr{V}\} = V \subset U$, and V is open and connected.

Conversely, suppose that for each open set U containing M there exists a connected open set V such that $M \subset V \subset U$. Let $N(U_1, \cdots, U_n)$ be a basic open set such that $M \in N(U_1, \cdots, U_n)$ and let $U = \bigcup_{i=1}^n U_i$. Then there exists a connected open set V such that $M \subset V \subset U$. Let $V_i = V \cap U_i$. Then $M \in N(V_1, \cdots, V_n) \subset N(U_1, \cdots, U_n)$, and by Lemma 1, $N(V_1, \cdots, V_n)$ is connected.

COROLLARY 2. *Let $x \in X$. Then 2^X is l.c. at $\{x\}$ if and only if X is l.c. at x.*

We remark that if $M \in C(X)$ and 2^X is l.c. at M, then Lemma 1 and Theorem 2 imply the existence of a local base of connected sets at M, each of which is of the form $N(U_1, \cdots, U_n)$.

The next several results concern the relationships between 2^X and $C(X)$ with respect to local connectedness and connectedness im kleinen at points of $C(X)$.

THEOREM 3. *Let $M \in C(X)$. If 2^X is c.i.k. at M, then $C(X)$ is c.i.k. at M.*

Proof. Let $N(U_1, \cdots, U_n) \cap C(X)$ be an open set containing M. Let $N(V_1, \cdots, V_m)$ be an open set such that $M \in N(V_1, \cdots, V_m) \subset$

$N(V_1, \cdots, V_m)^* \subset N(U_1, \cdots, U_n)$. Since 2^X is c.i.k. at M, there exists an open set $N(W_1, \cdots, W_k)$ such that

$$M \in N(W_1, \cdots, W_k) \subset N(V_1, \cdots, V_m)$$

and with the property that $B \in N(W_1, \cdots, W_k)$ implies $N(V_1, \cdots, V_m)$ contains a connected set containing B and M. Then $N(U_1, \cdots, U_n)$ contains a continuum containing B and M.

Let $K \in N(W_1, \cdots, W_k) \cap C(X)$. Then there exists a continuum $\mathscr{L}$ in $N(U_1, \cdots, U_n)$ containing K and M. Now $\bigcup \{A \mid A \in \mathscr{L}\} = L \in C(X)$, and $L \in N(U_1, \cdots, U_n)$, since $\mathscr{L} \subset N(U_1, \cdots, U_n)$. It follows that there exist order arcs $\mathscr{L}_K$ and $\mathscr{L}_M$ in $N(U_1, \cdots, U_n) \cap C(X)$ from K to L and from M to L. So $\mathscr{L}_K \cup \mathscr{L}_M$ is a continuum in $N(U_1, \cdots, U_n) \cap C(X)$ containing K and M. Hence $C(X)$ is c.i.k. at M.

COROLLARY 3. *Let $M \in C(X)$. If for each open set U containing M there is a component of U which contains M in its interior, then $C(X)$ is c.i.k. at M.*

Corollary 3 is a generalization of Theorem 6 of [6]. The example following Theorem 6 of [6] shows that the converse of Corollary 3 is false. It also shows that the converse of Question 1 below is false.

Question 1. Let $M \in C(X)$. If 2^X is l.c. at M, is $C(X)$ l.c. at M?

COROLLARY 4. *Let $x \in X$. Then X is c.i.k. at x if and only if $C(X)$ is c.i.k. at $\{x\}$.*

Proof. If X is c.i.k. at x, then by Corollary 1, 2^X is c.i.k. at $\{x\}$, and by Theorem 3, $C(X)$ is c.i.k. at $\{x\}$.

Suppose $C(X)$ is c.i.k. at $\{x\}$. Let U be an open set containing x. Then $\{x\} \in N(U) \cap C(X)$, so there exists an open set $N(V) \cap C(X)$, $\{x\} \in N(V) \cap C(X) \subset N(U) \cap C(X)$, with the property that $M \in N(V) \cap C(X)$ implies $N(U) \cap C(X)$ contains a connected set containing M and $\{x\}$.

