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**EXISTENCE, UNIQUENESS AND LIMITING BEHAVIOR OF
SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS IN
BANACH SPACE**

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Let X be a Banach space (real or complex) and A_n and B be linear operators in X with $D(B) \subseteq D(A_n)$, $n = 1, 2, \dots$. The following note is concerned with existence and uniqueness of solutions of the problem

$$(1.1) \quad \frac{d}{dt} [(I - A_n)u(t)] - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0,$$

and the limiting behavior of these solutions as the operators A_n tend to zero in a sense to be specified. We will show that for a large class of operators the problem (1.1) is well posed and that its solutions tend to the solution of the problem

$$(1.2) \quad \frac{du(t)}{dt} - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0.$$

In particular, we obtain an extension to Banach spaces of a result of R. E. Showalter [5] to the effect that (1.1) is well posed when X is a Hilbert space and A_n and B are maximal dissipative operators in X which satisfy the algebraic condition

$$(1.3) \quad \operatorname{Re}((I - A_n)x, Bx) \leq 0, \quad x \in D(B) \subseteq D(A_n).$$

In the next section we give sufficient conditions for (1.1) to be well posed. We note that these conditions do *not* guarantee that (1.2) is well posed. In §3 we show that if, in addition, $\{A_n\}$ tends to zero in a certain sense, then (1.2) is well posed and the solutions u_n of (1.1) tend to the solution of (1.2). In particular, it will follow that if A and B are densely defined maximal dissipative operators in a Hilbert space and if (1.3) is satisfied with $A_n = n^{-1}A$, then

$$\frac{d}{dt} [(I - n^{-1}A)u_n(t)] - Bu_n(t) = 0, \quad (t > 0), \quad u_n(0) = u_n \in D(B),$$

is well posed and as $n \rightarrow \infty$, u_n converges strongly to the unique solution of (1.2). Two examples are discussed in §4.

We emphasize that throughout this paper it is assumed that $D(B) \subseteq D(A_n)$. The question of limiting behavior of solutions of (1.1) when X is a Hilbert space, $A_n = n^{-1}A$ and $D(A) \subseteq D(B)$ has been considered previously [2], and it is interesting to compare the results of [2] with those of the present note in the case $D(A) = D(B)$. In [2] it was assumed that A and B were maximal dissipative operators

arising from certain densely defined, strongly coercive sesquilinear forms and that A was self-adjoint. On the other hand the algebraic condition (1.3) which is the most restrictive assumption of the present note, was not assumed in [2] and the convergence results are somewhat stronger than those obtained here. Thus while the results of [2] do not apply to perturbations of hyperbolic problems, they are in some respects more satisfactory as far as perturbations of parabolic problems are concerned when $D(A) = D(B)$. We note that the methods used here are completely different from those of [2].

2. Existence and uniqueness of solutions. A solution of the problem (1.1) is a function $u: [0, \infty) \rightarrow D(B)$ such that $(I - A_n)u \in C([0, \infty); X) \cap C'((0, \infty), X)$ and (1.1) is satisfied. The initial condition in (1.1) is supposed to hold in the sense that $(I - A_n)u(t) \rightarrow (I - A_n)u_0$ strongly in X as $t \rightarrow 0_+$. While we will always assume that $I - A_n$ is invertible, the inverse need not be bounded and so we do not know in general that $u(t) \rightarrow u_0$ strongly in X .

THEOREM 2.1. *Let X be a Banach space and A_n and B linear operators in X which satisfy the following*

$$(2.1) \quad I - A_n \text{ is one-to-one.}$$

$$(2.2) \quad D(B) \subseteq D(A_n).$$

$$(2.3) \quad \|x - A_n x - \zeta Bx\| \geq \|x - A_n x\| \text{ for all } x \in D(B) \text{ and } \zeta > 0.$$

$$(2.4) \quad \text{For some } \zeta_n > 0, \operatorname{Rg}(I - A_n - \zeta_n B) = X.$$

