

Pacific Journal of Mathematics

**A QUASI ORDER CHARACTERIZATION OF SMOOTH
CONTINUA**

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L. E. Ward, Jr. characterized a generalized tree as a compact Hausdorff space which admits a partial order satisfying certain conditions. An analogous characterization of smooth continua, in terms of quasi ordered topological spaces, is obtained.

A *quasi order* on a topological space X is a reflexive and transitive binary relation \leq . If this relation is also antisymmetric it is called a *partial order*. The quasi order \leq is *closed* if $\{(x, y) \in X \times X \mid x \leq y\}$ is a closed subset of the product space $X \times X$.

For each $x \in X$, the set $L(x) = \{y \in X \mid y \leq x\}$ (respectively, $M(x) = \{y \in X \mid x \leq y\}$) is called the *set of predecessors* (respectively, *successors*) of x . Let $E(x) = L(x) \cap M(x)$ and note that \leq is a partial order if and only if each $E(x)$ is a singleton. In case \leq is closed, the sets $L(x)$, $M(x)$, and $E(x)$ are closed subsets of X .

If $x \leq y$ and $x \notin E(y)$ we write $x < y$. The quasi order \leq is *order dense* if whenever $x < y$, there exists $z \in X$ such that $x < z < y$.

Let S be a subset of X . An element $z \in S$ is a *zero* of S if $z \leq x$ for each $x \in S$. If $x \leq y$ or $y \leq x$ for all $x, y \in S$, then S is called a *chain*.

We define the equivalence relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } E(x) = E(y).$$

Let $\phi: X \rightarrow X/\rho$ denote the natural quotient map.

A continuum (= compact connected Hausdorff space) X is *hereditarily unicoherent* at the point p [2] if for each $x \in X$, there exists a unique subcontinuum of X , denoted $[p, x]$, irreducible between p and x . We say X is *hereditarily unicoherent* if it is hereditarily unicoherent at each of its points.

If the continuum X is hereditarily unicoherent at p then X admits a very natural quasi order \leq_p , called the *weak cut point order with respect to p* :

$$x \leq_p y \text{ if and only if } x \in [p, y].$$

Note that for each $x \in X$, $L(x) = [p, x]$.

The continuum X is *smooth* if there exists a point $p \in X$ such that X is hereditarily unicoherent at p and the quasi order \leq_p is closed. By [1], Theorem 3.1, p. 65, this definition is equivalent to

Gordh's original definition [2]. To emphasize the point p we will often write " X is smooth at p ". A *generalized tree* is a hereditarily unicoherent, arcwise connected¹ smooth continuum. Ward's original definition [6] is stated here as Theorem 1. According to [4] the definitions are equivalent.

THEOREM 1. *The compact Hausdorff space X is a generalized tree if and only if X admits a partial order \leq such that*

- (1) \leq is closed;
- (2) \leq is order dense;
- (3) if $x, y \in X$, then $L(x) \cap L(y)$ is a nonempty chain;
- (4) if Y is a closed and connected subset of X , then Y contains a zero.

It follows that \leq is the weak cut point order with respect to p where $\{p\} = \bigcap \{L(x) \mid x \in X\}$ and $L(x) = [p, x]$.

It is the purpose of this paper to establish an analogous characterization for smooth continua.

Consider the following properties that a quasi order \leq on a space X may possess:

- (i) \leq is closed;
- (ii) \leq is order dense;
- (iii) there exists $p \in \bigcap \{L(x) \mid x \in X\}$ and each $L(x)$ is a chain;
- (iv) if Y is a closed connected subset of X , then Y contains a zero;
- (v) $E(x)$ is connected for each $x \in X$;
- (vi) if Y is a closed connected subset of X and $p \in Y$, then $E(y) \subseteq Y$ for each $y \in Y$.

THEOREM 2. *Let X be a compact Hausdorff space which admits a quasi order \leq satisfying (i)-(vi). Then X is a continuum which is smooth at p .*

The theorem will be proved via a series of lemmas. Unless otherwise stated assume X , \leq , and p are as above. Observe that (vi) implies p is the unique zero of X .

LEMMA 1. *The space X/ρ is compact Hausdorff and the map $\phi: X \rightarrow X/\rho$ is monotone.*

Proof. First note that $\{E(x) \mid x \in X\}$ is a pairwise disjoint closed covering of X . From Theorem 2, [7], p. 147, and [3], p. 132, we infer $\{E(x) \mid x \in X\}$ is an upper semicontinuous decomposition of X .