Now $x \in V \subset U$. Let $y \in V$. Then $\{y\} \in N(V) \cap C(X)$, so $N(U) \cap C(X)$ contains a connected set $\mathscr{L}$ containing $\{y\}$ and $\{x\}$. It follows that $\bigcup \{L \mid L \in \mathscr{L}\}$ is a connected subset of U containing x and y. Hence X is c.i.k. at x.

COROLLARY 5. *Let $x \in X$. If X is l.c. at x, then $C(X)$ is l.c. at $\{x\}$.*

Proof. This follows from the observation that if V is connected, then $N(V) \cap C(X)$ is connected, since each point of $(N(V) \cap C(X)) - F_1(V)$ can be joined by an order arc in $N(V) \cap C(X)$ to a point of $F_1(V)$, and $F_1(V)$ is connected.

The next example shows that the converse of Corollary 5 is false.

EXAMPLE 1. This example is from page 113 of [2]. For each positive integer n and each positive integer m let $L_{n,m}$ denote the line segment in the plane from $(1/(n+1), (-1)^{n+1}1/m(n+1))$ to $(1/n, 0)$. Let $A_n = (\bigcup_{m=1}^{\infty} L_{n,m})^*$ and let $X = (\bigcup_{n=1}^{\infty} A_n)^*$. Then X is c.i.k. at $(0, 0)$ but is not l.c. at $(0, 0)$.

We now give a brief argument that $C(X)$ is l.c. at $\{(0, 0)\}$. For each $n \geqq 2$ choose q_n, r_n, and s_n so that $1/(n+1) < q_n < r_n < 1/n < s_n < 1/(n-1)$. Let $U_n = \{(x, y) \mid x < r_n\}$ and $V_n = \{(x, y) \mid q_n < x < s_n\}$. Then $N(U_n) \cup N(U_n, V_n)$ is a continuum-wise connected open set in $C(X)$ containing $\{(0, 0)\}$, for if M, $N \in N(U_n) \cup N(U_n, V_n)$, then $M, N \subset \{(x, y) \mid x < 1/n\} \cup \{(x, 0) \mid 1/n \leqq x < s_n\}$ and a continuum can be constructed in $C(X)$ containing M and N and lying in $N(U_n) \cup N(U_n, V_n)$. Clearly $\{N(U_n) \cup N(U_n, V_n) \mid n = 2, 3, \cdots\}$ is a neighborhood base at $\{(0, 0)\}$.

The following definition and Lemma 2 concern the finite topology and will be used in proving our main results, in which we obtain necessary and sufficient conditions that 2^X be l.c. (c.i.k.) at an arbitrary point.

Let $A \in 2^X$. A basic open set $N(U_1, \cdots, U_n)$ is *essential with respect to* A if $A \in N(U_1, \cdots, U_n)$ and for each $i = 1, \cdots, n$, $A - \bigcup_{j \neq i} U_j \neq \emptyset$.

LEMMA 2. *Let $A \in 2^X$ and $N(U_1, \cdots, U_n)$ be an open set containing A. Then there exists an open set $N(V_1, \cdots, V_m)$ such that $A \in N(V_1, \cdots, V_m) \subset N(U_1, \cdots, U_n)$ and $N(V_1, \cdots, V_m)$ is essential with respect to A.*