Then for any $u_0 \in D(B)$ the problem (1.1) has a unique solution $u(t)$ and

$$(2.5) \quad \|(I - A_n)u(t)\| \leq \|(I - A_n)u_0\|, t \geq 0.$$

Proof. Set $\tilde{A}_n = A_n|_{D(B)}$ and $B_n = B(I - \tilde{A}_n)^{-1}$ with $D(B_n) = \operatorname{Rg}(I - \tilde{A}_n)$. A function u is a solution of (1.1) if and only if $(I - A_n)u = v \in C([0, \infty); X) \cap C'((0, \infty); X)$ and

$$(2.6) \quad \frac{dv(t)}{dt} - B_n v(t) = 0, (t > 0), v(0) = v_0$$

where $v_0 = (I - A_n)u_0 \in D(B_n)$. From (2.3) we obtain

$$\|y - \zeta B_n y\| \geq \|y\|, y \in D(B_n), \zeta > 0,$$

which means that B_n is a dissipative operator in X , and from (2.4) we have $\operatorname{Rg}(I - \zeta B_n) = X$ from some $\zeta > 0$ (hence for all $\zeta > 0$). From these facts it follows that $D(B_n)$ is dense in X (Goldstein [1]; c.f. [4]). We may now apply the Lumer-Phillips theorem [3] to the

effect that B_n is the infinitesimal generator of a (C_0) -semigroup $\{e^{tB_n}; t \geq 0\}$ of contractions on X . Thus for any $v_0 \in D(B_n)$, (2.6) has a unique solution given by $v(t) = e^{tB_n}v_0$ and $|v(t)| \leq |v_0|$. The conclusions of the theorem now follow by setting

$$(2.7) \quad u(t) = (I - A_n)^{-1}e^{tB_n}(I - A_n)u_0.$$

COROLLARY 2.1. *Let X be a Hilbert space and A_n and B be densely defined, maximal dissipative linear operators in X such that $D(B) \subseteq D(A_n)$ and which satisfy (1.3). Then the conclusions of Theorem 2.1 hold. Moreover, $B \in C([0, \infty), X) \cap C'((0, \infty); X)$ and $u(t) \rightarrow u_0$ strongly in X as $t \rightarrow 0_+$.*

Proof. Since A_n is densely defined and maximal dissipative, $(I - A_n)$ is a bijection of $D(A_n)$ onto X and $\|(I - A_n)^{-1}\| \leq 1$. Also, R. E. Showalter proved [5] that under the stated hypotheses, $A_n + B$ is a densely defined, maximal dissipative operator in X . From this fact follows that $\text{Rg}(I - A_n - B) = X$. For a Hilbert space, conditions (1.3) and (2.3) are equivalent. The conclusions of the corollary now follow from (2.7) and Theorem 2.1.

REMARK. Suppose (2.1)–(2.4) hold and that in addition there is a constant $C > 0$ such that

$$(2.8) \quad \|x - A_n x - \zeta Bx\| \geq C\|x - A_n x\|$$

for each $x \in D(B)$ and all ζ with $\text{Re}(\zeta) > 0$. Then the semigroup $\{e^{tB_n}; t \geq 0\}$ has a strong holomorphic extension into some sector $|\arg t| < \alpha$, and therefore (2.6) (respectively, (1.1)) is uniquely solvable for any $v_0 \in X$ (respectively, $u_0 \in D(A_n)$). In fact, since B_n generates a (C_0) -semigroup of contractions, the open right half-plane lies in the resolvent set of B_n and from (2.8) we obtain $\|(\lambda - B_n)^{-1}\| \leq (C|\lambda|)^{-1}$ whenever $\text{Re} \lambda > 0$, which implies the desired conclusion. When X is a Hilbert space, a sufficient condition for (2.8) is that all of the values of $z = (x - A_n x, Bx)$ lie in some fixed sector

$$|\arg z - \pi| \leq \frac{\pi}{2} - \varepsilon, \quad \varepsilon > 0.$$

To prove this, write $z = |z|e^{i\theta}$ and $\zeta = |\zeta|e^{i\phi}$. (2.8) is equivalent to

$$(1 - C^2)\|x - A_n x\|^2 - 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \geq 0.$$

If $|\theta - \pi| \leq \pi/2 - \varepsilon$, there is a $\delta > 0$ such that $\cos(\phi - \theta) \leq 1 - \delta$ for all $\phi \in (-\pi/2, \pi/2)$ and therefore

$$\begin{aligned} & (1 - \delta)^2\|x - A_n x\|^2 - 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \\ & \geq (1 - \delta)^2\|x - A_n x\|^2 - 2|\zeta|(1 - \delta)\|x - A_n x\|\|Bx\| \\ & \quad + |\zeta|^2\|Bx\|^2 = [(1 - \delta)\|x - A_n x\| - |\zeta|\|Bx\|]^2 \geq 0. \end{aligned}$$

Thus (2.8) holds with $C^2 = 2\delta - \delta^2$.