¹ An *arc* is a continuum (not necessarily metrizable) with exactly two noncut points.

By Theorem 3-33, [3], p. 133, X/ρ is compact Hausdorff. Finally, it follows from (i) and (v) that $\phi^{-1}(\phi(x)) = E(x)$ is closed and connected; hence $\phi: X \rightarrow X/\rho$ is monotone.

The quasi order \leq on X induces a relation \leq' on X/ρ defined by

$$\phi(x) \leq' \phi(y) \text{ if and only if } x \leq y .$$

For the sake of clarity let $L'(\phi(x))$ denote the set of predecessors of $\phi(x)$ in X/ρ .

LEMMA 2. *The space X/ρ is a generalized tree which is smooth at $\phi(p)$. Moreover, \leq' is the weak cut point order with respect to $\phi(p)$ and $L'(\phi(x))$ is the unique subcontinuum of X/ρ irreducible between $\phi(p)$ and $\phi(x)$.*

Proof. It is straightforward to verify that \leq' is a partial order satisfying the hypotheses of Theorem 1.

LEMMA 3. *The space X is a continuum. In particular, $L(x)$ is closed and connected for each $x \in X$.*

Proof. Since $L(x)$ is the inverse image of $L'(\phi(x)) \subseteq X/\rho$ under the monotone map $\phi: X \rightarrow X/\rho$ it follows from Theorem 9, [5], p. 131, that $L(x)$ is closed and connected. Since $p \in \bigcap \{L(x) \mid x \in X\}$ and $X = \bigcup \{L(x) \mid x \in X\}$, the lemma is proved.

LEMMA 4. *If Y is a subcontinuum of X and $p \in Y$, then $\phi^{-1}(\phi(Y)) = Y$.*

Proof. We show only $\phi^{-1}(\phi(Y)) \subseteq Y$. If $z \in \phi^{-1}(\phi(Y))$ there exists $y \in Y$ such that $\phi(y) = \phi(z)$. By (vi)

$$z \in E(z) = E(y) \subseteq Y .$$

LEMMA 5. *The continuum X is hereditarily unicoherent at p .*

Proof. Let x be a fixed, but arbitrary, point in X and let $Y \subseteq X$ be a subcontinuum irreducible between p and x . Then $\phi(Y) \subseteq X/\rho$ is a subcontinuum containing $\phi(p)$ and $\phi(x)$. Since X/ρ is a generalized tree, $L'(\phi(x)) \subseteq \phi(Y)$. It follows from

$$L(x) = \phi^{-1}(L'(\phi(x)) \subseteq \phi^{-1}(\phi(Y)) = Y$$

and Lemma 3 that $L(x) = Y$. That is, $L(x)$ is the unique subcontinuum of X irreducible between p and x .

We have shown that the space X is a continuum which is here-

ditarily unicoherent at p . Moreover, $[p, x] = L(x)$ for each $x \in X$. It follows immediately that \leq is the weak cut point order with respect to p . Since \leq is closed by hypothesis, the proof of Theorem 2 is complete.

The converse of Theorem 2 is also true. Before proceeding, however, we need a few results about smooth continua. The reader is referred to [2] for the details.

THEOREM 3. *If the continuum X is smooth at p then X/ρ is a generalized tree which is smooth at $\phi(p)$, the map $\phi: X \rightarrow X/\rho$ is monotone, and $\text{int}_x E(x) = \square$.²*

LEMMA 6. *If the continuum X is smooth at p then $x \leq_p y$ (respectively, $x <_p y$) if and only if $\phi(x) \leq_{\phi(p)} \phi(y)$ (respectively, $\phi(x) <_{\phi(p)} \phi(y)$). Moreover, if Y is a subcontinuum of X and $p \in Y$, then $\phi^{-1}(\phi(Y)) = Y$.*

THEOREM 4. *If the continuum X is smooth at p then \leq_p satisfies (i)-(vi).*

Proof. It is immediate that (i) and (vi) hold. Since $E(x)$ is the inverse image of the point $\phi(x)$ under the monotone map $\phi: X \rightarrow X/\rho$, (v) holds. Conditions (ii) and (iii) follow from Lemma 6 and the fact that $L(x) = \phi^{-1}(L'(\phi(x)))$. Finally to show (iv) holds, let Y be a subcontinuum of X . Then $\phi(Y)$ is a subcontinuum of the generalized tree X/ρ . Let $z \in X$ be such that $\phi(z)$ is a zero of $\phi(Y)$. Choose any

$$y \in \phi^{-1}(\phi(z)) \cap Y = E(z) \cap Y.$$

It follows from Lemma 6 that y is a zero of Y .