Proof. Choose $x_1, \cdots, x_n \in A$ such that $x_i \in U_i$. Let $V_1, \cdots, V_n$ be open sets such that for each $i = 1, \cdots, n$, $x_i \in V_i \subset \bigcap \{U_j \mid x_i \in U_j\}$ and with the additional property that $V_i = V_j$ if $x_i = x_j$ and $V_i \cap V_j = \emptyset$ if $x_i \neq x_j$. Let $\{V_1, \cdots, V_k\}$ (relabeling if necessary) be the set of V_i's which are distinct. For each $y \in A - \bigcup_{i=1}^{k} V_i$ let O_y be an open set such that $y \in O_y \subset \bigcap \{U_j \mid y \in U_j\}$ and such that $O_y \cap \{x_1, \cdots, x_n\} = \emptyset$. Since $A - \bigcup_{i=1}^{k} V_i$ is compact, there exist $y_1, \cdots, y_p$ such that $A - \bigcup_{i=1}^{k} V_i \subset \bigcup_{i=1}^{p} O_{y_i}$. We may assume that all the O_{y_i}'s are distinct. Let $\{O_{y_1}, \cdots, O_{y_q}\}$ (relabeling if necessary) be the subset of

$\{O_{y_1}, \cdots, O_{y_p}\}$ consisting of all the O_{y_i}'s with the property that $(A - \bigcup_{i=1}^k V_i) - \bigcup_{j \neq i} O_{y_j} \neq \varnothing$.

For notational purposes, for each $j = 1, \cdots, q$ let $O_{y_j} = V_{k+j}$ and let $k + q = m$. Then $A \in N(V_1, \cdots, V_k, V_{k+1}, \cdots, V_m)$. Clearly

$$N(V_1, \cdots, V_k, V_{k+1}, \cdots, V_m) \subset N(U_1, \cdots, U_n) .$$

For each $j = 1, \cdots, k$ there exists $x_i \in A$ such that $x_i \in V_j$ and $x_i \notin (\bigcup_{p=1}^m V_p) - V_j$. For each $j = k + 1, \cdots, m$,

$$\left(A - \bigcup_{i=1}^{k} V_i\right) - \bigcup_{\substack{i=k+1 \\ i \neq j}}^{m} V_i \neq \varnothing ,$$

so there exists $a_j \in V_j \cap (A - \bigcup_{i=1}^k V_i)$ such that $a_j \notin \bigcup_{i \neq j} V_i$. It follows that $N(V_1, \cdots, V_m)$ is essential with respect to A.

THEOREM 4. *Let $A \in 2^X$. Then 2^X is c.i.k. at A if and only if 2^X is c.i.k. at each component of A.*

Proof. Suppose that 2^X is c.i.k. at A. Let A_1 be a component of A and let W be an open set containing A_1. Let U be an open set such that $A_1 \subset U \subset U^* \subset W$ and such that $(\text{bd } U) \cap A = \varnothing$. Let $\{U_1, \cdots, U_n\}$ be a finite cover of $A - U$ by open sets such that for each $i = 1, \cdots, n$, $U \cap U_i = \varnothing$ and $A \cap U_i \neq \varnothing$. Then $A \in N(U, U_1, \cdots, U_n)$.

Let $\mathscr{C}$ be a component of $N(U, U_1, \cdots, U_n)$ which contains A in its interior. Define $f: \mathscr{C} \to N(U)$ by $f(B) = B \cap U$. If $N(V_1, \cdots, V_p) \subset N(U)$, then $f^{-1}(N(V_1, \cdots, V_p)) = N(V_1, \cdots, V_p, U_1, \cdots, U_n) \cap \mathscr{C}$, so f is continuous. Hence $f(\mathscr{C})$ is connected.

Let $N(V_1, \cdots, V_q)$ be an open set such that $A \in N(V_1, \cdots, V_q) \subset \mathscr{C}$. Let $\{V_1, \cdots, V_m\}$ (relabeling if necessary) be the largest subset of $\{V_1, \cdots, V_q\}$ with the property that for each $j = 1, \cdots, m$, $V_j \cap U \neq \varnothing$. Let $\{V_1, \cdots, V_k\}$ (relabeling if necessary) be the largest subset of $V_1, \cdots, V_m$ with the property that for each $j = 1, \cdots, k$, $V_j \cap (\bigcup_{i=1}^n U_i) = \varnothing$. For each $j = 1, \cdots, k$, let $V_j^1 = V_j \cap U$ and $V_j^2 = V_j \cap (\bigcup_{i=1}^n U_i)$. Then

$$\begin{aligned} A \in N(V_1^1, \cdots, V_k^1, V_{k+1}, \cdots, V_m, V_1^2, \cdots, V_k^2, V_{m+1}, \cdots, V_q) \\ = \mathscr{V} \subset N(V_1, \cdots, V_q) \subset \mathscr{C} . \end{aligned}$$