3. **Limiting behavior of solutions.** We first prove that if B is closed and A_n tends to zero in a certain way then (1.2) is well posed.

THEOREM 3.1. *Let X be a Banach space and A_n and B be linear operators in X which satisfy (2.1)–(2.4). Suppose in addition*

(3.1) B is closed .

(3.2)
$$\lim_{n \rightarrow \infty} \sup_{\substack{x \in D(B) \\ x \neq 0}} \|A_n x\| / (\|Bx\| + \|x\|) = 0 .$$

Then B is the infinitesimal generator of a (C_0) -semigroup of contractions on X .

Proof. We have to show that B is a dissipative operator such that $\text{Rg}(I - B) = X$. From (2.3) and (3.2) we obtain, upon letting $n \rightarrow \infty$,

(3.3)
$$\|x - \zeta Bx\| \geq \|x\|, \quad x \in D(B), \quad \zeta > 0 ,$$

and so B is dissipative. For each n and $\zeta > 0$, B_n is dissipative and $\text{Rg}(I - \zeta B_n) = X$. Let $y \in X$ and $x_n \in D(B)$ such that

$$x_n - A_n x_n - Bx_n = y, \quad n = 1, 2, \dots .$$

By (2.3), $\|x_n - A_n x_n\| \leq \|y\|$ and therefore $\{Bx_n\}$ is bounded. Let

$$C_n = \sup_{\substack{x \in D(B) \\ x \neq 0}} \|A_n x\| / (\|Bx\| + \|x\|) .$$

$C_n \rightarrow 0$ as $n \rightarrow \infty$ according to (3.2). From (3.3)

$$\begin{aligned} \|x_n\| &\leq \|x_n - Bx_n\| = \|y + A_n x_n\| \\ &\leq \|y\| + C_n(\|Bx_n\| + \|x_n\|) \end{aligned}$$

so that

$$(1 - C_n)\|x_n\| \leq \|y\| + C_n\|Bx_n\| .$$

Hence $\{x_n\}$ is also bounded. It follows from (3.2) that $A_n x_n \rightarrow 0$ strongly in X as $n \rightarrow \infty$. Therefore

$$\begin{aligned} \|x_n - x_m\| &\leq \|(x_n - x_m) - B(x_n - x_m)\| \\ &\leq \|A_n x_n - A_m x_m\| \longrightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$. Let $x = \lim x_n$. We have that $x_n \rightarrow x$, $Bx_n \rightarrow x - y$. Since B is closed, $x \in D(B)$ and $x - Bx = y$, that is, $\text{Rg}(I - B) = X$. This fact, together with the dissipativity of B , implies $D(B)$ is dense

in X . The conclusion of the theorem now follows from the Lumer-Phillips theorem.

THEOREM 3.2. *Let X be a Banach space and A_n and B be linear operators in X which satisfy (2.1)–(2.4), (3.1) and (3.2). Then as $n \rightarrow \infty$, $e^{tB_n} \rightarrow e^{tB}$ strongly and uniformly on bounded subsets of $[0, \infty)$.*

Proof. We apply the Trotter convergence theorem [6]. To do this we show that for each $\zeta > 0$,

$$(3.4) \quad \lim_{n \rightarrow \infty} (I - \zeta B_n)^{-1} = (I - \zeta B)^{-1}$$

in the uniform operator topology of $\mathcal{L}(X)$ (= the linear space of bounded linear operators on X).

We may write

$$(3.5) \quad \begin{aligned} (I - \zeta B_n)^{-1} &= (I - \zeta B(I - \tilde{A}_n)^{-1})^{-1} \\ &= (I - A_n)(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1}. \end{aligned}$$

For each $x \in X$,

$$(3.6) \quad \begin{aligned} \|A_n(I - \zeta B)^{-1}x\| &\leq C_n(\|B(I - \zeta B)^{-1}x\| + \|(I - \zeta B)^{-1}x\|) \\ &\leq C_n\left(1 + \frac{2}{\zeta}\right)\|x\|. \end{aligned}$$

