

# Pacific Journal of Mathematics

**A QUASI ORDER CHARACTERIZATION OF SMOOTH  
CONTINUA**

LEWIS LUM

## A QUASI ORDER CHARACTERIZATION OF SMOOTH CONTINUA

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**L. E. Ward, Jr. characterized a generalized tree as a compact Hausdorff space which admits a partial order satisfying certain conditions. An analogous characterization of smooth continua, in terms of quasi ordered topological spaces, is obtained.**

A *quasi order* on a topological space  $X$  is a reflexive and transitive binary relation  $\leq$ . If this relation is also antisymmetric it is called a *partial order*. The quasi order  $\leq$  is *closed* if  $\{(x, y) \in X \times X \mid x \leq y\}$  is a closed subset of the product space  $X \times X$ .

For each  $x \in X$ , the set  $L(x) = \{y \in X \mid y \leq x\}$  (respectively,  $M(x) = \{y \in X \mid x \leq y\}$ ) is called the *set of predecessors* (respectively, *successors*) of  $x$ . Let  $E(x) = L(x) \cap M(x)$  and note that  $\leq$  is a partial order if and only if each  $E(x)$  is a singleton. In case  $\leq$  is closed, the sets  $L(x)$ ,  $M(x)$ , and  $E(x)$  are closed subsets of  $X$ .

If  $x \leq y$  and  $x \notin E(y)$  we write  $x < y$ . The quasi order  $\leq$  is *order dense* if whenever  $x < y$ , there exists  $z \in X$  such that  $x < z < y$ .

Let  $S$  be a subset of  $X$ . An element  $z \in S$  is a *zero* of  $S$  if  $z \leq x$  for each  $x \in S$ . If  $x \leq y$  or  $y \leq x$  for all  $x, y \in S$ , then  $S$  is called a *chain*.

We define the equivalence relation  $\rho$  on  $X$  by

$$(x, y) \in \rho \text{ if and only if } E(x) = E(y).$$

Let  $\phi: X \rightarrow X/\rho$  denote the natural quotient map.

A continuum (= compact connected Hausdorff space)  $X$  is *hereditarily unicoherent* at the point  $p$  [2] if for each  $x \in X$ , there exists a unique subcontinuum of  $X$ , denoted  $[p, x]$ , irreducible between  $p$  and  $x$ . We say  $X$  is *hereditarily unicoherent* if it is hereditarily unicoherent at each of its points.

If the continuum  $X$  is hereditarily unicoherent at  $p$  then  $X$  admits a very natural quasi order  $\leq_p$ , called the *weak cut point order with respect to  $p$* :

$$x \leq_p y \text{ if and only if } x \in [p, y].$$

Note that for each  $x \in X$ ,  $L(x) = [p, x]$ .

The continuum  $X$  is *smooth* if there exists a point  $p \in X$  such that  $X$  is hereditarily unicoherent at  $p$  and the quasi order  $\leq_p$  is closed. By [1], Theorem 3.1, p. 65, this definition is equivalent to

Gordh's original definition [2]. To emphasize the point  $p$  we will often write " $X$  is smooth at  $p$ ". A *generalized tree* is a hereditarily unicoherent, arcwise connected<sup>1</sup> smooth continuum. Ward's original definition [6] is stated here as Theorem 1. According to [4] the definitions are equivalent.

**THEOREM 1.** *The compact Hausdorff space  $X$  is a generalized tree if and only if  $X$  admits a partial order  $\leq$  such that*

- (1)  $\leq$  is closed;
- (2)  $\leq$  is order dense;
- (3) if  $x, y \in X$ , then  $L(x) \cap L(y)$  is a nonempty chain;
- (4) if  $Y$  is a closed and connected subset of  $X$ , then  $Y$  contains a zero.

It follows that  $\leq$  is the weak cut point order with respect to  $p$  where  $\{p\} = \bigcap \{L(x) \mid x \in X\}$  and  $L(x) = [p, x]$ .

It is the purpose of this paper to establish an analogous characterization for smooth continua.

Consider the following properties that a quasi order  $\leq$  on a space  $X$  may possess:

- (i)  $\leq$  is closed;
- (ii)  $\leq$  is order dense;
- (iii) there exists  $p \in \bigcap \{L(x) \mid x \in X\}$  and each  $L(x)$  is a chain;
- (iv) if  $Y$  is a closed connected subset of  $X$ , then  $Y$  contains a zero;
- (v)  $E(x)$  is connected for each  $x \in X$ ;
- (vi) if  $Y$  is a closed connected subset of  $X$  and  $p \in Y$ , then  $E(y) \subseteq Y$  for each  $y \in Y$ .