Now if $B \in \mathscr{V}$, then

$$\begin{aligned} f(B) &= B \cap U \\ &= B \cap \left[\left(\bigcup_{j=1}^{k} V_j^1\right) \cup \left(\bigcup_{j=k+1}^{m} V_j\right)\right] \in N(V_1^1, \cdots, V_k^1, V_{k+1}, \cdots, V_m) . \end{aligned}$$

Conversely, suppose $C \in N(V_1^1, \cdots, V_k^1, V_{k+1}, \cdots, V_m)$. For each $j = 1, \cdots, k$, let $x_j \in V_j^2$ and for each $j = m + 1, \cdots, q$ let $x_j \in V_j$. Then

$C \cup \{x_1, \cdots, x_k, x_{m+1}, \cdots, x_q\} \in \mathscr{Y}$ and $f(C \cup \{x_1, \cdots, x_k, x_{m+1}, \cdots, x_q\}) = C \in f(\mathscr{Y})$. Hence $f(\mathscr{Y}) = N(V_1^1, \cdots, V_k^1, V_{k+1}, \cdots, V_m)$. So $f(\mathscr{C})$ contains an open set containing $A \cap U$.

Let $C = \bigcup \{f(B) \mid B \in \mathscr{C}\}$. Then $C^* \subset U^* \subset W$. Let $C(A_1)$ be the component of C^* which contains A_1. Let $N(V_1, \cdots, V_m, V_{m+1}, \cdots, V_p)$ be an open set such that $A \in N(V_1, \cdots, V_m, V_{m+1}, \cdots, V_p) \subset \mathscr{C}$ and such that $\bigcup_{i=1}^m V_i \subset U$ and $\bigcup_{i=m+1}^p V_i \subset \bigcup_{i=1}^n U_i$. Let $\{V_1, \cdots, V_k\}$ (relabeling if necessary) be the largest subset of $\{V_1, \cdots, V_m\}$ with the property that for each $i = 1, \cdots, k$, $V_i^* \cap C(A_1) = \emptyset$. (A slight modification of the following argument is necessary in the case that $\{V_1, \cdots, V_k\} = \emptyset$.) Let O be an open set containing $C(A_1)$ such that $O \cap (\bigcup_{i=1}^k V_i^*) = \emptyset$ and such that $(\text{bd } O) \cap C^* = \emptyset$.

Let $x \in A_1$. Suppose $x \notin \text{int } C(A_1)$. Let O_x be an open set containing x such that $O_x \subset O \cap (\bigcap \{V_i \mid x \in V_i\})$. Let $y \in O_x$ such that $y \notin C(A_1)$ and let $C(y)$ be the component of C^* which contains y. Since $(\text{bd } O) \cap C^* = \emptyset$, $C(y) \subset O$. Let O_y be an open set containing $C(y)$ such that $O_y \subset O$, $O_y \cap C(A_1) = \emptyset$, and such that $(\text{bd } O_y) \cap C^* = \emptyset$.

Now $O_y, O - O_y^*$, and $X - O^*$ are disjoint open sets, and $C^* \subset O_y \cup (O - O_y^*) \cup (X - O^*)$. Consequently the sets $N(O_y)$, $N(O - O_y^*)$, $N(X - O^*)$, $N(O_y, O - O_y^*)$, $N(O_y, X - O^*)$, $N(O - O_y^*, X - O^*)$, and $N(O_y, O - O_y^*, X - O^*)$ are pairwise disjoint, and $f(\mathscr{C})^*$ is contained in the union of these sets.