Thus for all sufficiently large n ,

$$(I - A_n(I - \zeta B)^{-1})^{-1} = \sum_{k=0}^{\infty} (A_n(I - \zeta B)^{-1})^k$$

and

$$\|(I - A_n(I - \zeta B)^{-1})^{-1} - I\| \leq \sum_{k=0}^{\infty} C_n^{k+1} \left(1 + \frac{2}{\zeta}\right)^{k+1}$$

which tends to zero as $n \rightarrow \infty$. Therefore

$$(3.7) \quad \lim_{n \rightarrow \infty} (I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1} = (I - \zeta B)^{-1}$$

in the uniform operator topology of $\mathcal{L}(X)$. From (3.6) we have

$$(3.8) \quad \begin{aligned} &\|A_n(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1}\| \\ &\leq C_n\left(1 + \frac{2}{\zeta}\right)\|(I - A_n(I - \zeta B)^{-1})^{-1}\| \\ &\leq \sum_{k=0}^{\infty} C_n^{k+1} \left(1 + \frac{2}{\zeta}\right)^{k+1} \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. (3.4) now follows from (3.5), (3.7), and (3.8).

THEOREM 3.3. *Let X be a Banach space and A_n and B be linear operators in X which satisfy (2.1)–(2.4), (3.1) and (3.2). Suppose in*

addition that $\text{Rg}(I - A_n) = X$, $n = 1, 2, \dots$, and $\sup_n \|(I - A_n)^{-1}\| < \infty$. Let $u_0 \in D(B)$ and $u_n(t)$ be the unique solution of (1.1). Then as $n \rightarrow \infty$, $u_n(t) \rightarrow e^{tB}u_0$ strongly in X , uniformly on bounded subsets of $[0, \infty)$.

Proof. From (2.7) we obtain

$$\begin{aligned} \|u_n(t) - e^{tB}u_0\| &\leq \|(I - A_n)^{-1}(e^{tB_n} - e^{tB})u_0\| \\ &\quad + \|(I - A_n)^{-1}A_n e^{tB}u_0\| + \|(I - A_n)^{-1}e^{tB_n}A_n u_0\| \\ &\leq (\text{const.}) [\|e^{tB_n}u_0 - e^{tB}u_0\| + C_n(\|Bu_0\| + \|u_0\|)] \end{aligned}$$

and the right side tends to zero as $n \rightarrow \infty$, uniformly on bounded subsets of $[0, \infty)$.

COROLLARY 3.1. *Let X be a Hilbert space and A_n and B be densely defined, maximal dissipative operators in X such that $D(B) \subseteq D(A_n)$ and which satisfy (1.3) and (3.2). Then for each $u_0 \in D(B)$ the problem (1.1) has a unique solution u_n and $u_n(t) \rightarrow e^{tB}u_0$ as $n \rightarrow \infty$, uniformly on bounded subsets of $[0, \infty)$.*

Proof. As noted in the proof of Corollary 2.1, A_n and B satisfy (2.1)–(2.4) and moreover, $\text{Rg}(I - A_n) = X$ with $\|(I - A_n)^{-1}\| \leq 1$. In addition B , being a densely defined, maximal dissipative operator in a Hilbert space, is closed. The corollary now follows from Theorem 3.3.

REMARK. When $A_n = n^{-1}A$, (3.2) is automatically satisfied provided A and B are closed operators with $D(B) \subseteq D(A)$. Thus in this case hypothesis (3.2) may be omitted in Corollary 3.1. In fact, as a rather well-known consequence of the closed graph theorem we have

$$\|Ax\| \leq C(\|Bx\| + \|x\|), \quad x \in D(B)$$

where the constant C does not depend on x . Therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \|A_n u\| / (\|Bx\| + \|x\|) \leq Cn^{-1}.$$

4. Examples. As a first example we consider the problem

$$(4.1) \quad \frac{\partial}{\partial t} \left(u - a_n^1(x)u - a_n(x) \frac{\partial u}{\partial x} \right) - \left(b^1(x)u + b(x) \frac{\partial u}{\partial x} \right) = 0, \\ 0 < x < 1, \quad t > 0,$$

$$(4.2) \quad u(x, 0) = u_0(x), \quad 0 < x < 1; \quad u(0, t) = cu(1, t), \quad t > 0,$$

where c is a complex constant satisfying certain conditions and the coefficients in (4.1) are real-valued and of class $C'([0, 1])$. Let X be the complex Hilbert space $L_2(0, 1)$ and $H_2^1(0, 1)$ be the subclass of