**THEOREM 2.** *Let  $X$  be a compact Hausdorff space which admits a quasi order  $\leq$  satisfying (i)-(vi). Then  $X$  is a continuum which is smooth at  $p$ .*

The theorem will be proved via a series of lemmas. Unless otherwise stated assume  $X$ ,  $\leq$ , and  $p$  are as above. Observe that (vi) implies  $p$  is the unique zero of  $X$ .

**LEMMA 1.** *The space  $X/\rho$  is compact Hausdorff and the map  $\phi: X \rightarrow X/\rho$  is monotone.*

*Proof.* First note that  $\{E(x) \mid x \in X\}$  is a pairwise disjoint closed covering of  $X$ . From Theorem 2, [7], p. 147, and [3], p. 132, we infer  $\{E(x) \mid x \in X\}$  is an upper semicontinuous decomposition of  $X$ .

<sup>1</sup> An *arc* is a continuum (not necessarily metrizable) with exactly two noncut points.

By Theorem 3-33, [3], p. 133,  $X/\rho$  is compact Hausdorff. Finally, it follows from (i) and (v) that  $\phi^{-1}(\phi(x)) = E(x)$  is closed and connected; hence  $\phi: X \rightarrow X/\rho$  is monotone.

The quasi order  $\leq$  on  $X$  induces a relation  $\leq'$  on  $X/\rho$  defined by

$$\phi(x) \leq' \phi(y) \text{ if and only if } x \leq y .$$

For the sake of clarity let  $L'(\phi(x))$  denote the set of predecessors of  $\phi(x)$  in  $X/\rho$ .

**LEMMA 2.** *The space  $X/\rho$  is a generalized tree which is smooth at  $\phi(p)$ . Moreover,  $\leq'$  is the weak cut point order with respect to  $\phi(p)$  and  $L'(\phi(x))$  is the unique subcontinuum of  $X/\rho$  irreducible between  $\phi(p)$  and  $\phi(x)$ .*

*Proof.* It is straightforward to verify that  $\leq'$  is a partial order satisfying the hypotheses of Theorem 1.

**LEMMA 3.** *The space  $X$  is a continuum. In particular,  $L(x)$  is closed and connected for each  $x \in X$ .*

*Proof.* Since  $L(x)$  is the inverse image of  $L'(\phi(x)) \subseteq X/\rho$  under the monotone map  $\phi: X \rightarrow X/\rho$  it follows from Theorem 9, [5], p. 131, that  $L(x)$  is closed and connected. Since  $p \in \bigcap \{L(x) \mid x \in X\}$  and  $X = \bigcup \{L(x) \mid x \in X\}$ , the lemma is proved.

**LEMMA 4.** *If  $Y$  is a subcontinuum of  $X$  and  $p \in Y$ , then  $\phi^{-1}(\phi(Y)) = Y$ .*

*Proof.* We show only  $\phi^{-1}(\phi(Y)) \subseteq Y$ . If  $z \in \phi^{-1}(\phi(Y))$  there exists  $y \in Y$  such that  $\phi(y) = \phi(z)$ . By (vi)

$$z \in E(z) = E(y) \subseteq Y .$$

**LEMMA 5.** *The continuum  $X$  is hereditarily unicoherent at  $p$ .*

*Proof.* Let  $x$  be a fixed, but arbitrary, point in  $X$  and let  $Y \subseteq X$  be a subcontinuum irreducible between  $p$  and  $x$ . Then  $\phi(Y) \subseteq X/\rho$  is a subcontinuum containing  $\phi(p)$  and  $\phi(x)$ . Since  $X/\rho$  is a generalized tree,  $L'(\phi(x)) \subseteq \phi(Y)$ . It follows from

$$L(x) = \phi^{-1}(L'(\phi(x)) \subseteq \phi^{-1}(\phi(Y)) = Y$$

and Lemma 3 that  $L(x) = Y$ . That is,  $L(x)$  is the unique subcontinuum of  $X$  irreducible between  $p$  and  $x$ .

We have shown that the space  $X$  is a continuum which is here-

ditarily unicoherent at  $p$ . Moreover,  $[p, x] = L(x)$  for each  $x \in X$ . It follows immediately that  $\leq$  is the weak cut point order with respect to  $p$ . Since  $\leq$  is closed by hypothesis, the proof of Theorem 2 is complete.

The converse of Theorem 2 is also true. Before proceeding, however, we need a few results about smooth continua. The reader is referred to [2] for the details.