For each $i = 1, \cdots, k$, let $x_i \in V_i$. For each $i = k + 1, \cdots, m$, $C(A_1) \cap V_i^* \neq \emptyset$, and since $O - O_y^*$ is an open set containing $C(A_1)$, there exists $x_i \in O - O_y^*$ such that $x_i \in V_i$. Then $\{x_1, \cdots, x_m\}$, $\{x_1, \cdots, x_m, y\} \in N(V_1, \cdots, V_m) \subset f(\mathscr{C})$. Furthermore, $\{x_1, \cdots, x_m\} \in N(O - O_y^*, X - O^*)$ and $\{x_1, \cdots, x_m, y\} \in N(O_y, O - O_y^*, X - O^*)$. Hence $f(\mathscr{C})^*$ is not connected, so $f(\mathscr{C})$ is not connected, a contradiction. Thus the assumption that $x \notin \text{int } C(A_1)$ was false. It follows that $C(A_1)$ is a connected subset of C^* which contains A_1 in its interior. Hence, by Theorem 1, 2^X is c.i.k. at A_1.

For the converse, suppose that 2^X is c.i.k. at each component of A. Let $\mathscr{U}$ be an open set containing A and $N(U_1, \cdots, U_n)$ be a basic open set such that $A \in N(U_1, \cdots, U_n) \subset N(U_1, \cdots, U_n)^* \subset \mathscr{U}$. By Lemma 2 we may assume that $N(U_1, \cdots, U_n)$ is essential with respect to A. For each component A_α of A let $\{U_{i_1}, \cdots, U_{i_{n_\alpha}}\}$ be the largest subset of $\{U_1, \cdots, U_n\}$ with the property that for $j = 1, \cdots, n_\alpha$, $A_\alpha \cap U_{i_j} \neq \emptyset$. Then $A_\alpha \in N(U_{i_1}, \cdots, U_{i_{n_\alpha}})$. Let $U_\alpha = \bigcup_{j=1}^{n_\alpha} U_{i_j}$. By Theorem 1 there is a component M_α of U_α which contains A_α in its interior. For each $j = 1, \cdots, n_\alpha$ let $V_j^\alpha = (\text{int } M_\alpha) \cap U_{i_j}$. Then $A_\alpha \in N(V_1^\alpha, \cdots, V_{n_\alpha}^\alpha) \subset N(U_{i_1}, \cdots, U_{i_{n_\alpha}})$.

Now $A \subset \bigcup_\alpha (\bigcup_{j=1}^{n_\alpha} V_j^\alpha)$ and since A is compact there exists a finite subcover of A of the form $\bigcup_{i=1}^m (\bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i})$. Then

$$A \in N(V_1^{\alpha_1}, \cdots, V_{n_{\alpha_1}}^{\alpha_1}, \cdots, V_1^{\alpha_m}, \cdots, V_{n_{\alpha_m}}^{\alpha_m}) \subset N(U_1, \cdots, U_n) .$$

The last inclusion follows from the construction and the fact that $N(U_1, \cdots, U_n)$ is essential with respect to A. Let $M = \bigcup_{i=1}^m M_{\alpha_i}^*$. Then $M \in N(U_1, \cdots, U_n)^*$.

Let $B \in N(V_1^{\alpha_1}, \cdots, V_{n_{\alpha_1}}^{\alpha_1}, \cdots, V_1^{\alpha_m}, \cdots, V_{n_{\alpha_m}}^{\alpha_m})$. Note that $B = \bigcup_{i=1}^m (B \cap \bigcup_{j=1}^{n_{\alpha_j}} V_j^{\alpha_i *})$. Now $B \cap (\bigcup_{j=1}^{n_{\alpha_1}} V_j^{\alpha_1 *}) \subset M_{\alpha_1}^*$, so there exists an order arc $\mathscr{B}_{\alpha_1}$ from $B \cap (\bigcup_{j=1}^{n_{\alpha_1}} V_j^{\alpha_1 *})$ to $M_{\alpha_1}^*$. Define $f_1 \colon \mathscr{B}_{\alpha_1} \to \mathscr{U}$ by $f_1(C) = C \cup (\bigcup_{i=2}^m (B \cap \bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i *}))$. Since union is continuous, $f_1(\mathscr{B}_{\alpha_1})$ is connected, and B, $M_{\alpha_1}^* \cup (\bigcup_{i=2}^m (B \cap \bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i *})) \in f_1(\mathscr{B}_{\alpha_1})$. For each $i = 2, \cdots, m$, there exists an order arc $\mathscr{B}_{\alpha_i}$ from $B \cap (\bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i *})$ to $M_{\alpha_i}^{\alpha}$. For each $i = 2, \cdots, m-1$, define $f_i(\mathscr{B}_{\alpha_i}) \to \mathscr{U}$ by

