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**ON THE FROBENIUS RECIPROCITY THEOREM FOR
SQUARE-INTEGRABLE REPRESENTATIONS**

RAY ALDEN KUNZE

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In this paper, a global version of the Frobenius reciprocity theorem is established for irreducible square-integrable representations of locally compact unimodular groups. As in the classical compact case, it asserts that certain intertwining spaces are canonically and isometrically isomorphic. The proof is elementary, and the appropriate isomorphism is exhibited explicitly. The essential point is that square-integrability implies the continuity of functions in certain subspaces of L^2 spaces on which the group acts and leads to a characterization of the subspaces in terms of reproducing kernels.

The preliminary results on reproducing kernels are contained in Theorems 1 and 2 in § 2. Our main result on reciprocity, Theorem 3 in § 3, does not require direct integral decomposition theory as in [2] and [4] and is formally similar to the version of the reciprocity theorem proved by C. C. Moore in [5]; however, we only consider unitary representations, and do not need to formulate the result in terms of summable induced representations on L^1 -spaces.

After this paper was initially submitted, we learned that A. Wawrzyńczyk [6] had already proved a result, similar but not identical to our Theorem 3. His proof is based on a general duality theorem for automorphic forms due to K. and L. Maurin [3], and he does not prove results corresponding to our Theorems 1 and 2.

Let G be a locally compact unimodular group and S a continuous irreducible square-integrable unitary representation of G on a complex Hilbert space \mathcal{H} . We recall that this implies

$$x \rightarrow (S(x)\varphi | \psi), x \in G$$

is square-integrable on G for all φ and ψ in \mathcal{H} and the existence of a positive constant d (the formal degree) such that

$$(1.1) \quad \int_G (S(x)\varphi | \alpha) \overline{(S(x)\psi | \beta)} dx = d^{-1}(\varphi | \psi) \overline{(\alpha | \beta)}$$

for all $\varphi, \alpha, \psi, \beta$ in \mathcal{H} .

Let K be a compact subgroup of G and λ a continuous irreducible unitary representation of K on a complex-Hilbert space \mathcal{H} . Let $T = T(\cdot, \lambda)$ be the continuous unitary representation of G induced by λ . By definition, $T(y)(y \in G)$ is right translation by y on the space $L^2(G, \lambda)$ of all square-integrable maps $f: G \rightarrow \mathcal{H}$ such that

$$(1.2) \quad f(kx) = \lambda(k)f(x)$$

for all (k, x) in $K \times G$.

Now let $\mathcal{S}(S, T)$ denote the Banach space of bounded linear maps $U: \mathcal{H} \rightarrow L^2(G, \lambda)$ which intertwine S and T , i.e., are such that

$$(1.3) \quad US(x) = T(x)U$$

for all x in G . Similarly, let $\mathcal{S}(S_K, \lambda)$ denote the space of operators intertwining S_K (the restriction of S to K) and λ .

In § 2, we obtain certain properties of the spaces $U(\mathcal{H})$ for U in $\mathcal{S}(S, T)$, and using these properties, we then show in § 3 that there is a canonical isometric isomorphism of $\mathcal{S}(S_K, \lambda)$ onto $\mathcal{S}(S, T)$. From this we conclude that T contains S (discretely) exactly as many times as S_K contains λ .

2. The spaces $U(\mathcal{H})$, $U \in \mathcal{S}(S, T)$. Because S is irreducible, it is easy to see that each U in $\mathcal{S}(S, T)$ is a scalar multiple of an isometry (cf. the argument proving (3.3)). Hence, $U(\mathcal{H})$ is a closed subspace (possibly 0) of $L^2(G, \lambda)$. Less obvious and much more important is the fact that each function class $U\varphi$ ($U \in \mathcal{S}(S, T)$, $\varphi \in \mathcal{H}$) contains a unique continuous function.

THEOREM 1. *Let U be any operator intertwining S and T . Then $U(\mathcal{H})$ is a closed subspace of $L^2(G, \lambda)$ consisting of continuous functions in which point evaluations*

$$f \mapsto f(x), \quad f \in U(\mathcal{H})$$

are continuous linear maps of $U(\mathcal{H})$ into \mathcal{H} for every x in G .