**THEOREM 3.** *If the continuum  $X$  is smooth at  $p$  then  $X/\rho$  is a generalized tree which is smooth at  $\phi(p)$ , the map  $\phi: X \rightarrow X/\rho$  is monotone, and  $\text{int}_X E(x) = \square$ .<sup>2</sup>*

**LEMMA 6.** *If the continuum  $X$  is smooth at  $p$  then  $x \leq_p y$  (respectively,  $x <_p y$ ) if and only if  $\phi(x) \leq_{\phi(p)} \phi(y)$  (respectively,  $\phi(x) <_{\phi(p)} \phi(y)$ ). Moreover, if  $Y$  is a subcontinuum of  $X$  and  $p \in Y$ , then  $\phi^{-1}(\phi(Y)) = Y$ .*

**THEOREM 4.** *If the continuum  $X$  is smooth at  $p$  then  $\leq_p$  satisfies (i)-(vi).*

*Proof.* It is immediate that (i) and (vi) hold. Since  $E(x)$  is the inverse image of the point  $\phi(x)$  under the monotone map  $\phi: X \rightarrow X/\rho$ , (v) holds. Conditions (ii) and (iii) follow from Lemma 6 and the fact that  $L(x) = \phi^{-1}(L'(\phi(x)))$ . Finally to show (iv) holds, let  $Y$  be a subcontinuum of  $X$ . Then  $\phi(Y)$  is a subcontinuum of the generalized tree  $X/\rho$ . Let  $z \in X$  be such that  $\phi(z)$  is a zero of  $\phi(Y)$ . Choose any

$$y \in \phi^{-1}(\phi(z)) \cap Y = E(z) \cap Y.$$

It follows from Lemma 6 that  $y$  is a zero of  $Y$ .

Observe that condition (iii) is equivalent to condition (3) of Ward's theorem. The paraphrase was inserted as a matter of convenience, since the point  $p$  appears in condition (vi).

We remark that each of conditions (i)-(vi) is independent of the remaining five. We include here examples to clarify the necessity of the last two conditions. The omitted details are left to the reader. Let  $\leq_0$  denote the natural partial order on the real numbers.

**EXAMPLE 1.** (Due to J. Ladwig.) Let  $X$  denote the Cantor Set and let  $\{(a_n, b_n) \mid n = 1, 2, \dots\}$  be the collection of "deleted intervals"; i.e.,

$$X = [0, 1] - \bigcup_{n=1}^{\infty} (a_n, b_n)$$

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<sup>2</sup> "int<sub>X</sub>" denotes interior in the space  $X$  and " $\square$ " denotes the empty set.

and for  $n = 1, 2, \dots$

$$[a_n, b_n] \cap X = \{a_n, b_n\}.$$

Define  $x \leq y$  if and only if  $x \leq_0 y$  or  $x$  and  $y$  are endpoints of a common deleted interval. The quasi order  $\leq$  on  $X$  satisfies (i)-(iv) and (vi) but not (v).

EXAMPLE 2. In the plane let  $X$  be the triangle with vertices  $p = (0, 0), (1, 0),$  and  $(1, 1)$ . Define  $(x, y) \leq (u, v)$  if and only if  $x \leq_0 u$ . Then  $\leq$  on  $X$  satisfies (i)-(v) but not (vi); e.g., take  $Y = [0, 1] \times \{0\}$ .

COROLLARY 1. Let  $X$  be a continuum which is smooth at  $p$ . Then  $\leq_p$  is a partial order if and only if  $X$  is a generalized tree which is smooth at  $p$ .

Proof. If  $\leq_p$  is a partial order then each  $E(x)$  is degenerate and conditions (i)-(vi) reduce to (1)-(4) of Theorem 1. The converse is trivial since each  $L(x)$  is an arc for each  $x \in X$ .

It is necessary that the continuum  $X$  in Corollary 1 be smooth at  $p$  as the example below shows.

EXAMPLE 3. In the plane let

$$A = \left\{ \left( x, \sin \frac{1}{x} \right) \mid 0 < x \leq 1 \right\},$$

$$B = \{0\} \times [-1, 1],$$

$$C = [-1, 0] \times \{-1\}.$$

The continuum  $X = A \cup B \cup C$  is clearly not a generalized tree. However,  $X$  is hereditarily unicoherent and  $\leq_p$  is a partial order for  $p = (-1, 1)$ .

Finally observe that in the presence of conditions (i) and (iii)-(vi), condition (ii) is equivalent to

$$(ii') \text{ int}_{L(x)} E(x) = \square \text{ for each } x \in X - \{p\}.$$

For if  $X$  is smooth at  $p$  then so is  $L(x)$ ; thus (ii') is a consequence of Theorem 3. Conversely, we show (i), (ii'), and (iii) imply (ii). Suppose  $x, y \in X$  are such that  $x < y$  and  $x < z < y$  for no  $z \in X$ . Then  $L(y) - L(x)$  is a nonempty open (in  $L(y)$ ) subset of  $E(y)$ , contradicting (ii').

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