$$f_i(C) = \left(\bigcup_{k=1}^{i-1} M_{\alpha_k}^*\right) \cup C \cup \left(\bigcup_{k=i+1}^{m} \left(B \cap \bigcup_{j=1}^{n_{\alpha_k}} V_j^{\alpha_k *}\right)\right).$$

Then $f_i(\mathscr{B}_{\alpha_i})$ is a connected subset of $\mathscr{U}$ containing $(\bigcup_{k=1}^{i-1} M_{\alpha_k}^*) \cup (\bigcup_{k=i}^m (B \cap \bigcup_{j=1}^{n_{\alpha_k}} V_j^{\alpha_k *}))$ and $(\bigcup_{k=1}^{i} M_{\alpha_k}^*) \cup (\bigcup_{k=i+1}^m (B \cap \bigcup_{j=1}^{n_{\alpha_k}} V_j^{\alpha_k *}))$. Define $f_m(\mathscr{B}_{\alpha_m}) \to \mathscr{U}$ by $f_m(C) = (\bigcup_{k=1}^{m-1} M_{\alpha_k}^*) \cup C$. Then $f_m(\mathscr{B}_{\alpha_m})$ is a connected subset of $\mathscr{U}$ containing $(\bigcup_{k=1}^{m-1} M_{\alpha_k}^*) \cup (B \cap \bigcup_{j=1}^{n_{\alpha_m}} V_j^{\alpha_m *})$ and M. Hence $\bigcup_{i=1}^m f_i(\mathscr{B}_{\alpha_i})$ is a connected subset of $\mathscr{U}$ containing B and M. It follows that 2^X is c.i.k. at A.

THEOREM 5. *Let $A \in 2^X$. Then 2^X is l.c. at A if and only if 2^X is l.c. at each component of A.*

Proof. Suppose that 2^X is l.c. at A. Let A_1 be a component of A and let W be an open set containing A_1. Let U be an open set such that $A_1 \subset U \subset W$ aud such that $(\text{bd}\, U) \cap A = \varnothing$. Let $\{U_1, \cdots, U_n\}$ be a finite cover of $A - U$ by open sets such that for each $i = 1, \cdots, n$, $U \cap U_i = \varnothing$ and $A \cap U_i \neq \varnothing$. Then $A \in N(U, U_1, \cdots, U_n)$.

Let $\mathscr{V}$ be a connected open set such that $A \in \mathscr{V} \subset N(U, U_1, \cdots, U_n)$. Define $f \colon \mathscr{V} \to N(U)$ by $f(B) = B \cap U$. An argument similar to the one used in Theorem 5 will establish that f is both continuous and open. Hence $f(\mathscr{V})$ is connected and open.

Let $V = \bigcup \{f(B) \mid B \in \mathscr{V}\}$. Then $V \subset U$. Let $Q(A_1)$ be the quasicomponent of V which contains A_1 and let $x \in Q(A_1)$. Let $B \in \mathscr{V}$ such that $x \in f(B)$. Then there exists an open set $N(V_1, \cdots, V_m, V_{m+1}, \cdots, V_p)$ such that $B \in N(V_1, \cdots, V_m, V_{m+1}, \cdots, V_p) \subset N(V_1^*, \cdots, V_m^*, V_{m+1}, \cdots, V_p) \subset \mathscr{V}$ and such that $\bigcup_{i=1}^m V_i^* \subset U$ and $\bigcup_{i=m+1}^p V_i \subset \bigcup_{i=1}^n U_i$. Let $\{V_i, \cdots, V_k\}$ be the largest subset of $\{V_1, \cdots, V_m\}$ with the property that for each $i = 1, \cdots, k$, $V_i^* \cap Q(A_1) = \varnothing$. (A slight modification of the following argument is necessary in the case that