$L_2(0, 1)$ consisting of those functions whose first derivative in the sense of distributions is again in $L_2(0, 1)$. The norms in X and in $H_2^1(0, 1)$ will be denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively and the inner product in X by (\cdot, \cdot) ; we have

$$\|u\|_1 = \left(\|u\|_0^2 + \left\| \frac{du}{dx} \right\|_0^2 \right)^{1/2}, \quad u \in H_2^1(0, 1).$$

Each function in $H_2^1(0, 1)$ is continuous, i.e., coincides with a function in $C([0, 1])$ up to a set of Lebesgue measure zero, and the injection of $H_2^1(0, 1)$ into $C([0, 1])$ is continuous.

We define operators A_n and B in X as follows:

$$D(A_n) = D(B) = \{u \in H_2^1(0, 1): u(0) = cu(1)\}$$

and for $u \in D(A_n) = D(B)$,

$$A_n u = a_n^1 u + a_n \frac{du}{dx}, \quad Bu = b^1 u + b \frac{du}{dx}.$$

From our preceding remarks it is easy to see that $D(B)$ is a closed subspace of $H_2^1(0, 1)$ and $D(B)$ is dense in $L_2(0, 1)$.

By a solution of (4.1), (4.2) we mean a solution of (1.1) in which A_n and B are the operators defined above. In order to apply the theory developed in §§ 2 and 3 to the problem (4.1) and (4.2) we shall have to verify in particular condition (1.3). Concerning this we have

LEMMA 4.1. Suppose $a_n b \geq 0$ and that

$$(4.3) \quad b^1 - \frac{1}{2} \frac{db}{dx} - a_n^1 b^1 + \frac{1}{2} \frac{d}{dx} (a_n b^1 + a_n^1 b) \leq 0;$$

$$(4.4) \quad \alpha_n(1) - |c|^2 \alpha_n(0) \leq 0$$

where $\alpha_n = b - a_n b^1 - a_n^1 b$. Then (1.3) is satisfied.

Proof. For $u \in D(B)$ we have

$$\begin{aligned} \operatorname{Re} (u - A_n u, Bu) &= \int_0^1 (b^1 - a_n^1 b^1) |u|^2 dx - \int_0^1 a_n b \left| \frac{du}{dx} \right|^2 dx \\ &\quad + \operatorname{Re} \int_0^1 (b - a_n^1 b - a_n b^1) \bar{u} \frac{du}{dx} dx. \end{aligned}$$

The following identity is easily obtained by an integration by parts: For $u \in D(B)$ and $f \in C'([0, 1])$,

$$\operatorname{Re} \int_0^1 f \bar{u} \frac{du}{dx} dx = \frac{1}{2} (f(1) - |c|^2 f(0)) |u(1)|^2 - \frac{1}{2} \int_0^1 \frac{df}{dx} |u|^2 dx.$$

From this identity we obtain for $u \in D(B)$

$$\begin{aligned} \operatorname{Re}(u - A_n u, Bu) &= \frac{1}{2}(\alpha_n(1) - |c|^2 \alpha_n(0)) |u(1)|^2 - \int_0^1 a_n b \left| \frac{du}{dx} \right|^2 dx \\ &+ \int_0^1 \left[b^1 - \frac{1}{2} \frac{db}{dx} - a_n^1 b^1 + \frac{1}{2} \frac{d}{dx} (a_n^1 b + a_n b^1) \right] |u|^2 dx \leq 0. \end{aligned}$$

REMARK. If $\{a_n\}$ and $\{a_n^1\}$ tend to zero in the topology of $C'([0, 1])$ and if for some $\varepsilon > 0$ we have

$$b^1 - \frac{1}{2} \frac{db}{dx} \leq -\varepsilon, \quad b(1) - |c|^2 b(0) \leq -\varepsilon,$$

then (4.3) and (4.4) are easily seen to be satisfied for all sufficiently large n .

THEOREM 4.1. Assume (4.3) and (4.4), that $a_n b \geq 0$ and $a_n^2 + b^2 > 0$. In addition suppose

$$(4.5) \quad a_n(1) - |c|^2 a_n(0) \leq 0;$$

$$(4.6) \quad a_n^1 - \frac{1}{2} \frac{da_n}{dx} < 1, \quad 0 \leq x \leq 1;$$

$$(4.7) \quad \text{There exists } \zeta_n > 0 \text{ such that}$$

$$c \exp \left[\int_0^1 \frac{1 - a_n^1(\xi) - \zeta_n b^1(\xi)}{a_n(\xi) + \zeta_n b(\xi)} d\xi \right] \neq 1.$$

Then the hypotheses of Theorem 2.1 are satisfied.