Proof. Let $\varphi \in \mathcal{H}$ and f any function in the class $U\varphi$. Set

$$e(y) = d(\varphi | S(y)\varphi), \quad y \in G.$$

Then, because G is unimodular and in view of (1.1), it follows that

$$\begin{aligned} \int_G |e(y)| \|f(xy)\| dy &\leq \left(\int_G |e(y)|^2 dy \right)^{1/2} \left(\int_G \|f(xy)\|^2 dy \right)^{1/2} \\ &= d^{1/2} \|\varphi\|^2 \|f\|_2. \end{aligned}$$

Thus, we can define a bounded function $g: G \rightarrow \mathcal{H}$ which satisfies (1.2) by setting

$$(2.1) \quad g(x) = \int_G e(y)f(xy)dy.$$

Moreover, g is continuous. For

$$\begin{aligned} \|g(x) - g(zx)\| &\leq \int_G |e(y)| \|f(xy) - f(zxy)\| dy \\ &\leq d^{1/2} \|\varphi\|^2 \left(\int_G \|f(y) - f(zy)\|^2 dy \right)^{1/2} \rightarrow 0 \text{ as } z \rightarrow 1. \end{aligned}$$

Now let h be any function with compact support in $L^2(G, \lambda)$. Then

$$\begin{aligned} &\int_G (g(x) | h(x)) dx \\ &= \int_G \left(\int_G e(y) f(xy) dy | h(x) \right) dx = \int_G dx \int_G e(y) (f(xy) | h(x)) dy \\ &= \int_G dy \int_G e(y) (f(xy) | h(x)) dx = \int_G e(y) (T(y)f | h) dy \\ &= \int_G e(y) (US(y)\varphi | h) dy && \text{(by (1.3))} \\ &= d \int_G (S(y)\varphi | U^*h) \overline{(S(y)\varphi | \varphi)} dy \\ &= (\varphi | \varphi) \overline{(U^*h | \varphi)} && \text{(by (1.1))} \\ &= (\|\varphi\|^2 f | h). \end{aligned}$$

Since this holds for all such h , it follows that

$$(2.2) \quad g(x) = \|\varphi\|^2 f(x), \text{ a.e. .}$$

Because the complement of a set of Haar measure 0 is dense, it follows that each function class $U\varphi$ contains a unique continuous function; from now on that function will be denoted by $U\varphi$.

Suppose $\varphi \neq 0$. Then from (2.1), (2.2) and the computations above, we have

$$(2.3) \quad (U\varphi)(x) = \frac{d}{\|\varphi\|^2} \int_G (U\varphi)(xy) \overline{(S(y)\varphi | \varphi)} dy$$

and

$$(2.4) \quad \|U\varphi(x)\| \leq d^{1/2} \|U\varphi\|_2$$

for every x in G . Therefore, point evaluations are continuous.

Now suppose $U(\mathcal{H}) \neq 0$. Then, since the maps

$$E_x: f \rightarrow f(x), \quad f \in U(\mathcal{H}), \quad x \in G$$

are all continuous, $U(\mathcal{H})$ is completely determined by the positive definite kernel

$$(2.5) \quad Q(x, y) = E_x E_y^*$$

in the simple fashion described in [1]. On the other hand, it is easy to see directly that $Q(x, y) = P(xy^{-1})$ where

$$(2.6) \quad P(x) = E_1 T(x) E_1^*, \quad x \in G$$

and that following result is valid.

THEOREM 2. *The operator-valued function P is continuous and square-integrable on G . It has formal properties*

- (1) $P(x)^* = P(x^{-1})$
- (2) $P(xy^{-1}) = E_x E_y^*$
- (3) $P(k_1 x k_2) = \lambda(k_1) P(x) \lambda(k_2)$
- (4) $P(x) = \int_G P(xy^{-1}) P(y) dy$

which are valid for all k_1, k_2 in K and x, y in G . Moreover, left convolution by P is the orthogonal projection of $L^2(G, \lambda)$ on $U(\mathcal{H})$; in particular

$$(5) \quad f(x) = \int_G P(xy^{-1}) f(y) dy$$

for all f in $U(\mathcal{H})$ and x in G .