$\{V_1, \cdots, V_k\} = \varnothing$.) Since $\bigcup_{i=1}^k V_i^*$ is compact, there exist disjoint open-closed sets S and T such that $\bigcup_{i=1}^k V_i^* \subset S$, $Q(A_1) \subset T$ and $S \cup T = V$.

Suppose $x \notin \operatorname{int} Q(A_1)$. Let O be an open set containing x such that $O \subset T \cap (\bigcap \{V_i \mid x \in V_i\})$. Let $y \in O$ such that $y \notin Q(A_1)$. Then there exist disjoint open-closed sets T' and T'' such that $Q(A_1) \subset T'$, $y \in T''$, and $T' \cup T'' = T$.

Now T', T'', and S are disjoint open sets whose union is V. Consequently the sets $N(T')$, $N(T'')$, $N(S)$, $N(T', T'')$, $N(T', S)$, $N(T'', S)$, and $N(T', T'', S)$ are pairwise disjoint and $f(\mathscr{V})$ is contained in the union of these sets.

For each $i = 1, \cdots, k$, let $x_i \in V_i$. For each $i = k+1, \cdots, m$, $Q(A_1) \cap V_i^* \neq \varnothing$, and since T' is an open set containing $Q(A_1)$, there exists $x_i \in T'$ such that $x_i \in V_i$. Then

$$\{x_1, \cdots, x_m\},\ \{x_1, \cdots, x_m, y\} \in N(V_1, \cdots, V_m) \subset f(\mathscr{V}) .$$

Furthermore, $\{x_1, \cdots, x_m\} \in N(T', S)$ and $\{x_1, \cdots, x_m, y\} \in N(T', T'', S)$. Hence $f(\mathscr{V})$ is not connected, a contradiction, so the assumption that $x \notin \operatorname{int} Q(A_1)$ was false.

We have now established that $Q(A_1)$ is open. So $Q(A_1)$ and $V - Q(A_1)$ are disjoint open-closed subsets of V. If $Q(A_1)$ were not connected, there would exist a proper open-closed subset of $Q(A_1)$ (and hence of V) containing A_1, which is impossible. It follows that $Q(A_1)$ is an open connected subset of V containing A_1. Hence, by Theorem 2, 2^X is l.c. at A.

For the converse, suppose that 2^X is l.c. at each component of A. Let $N(U_1, \cdots, U_n)$ be a basic open set containing A. By Lemma 2 we may assume that $N(U_1, \cdots, U_n)$ is essential with respect to A. For each component A_α of A let $\{U_{i_1}, \cdots, U_{i_{n_\alpha}}\}$ be the largest subset of $\{U_1, \cdots, U_n\}$ with the property that for each $j = 1, \cdots, n_\alpha$, $U_{i_j} \cap A_\alpha \neq \varnothing$. Then $A_\alpha \in N(U_{i_1}, \cdots, U_{i_{n_\alpha}})$. Let $U_\alpha = \bigcup_{j=1}^{n_\alpha} U_{i_j}$. By Theorem 2 there is a connected open set V_α such that $A_\alpha \subset V_\alpha \subset U_\alpha$. For each $j = 1, \cdots, n_\alpha$ let $V_j^\alpha = V_\alpha \cap U_{i_j}$. Then

$$A_\alpha \in N(V_1^\alpha, \cdots, V_{n_\alpha}^\alpha) \subset N(U_{i_1}, \cdots, U_{i_{n_\alpha}})$$