Observe that condition (iii) is equivalent to condition (3) of Ward's theorem. The paraphrase was inserted as a matter of convenience, since the point p appears in condition (vi).

We remark that each of conditions (i)-(vi) is independent of the remaining five. We include here examples to clarify the necessity of the last two conditions. The omitted details are left to the reader. Let \leq_0 denote the natural partial order on the real numbers.

EXAMPLE 1. (Due to J. Ladwig.) Let X denote the Cantor Set and let $\{(a_n, b_n) \mid n = 1, 2, \dots\}$ be the collection of "deleted intervals"; i.e.,

$$X = [0, 1] - \bigcup_{n=1}^{\infty} (a_n, b_n)$$

² "int_x" denotes interior in the space X and " \square " denotes the empty set.

and for $n = 1, 2, \dots$

$$[a_n, b_n] \cap X = \{a_n, b_n\} .$$

Define $x \leq y$ if and only if $x \leq_0 y$ or x and y are endpoints of a common deleted interval. The quasi order \leq on X satisfies (i)-(iv) and (vi) but not (v).

EXAMPLE 2. In the plane let X be the triangle with vertices $p = (0, 0), (1, 0),$ and $(1, 1)$. Define $(x, y) \leq (u, v)$ if and only if $x \leq_0 u$. Then \leq on X satisfies (i)-(v) but not (vi); e.g., take $Y = [0, 1] \times \{0\}$.

COROLLARY 1. Let X be a continuum which is smooth at p . Then \leq_p is a partial order if and only if X is a generalized tree which is smooth at p .

Proof. If \leq_p is a partial order then each $E(x)$ is degenerate and conditions (i)-(vi) reduce to (1)-(4) of Theorem 1. The converse is trivial since each $L(x)$ is an arc for each $x \in X$.

It is necessary that the continuum X in Corollary 1 be smooth at p as the example below shows.

EXAMPLE 3. In the plane let

$$\begin{aligned} A &= \left\{ \left(x, \sin \frac{1}{x} \right) \mid 0 < x \leq 1 \right\} , \\ B &= \{0\} \times [-1, 1] , \\ C &= [-1, 0] \times \{-1\} . \end{aligned}$$

The continuum $X = A \cup B \cup C$ is clearly not a generalized tree. However, X is hereditarily unicoherent and \leq_p is a partial order for $p = (-1, 1)$.

Finally observe that in the presence of conditions (i) and (iii)-(vi), condition (ii) is equivalent to

$$(ii') \text{ int}_{L(x)} E(x) = \square \text{ for each } x \in X - \{p\} .$$

For if X is smooth at p then so is $L(x)$; thus (ii') is a consequence of Theorem 3. Conversely, we show (i), (ii'), and (iii) imply (ii). Suppose $x, y \in X$ are such that $x < y$ and $x < z < y$ for no $z \in X$. Then $L(y) - L(x)$ is a nonempty open (in $L(y)$) subset of $E(y)$, contradicting (ii').

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Received June 29, 1973 and in revised form October 19, 1973. This research constitutes a part of the author's doctoral dissertation written under L. E. Ward, Jr., at the University of Oregon.

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The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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Pacific Journal of Mathematics

Vol. 53, No. 2

April, 1974

Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i>	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i>	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i>	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i>	347
Stephen LaVern Campbell, <i>Linear operators for which T^*T and TT^* commute. II</i>	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i>	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i>	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i>	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in 2^X and $C(X)$</i>	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i>	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i>	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i>	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i>	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i>	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i>	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ...	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i>	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i>	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i>	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot</i>	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i>	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i>	539
Dorte Olesen, <i>Derivations of AW^*-algebras are inner</i>	555
Dorte Olesen and Gert Kjærgaard Pedersen, <i>Derivations of C^*-algebras have semi-continuous generators</i>	563
Duane O'Neill, <i>On conjugation cobordism</i>	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i>	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of $L^1(\mu; E)$</i>	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i>	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i>	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i>	627
Carl E. Swenson, <i>Direct sum subset decompositions of Z</i>	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i>	635
Robert S. Wilson, <i>Representations of finite rings</i>	643