Thus, in particular, for each $u_0 \in D(B)$ the problem (4.1), (4.2) has a unique solution.

THEOREM 4.2. Assume (4.3)-(4.5), that $a_n b \geq 0$ and $b \neq 0$. In addition suppose

$$(4.8) \quad \{a_n\} \text{ and } \{a_n^1\} \text{ tend to zero in the topology of } C'([0, 1]) \text{ as } n \rightarrow \infty.$$

Then the hypotheses of Theorem 3.2 are satisfied for all sufficiently large n .

THEOREM 4.3. Assume (4.3)-(4.5), (4.8) and that $a_n b > 0$. Then the hypotheses of Theorem 3.3 are satisfied for all sufficiently large n .

Thus if u_n is the unique solution of (4.1), (4.2), as $n \rightarrow \infty$ $u_n(t)$ converges in $L_2(0, 1)$ to the unique solution of

$$\frac{\partial u}{\partial t} - b(x) \frac{\partial u}{\partial x} - b^1(x) u = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1; \quad u(0, t) = cu(1, t), \quad t > 0,$$

uniformly on bounded subsets of $[0, \infty)$.

Proof of Theorem 4.1. We have already verified (2.3). To check (2.1) consider the equation

$$u - A_n u = f \in X.$$

Multiplying by \bar{u} and integrating gives

$$\int_0^1 (1 - a_n^1) |u|^2 dx - \operatorname{Re} \int_0^1 a_n \bar{u} \frac{du}{dx} dx = \operatorname{Re} \int_0^1 \bar{u} f dx$$

and this may be written

$$(4.9) \quad \int_0^1 \left(1 - a_n^1 + \frac{1}{2} \frac{da_n}{dx} \right) |u|^2 = \frac{1}{2} (a_n(1) - |c|^2 a_n(0)) |u(1)|^2 \\ + \operatorname{Re} \int_0^1 \bar{u} f dx.$$

From (4.5), (4.6) and (4.9) follows that $u = 0$ if $f = 0$.

We next verify (2.4). Let $f \in L_2(0, 1)$. We have to solve

$$(4.10) \quad u - A_n u - \zeta_n B u = (1 - a_n^1 - \zeta_n b^1) u - (a_n + \zeta_n b) \frac{du}{dx} = f$$

where $\zeta_n > 0$ is to be determined.

Since $a_n b \geq 0$ and $a_n^2 + b^2 > 0$, we have $a_n + \zeta b \neq 0$ for every $\zeta > 0$ and therefore (4.10) is equivalent to

$$u(x) = k_n \exp \int_0^x K_n(\xi) d\xi + \int_0^x \tilde{K}_n(x, \xi) f(\xi) d\xi$$

where

$$K_n(\xi) = (1 - a_n^1(\xi) - \zeta_n b^1(\xi)) / (a_n(\xi) + \zeta_n b(\xi)),$$

$$\tilde{K}_n(x, \xi) = - \left[\exp \int_\xi^x K_n(\eta) d\eta \right] / (a_n(\xi) + \zeta_n b(\xi)).$$

The constant k_n must be such that $u(0) = cu(1)$. This condition leads to

$$k_n = k_n c \exp \int_0^1 K_n(\xi) d\xi + c \int_0^1 \tilde{K}_n(1, \xi) f(\xi) d\xi$$

and this equation is solvable for k_n for arbitrary $f \in L_2(0, 1)$ if and only if

$$c \exp \int_0^1 K_n(\xi) d\xi \neq 1.$$

This last condition is satisfied provided ζ_n is chosen according to condition (4.7). Thus (2.4) is satisfied.