and by Lemma 1, $N(V_1^\alpha, \cdots, V_{n_\alpha}^\alpha)$ is connected. Now $A \subset \bigcup_\alpha (\bigcup_{j=1}^{n_\alpha} V_j^\alpha)$, and since A is compact, there exist $\alpha_1, \cdots, \alpha_m$ such that $A \subset \bigcup_{i=1}^m (\bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i})$. Then

$$A \in N(V_1^{\alpha_1}, \cdots, V_{n_{\alpha_1}}^{\alpha_1}, \cdots, V_1^{\alpha_m}, \cdots, V_{n_{\alpha_m}}^{\alpha_m}) = \mathscr{V} \subset N(U_1, \cdots, U_n) .$$

The last inclusion follows from the construction and the fact that $N(U_1, \cdots, U_n)$ is essential with respect to A.

Let $B, C \in \mathscr{V} \cap F(X)$ and for and $i = 1, \cdots, m$ let $B_i = B \cap (\bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i})$ and $C_i = C \cap (\bigcup_{j=1}^{n_{\alpha_i}} V_j^{\alpha_i})$. Then $B_i, C_i \in N(V_1^{\alpha_i}, \cdots, V_{n_{\alpha_i}}^{\alpha_i}) \cap F(X)$. As in the proof of Theorem 2, for each $i = 1, \cdots, m$ there exists a connected set $\mathscr{L}_i$ in $N(V_1^{\alpha_i}, \cdots, V_{n_{\alpha_i}}^{\alpha_i}) \cap F(X)$ which contains B_i and C_i. Define $f_1: \mathscr{L}_1 \to \mathscr{V} \cap F(X)$ by $F_1(D) = D \cup (\bigcup_{i=2}^m B_i)$. Since f_1 is continuous, $f_1(\mathscr{L}_1)$ is a connected subset of $\mathscr{V} \cap F(X)$ containing B and $C_1 \cup (\bigcup_{i=2}^m B_i)$. For each $i = 2, \cdots, m-1$ define $f_i: \mathscr{L}_i \to \mathscr{V} \cap F(X)$ by $f_i(D) = (\bigcup_{k=1}^{i-1} C_k) \cup D \cup (\bigcup_{k=i+1}^m B_k)$. Then $f_i(\mathscr{L}_i)$ is a connected subset of $\mathscr{V} \cap F(X)$ containing $(\bigcup_{k=1}^{i-1} C_i) \cup (\bigcup_{k=i}^m B_i)$ and $(\bigcup_{k=1}^{i} C_i) \cup (\bigcup_{k=i+1}^m B_i)$. Define $f_m: \mathscr{L}_m \to \mathscr{V} \cap F(X)$ by $f_m(D) = (\bigcup_{i=1}^{m-1} C_i) \cup D$. Then $f_m(\mathscr{L}_m)$ is a connected subset of $\mathscr{V} \cap F(X)$ containing $(\bigcup_{i=1}^{m-1} C_i) \cup B_m$ and C. Hence $\bigcup_{i=1}^m f_i(\mathscr{L}_i)$ is a connected subset of $\mathscr{V} \cap F(X)$ containing B and C. It follows that $\mathscr{V} \cap F(X)$ is connected, and since $\mathscr{V} \cap F(X)$ is dense in $\mathscr{V}$, $\mathscr{V}$ is connected. Hence 2^X is l.c. at A.

COROLLARY 6. *Let $A \in 2^X$. If X is c.i.k. (l.c.) at each point of A, then 2^X is c.i.k. (l.c.) at A.*

The converses of Corollary 6 are false. It is easy to verify (see Lemma 2 of [1]) that for any continuum X, 2^X is l.c. at X.

COROLLARY 7. *The following are equivalent:*
(1) *For each $i = 1, \cdots, n$, X is c.i.k. (l.c.) at p_i.*
(2) *For each $i = 1, \cdots, n$, 2^X is c.i.k. (l.c.) at $\{p_i\}$.*
(3) *2^X is c.i.k. (l.c.) at $\{p_1, \cdots, p_n\}$.*

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