Proof. Equation (1) follows from (2.6); (2) and (3) are consequences of the relations $E_x = E_1 T(x)$ ($x \in G$) and $E_1 T(k) = \lambda(k) E_1$ ($k \in K$). If α and β are vectors in \mathcal{H} , then

$$(P(x)\alpha | \beta) = (T(x)E_1^*\alpha | E_1^*\beta)$$

and since T is equivalent to S in $U(\mathcal{H})$, it follows that $x \rightarrow (P(x)\alpha | \beta)$ is not only continuous but square-integrable on G . We also have

$$\begin{aligned} (P(x)\alpha | \beta) &= (E_1^*\alpha | T(x^{-1})E_1^*\beta) \\ &= \int_G (E_1^*\alpha(y) | T(x^{-1})E_1^*\beta(y)) dy \\ &= \int_G (E_y E_1^*\alpha | E_y T(x^{-1})E_1^*\beta) dy \\ &= \int_G (P(y)\alpha | P(yx^{-1})\beta) dy && \text{(by (2))} \\ &= \int_G (P(xy^{-1})P(y)\alpha | \beta) dy && \text{(by (1))} \end{aligned}$$

for all α, β in \mathcal{H} ; hence, (4) is true.

Now suppose $f \in L^2(G, \lambda)$. Then for any x in G and α in \mathcal{H}

$$\begin{aligned} (f | T(x^{-1})E_1^*\alpha) &= \int_G (f(y) | (T(x^{-1})E_1^*\alpha)(y)) dy \\ &= \int_G (f(y) | E_y T(x^{-1})E_1^*\alpha) dy \\ &= \int_G (f(y) | P(yx^{-1})\alpha) dy = \int_G (P(xy^{-1})f(y) | \alpha) dy. \end{aligned}$$

If f is orthogonal to $U(\mathcal{H})$, then $(f | T(x^{-1})E_1^*\alpha) = 0$ for all x and α ; hence

$$\int_G (P(xy^{-1})f(y) | \alpha)dy = 0$$

for all x and α . Therefore

$$(2.7) \quad \int_G P(xy^{-1})f(y)dy = 0, \quad f \in U(\mathcal{H})^\perp.$$

On the other hand, if $f \in U(\mathcal{H})$, then

$$(f | T(x^{-1})E_1^*\alpha) = (E_1T(x)f | \alpha)$$

so that

$$(f(x) | \alpha) = \int_G (P(xy^{-1})f(y) | \alpha)dy$$

for all x and α ; hence, (5) is valid. To complete the proof it is enough to observe that (5) and (2.7) imply that for any f in $L^2(G, \lambda)$, the function

$$g(x) = \int_G P(xy^{-1})f(y)dy, \quad x \in G$$

is the orthogonal projection of f on $U(\mathcal{H})$.

3. The reciprocity theorem. In the statement of the next result, which is our version of the Frobenius reciprocity theorem for square-integrable representations, we retain the assumptions and notation used in §§ 1 and 2.

THEOREM 3. *The intertwining spaces $\mathcal{S}(S_K, \lambda)$ and $\mathcal{S}(S, T)$ are canonically isomorphic via an isometric linear map*

$$A \mapsto U_A, \quad A \in \mathcal{S}(S_K, \lambda)$$

that is defined by the equation

$$(3.1) \quad (U_A\varphi)(x) = cAS(x)\varphi, \quad \varphi \in \mathcal{H}, \quad x \in G$$

in which $c = (d/\dim(\mathcal{H}))^{1/2}$.

Proof. Let $A \in \mathcal{S}(S_K, \lambda)$, $\varphi \in \mathcal{H}$, and define f on G by

$$f(x) = AS(x)\varphi, \quad x \in G.$$

Then f is continuous, and

$$f(kx) = AS(k)S(x)\varphi = \lambda(k)AS(x)\varphi = \lambda(k)f(x)$$

for all (k, x) in $K \times G$. If $\alpha \in \mathcal{H}$ then

$$(f(x) | \alpha) = (AS(x)\varphi | \alpha) = (S(x)\varphi | A^*\alpha).$$

Since S is square-integrable, it follows that $x \rightarrow (f(x) | \alpha)$ is square-integrable for each α in \mathcal{H} . Hence, since \mathcal{H} is necessarily finite dimensional

$$\int_G \|f(x)\|^2 dx < \infty.$$

It follows that (3.1) defines an element $U_A\varphi$ in $L^2(G, \lambda)$, and $\varphi \rightarrow U_A\varphi$ ($\varphi \in \mathcal{H}$) is a linear map U_A of \mathcal{H} into $L^2(G, \lambda)$.