Proof of Theorem 4.2. We first note that (4.8) implies (4.6) for all sufficiently large n . Moreover, (4.7) is also satisfied for all large n if $\{\zeta_n\}$ is any sequence of positive numbers which tends to zero. Thus conditions (2.1)–(2.4) are satisfied. That (3.1) and (3.2) also hold is a consequence of the inequality

$$(4.11) \quad \|u\|_1 \leq K(\|Bu\|_0 + \|u\|_0), \quad u \in D(B)$$

where the constant K is independent of u . In fact, suppose (4.11) holds and $\{u_n\} \subset D(B)$, $u_n \rightarrow u$, $Bu_n \rightarrow v$ in $L_2(0, 1)$. By (4.11), $\{u_n\}$ converges in $H_2^1(0, 1)$. Since $D(B)$ is a closed subspace of $H_2^1(0, 1)$ and $\|Bu_n\|_0 \leq (\text{const.}) \|u_n\|_1$ it follows that $u \in D(B)$ and $Bu = v$, i.e., B is closed. Moreover, we have

$$\|A_n u\|_0 \leq \sup_{0 \leq x \leq 1} (|a_n(x)| + |a_n^1(x)|) \|u\|_1$$

and therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \|A_n u\|_0 / (\|Bu\|_0 + \|u\|_0) \leq K \sup_{0 \leq x \leq 1} (|a_n(x)| + |a_n^1(x)|)$$

which tends to zero as $n \rightarrow \infty$. Thus (3.2) is satisfied.

It only remains to prove (4.11). We have

$$\|Bu\|_0^2 = \int_0^1 \left(b \frac{du}{dx} + b^1 u \right) \left(b \frac{d\bar{u}}{dx} + b^1 \bar{u} \right) dx.$$

Using the inequality

$$2|yz| \leq \delta |y|^2 + \frac{1}{\delta} |z|^2, \quad \delta > 0$$

we obtain

$$\|Bu\|_0^2 \geq \inf_{0 \leq x \leq 1} |b(x)|^2 \left\| \frac{du}{dx} \right\|_0^2 - \epsilon \left\| \frac{du}{dx} \right\|_0^2 - K_\epsilon \|u\|_0^2.$$

Choosing $\epsilon = 1/2 \inf_{0 \leq x \leq 1} |b(x)|^2$ leads to (4.11).

Proof of Theorem 4.3. We have only to verify that $\text{Rg}(I - A_n) = L_2(0, 1)$, $n \geq N$, and

$$\sup_{n \geq N} \|(I - A_n)^{-1}\| < \infty.$$

From (4.9) and the present hypotheses it follows that for all sufficiently large n ,

$$\frac{1}{2} \|u\|_0^2 \leq \|u - A_n u\|, \quad u \in D(A_n).$$

Let $f \in L_2(0, 1)$. Since $a_n \neq 0$, the equation

$$u - A_n u = (1 - a_n^1)u - a_n \frac{du}{dx} = f$$

is equivalent to

$$u(x) = k_n \exp \int_0^x \frac{1 - a_n^1(\xi)}{a_n} d\xi + F_n(x)$$

where $F_n(x)$ is a known function and the constant k_n must be such that $u(0) = cu(1)$. This is possible for arbitrary $f \in L_2(0, 1)$ if and only if

$$c \exp \int_0^1 \frac{1 - a_n^1(\xi)}{a_n(\xi)} d\xi \neq 1$$

and this last condition is obviously satisfied for all sufficiently large n in view of (4.8). Thus $\text{Rg}(I - A_n) = X$, $n \geq N$, and the proof is complete.

EXAMPLE 2. We consider, for $n = 1, 2, \dots$, the problem

$$(4.12) \quad \frac{\partial}{\partial t} \left(u - \frac{1}{n} \frac{\partial u}{\partial x} \right) - \left(b(x)u + \frac{\partial^2 u}{\partial x^2} \right) = 0, \quad 0 < x < 1, \quad t > 0,$$

$$(4.13) \quad u(x, 0) = u_0(x), \quad 0 < x < 1,$$

$$(4.14) \quad u(0, t) = cu(1, t), \quad \bar{c} \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t), \quad t > 0.$$

The function b is real-valued and of class $C'([0, 1])$ and c is a complex constant. Let $X = L_2(0, 1)$, $D(A_n)$ be as in the first example and $A_n = 1/n \, d/dx$. Let $H_2^2(0, 1)$ be the set of functions in $L_2(0, 1)$ whose first and second distributional derivatives are in $L_2(0, 1)$ and set

$$D(B) = \left\{ u \in H_2^2(0, 1) : u(0) = cu(1), \bar{c} \frac{du}{dx}(0) = \frac{du}{dx}(1) \right\},$$

$$Bu = bu + \frac{d^2 u}{dx^2}, \quad u \in D(B).$$

The norm in $H_2^2(0, 1)$ is denoted by $\|\cdot\|_2$ and defined by

$$\|u\|_2 = \left(\|u\|_0^2 + \left\| \frac{du}{dx} \right\|_0^2 + \left\| \frac{d^2 u}{dx^2} \right\|_0^2 \right)^{1/2}.$$

Each function in $H_2^2(0, 1)$ is of class $C'([0, 1])$ and the injection of $H_2^2(0, 1)$ into $C'([0, 1])$ is continuous. It follows that $D(B)$ is a closed subspace of $H_2^2(0, 1)$, is dense in $L_2(0, 1)$ and as in the first example it is not difficult to verify that

$$(4.15) \quad \|u\|_2 \leq K(\|Bu\|_0 + \|u\|_0), \quad u \in D(B).$$

Let B^* be the adjoint of B . As is well-known, $D(B^*) \subset H_2^2(0, 1)$

and, since $b + d^2/dx^2$ is a formally self-adjoint differential operator,

$$B^*v = bv + \frac{d^2v}{dx^2}, \quad v \in D(B^*).$$

We show that $B^* = B$. If $v \in D(B^*)$ then for all $u \in D(B)$ we have

$$\begin{aligned} (Bu, v) &= \int_0^1 \left(bu + \frac{d^2u}{dx^2} \right) \bar{v} dx = \frac{du}{dx}(0)(\bar{c}\bar{v}(1) - \bar{v}(0)) \\ &\quad - u(1) \left(\frac{d\bar{v}}{dx}(1) - c \frac{d\bar{v}}{dx}(0) \right) + (u, B^*v). \end{aligned}$$

Since the first two terms on the right must vanish for all $u \in D(B)$ we have $v(0) = cv(1)$, $\bar{c}(dv/dx)(0) = (dv/dx)(1)$, that is, $v \in D(B)$. Thus $B^* \subseteq B$. On the other hand, $(Bu, v) = (u, Bv)$ for all u and v in $D(B)$ so that B is symmetric. Hence B is self-adjoint.

THEOREM 4.4. *Suppose $b \leq 0$, $db/dx \leq 0$ and*

$$b(1) - |c|^2 b(0) \geq 0.$$

Then the hypotheses of Corollary 3.1 are satisfied.

Thus for each n and $u_0 \in D(B)$ the problem (4.12)-(4.14) has a unique solution u_n and, as $n \rightarrow \infty$, $u_n(t)$ converges in $L_2(0, 1)$ to the solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - b(x)u = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1,$$

$$u(0, t) = cu(1, t), \quad \bar{c} \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t), \quad t > 0,$$

uniformly on bounded subsets of $[0, \infty)$.

Proof of Theorem 4.4. We have for $u \in D(B)$

$$(Bu, u) = \int_0^1 b|u|^2 dx - \int_0^1 \left| \frac{du}{dx} \right|^2 dx \leq 0.$$

B is therefore a self-adjoint, dissipative operator and, consequently, maximal dissipative. We also have

$$n \cdot \operatorname{Re}(A_n u, u) = \frac{1}{2}(1 - |c|^2)|u(1)|^2, \quad u \in D(A_n).$$

Since $b \leq 0$ and $db/dx \leq 0$ we have $b(1) \leq b(0) \leq 0$. Since also $b(1) \geq$

$|c|^2 b(0)$ it follows that $|c|^2 \geq 1$. Thus A_n is dissipative and one easily proves as in the first example that $Rg(I - A_n) = X$.

Next we verify (1.3). We have for $u \in D(B)$

$$\begin{aligned} n \cdot \operatorname{Re}(A_n u, Bu) &= \operatorname{Re} \int_0^1 \frac{du}{dx} \left(b\bar{u} + \frac{d^2 \bar{u}}{dx^2} \right) dx \\ &= \frac{1}{2} (b(1) - |c|^2 b(0)) |u(1)|^2 - \frac{1}{2} \int_0^1 \frac{db}{dx} |u|^2 dx \\ &\quad + \frac{1}{2} \left| \frac{du}{dx}(0) \right|^2 (|\bar{c}|^2 - 1) \geq 0. \end{aligned}$$

(1.3) follows from this inequality and the fact that B is dissipative. Finally, (3.2) is an immediate consequence of (4.15).

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