Now suppose A and B lie in $\mathcal{S}(S_K, \lambda)$, let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal base for \mathcal{H} , and let φ and ψ be vectors in \mathcal{H} . Then

$$(3.2) \quad (U_A\varphi | U_B\psi) = d^{-1}(\varphi | \psi) \sum_{i=1}^n (B^*\varepsilon_i | A^*\varepsilon_i).$$

In fact

$$\begin{aligned} \int_G (AS(x)\varphi | BS(x)\psi) dx &= \sum_i \int_G (S(x)\varphi | A^*\varepsilon_i) \overline{(S(x)\psi | B^*\varepsilon_i)} dx \\ &= d^{-1}(\varphi | \psi) \sum_i (B^*\varepsilon_i | A^*\varepsilon_i) \quad (\text{by (1.1)}). \end{aligned}$$

Because S and λ are unitary representations and $AS(k) = \lambda(k)A$, it follows that

$$AA^*\lambda(k) = \lambda(k)AA^*$$

for all $k \in K$. Since λ is irreducible this implies $AA^* = \|A\|^2 I$. Hence

$$(A^*\alpha | A^*B) = \|A\|^2 (\alpha | \beta)$$

for all α, β in \mathcal{H} . Using this and setting $B = A$ in (3.2), we find that

$$(3.3) \quad (U_A\varphi | U_A\psi) = \|A\|^2 (\varphi | \psi)$$

for all φ, ψ in \mathcal{H} . Therefore, U_A is a continuous linear map of \mathcal{H} into $L^2(G, \lambda)$, and $\|U_A\| = \|A\|$.

Next note that for φ in \mathcal{H} and x, y in G

$$(T(y)U_A\varphi)(x) = (U_A\varphi)(xy) = cAS(x)S(y)\varphi = (U_AS(y)\varphi)(x).$$

Hence, $T(y)U_A = U_AS(y)$ for all y in G . Therefore, $U_A \in \mathcal{S}(S, T)$. Since

$$U_{cA+B} = cU_A + U_B$$

it follows that $A \rightarrow U_A$ is an isometric linear map of $\mathcal{S}(S_K, T)$ into $\mathcal{S}(S, T)$.

Now suppose $U \in \mathcal{S}(S, T)$. Then by Theorem 1, we can define a continuous linear map A of \mathcal{H} into \mathcal{H} by setting

$$(3.4) \quad A\varphi = c^{-1}(U\varphi)(1), \quad \varphi \in \mathcal{H}.$$

Then for k in K and φ in \mathcal{H}

$$AS(k)\varphi = c^{-1}(US(k)\varphi)(1) = c^{-1}(T(k)U\varphi)(1) = c^{-1}(U\varphi)(k) = \lambda(k)A\varphi.$$

Thus $A \in \mathcal{S}(S_K, \lambda)$, and

$$U_A\varphi(x) = cAS(x)\varphi = (US(x)\varphi)(1) = (T(x)U\varphi)(1) = (U\varphi)(x)$$

for φ in \mathcal{H} and x in G . Hence, $U = U_A$ and $A \mapsto U_A$ ($A \in \mathcal{S}(S_K, \lambda)$) is an isometric linear map of $\mathcal{S}(S_K, \lambda)$ onto $\mathcal{S}(S, T)$.

COROLLARY. *The multiplicity of S in T is exactly the same as the multiplicity of λ in S_K .*

Proof. These multiplicities are just $\dim \mathcal{S}(S, T)$ and $\dim \mathcal{S}(S_K, \lambda)$, respectively.

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Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i>	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i>	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i>	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i>	347
Stephen LaVern Campbell, <i>Linear operators for which T^*T and TT^* commute. II</i>	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i>	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i>	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i>	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in 2^X and $C(X)$</i>	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i>	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i>	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i>	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i>	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i>	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i>	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ...	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i>	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i>	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i>	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot</i>	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i>	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i>	539
Dorte Olesen, <i>Derivations of AW^*-algebras are inner</i>	555
Dorte Olesen and Gert Kjærgaard Pedersen, <i>Derivations of C^*-algebras have semi-continuous generators</i>	563
Duane O'Neill, <i>On conjugation cobordism</i>	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i>	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of $L^1(\mu; E)$</i>	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i>	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i>	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i>	627
Carl E. Swenson, <i>Direct sum subset decompositions of Z</i>	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i>	635
Robert S. Wilson, <i>Representations of finite rings</i>	643