

Pacific Journal of Mathematics

**ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY
ELEMENTARY SURGERY ALONG A KNOT**

LOUISE ELIZABETH MOSER

ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY
 ELEMENTARY SURGERY ALONG A KNOT

LOUISE E. MOSER

Elementary surgery along a knot has been used in an attempt to construct a counterexample to the Poincaré Conjecture. Certain classes of knots have been examined, but no counterexample has yet been found. Another, and perhaps as interesting a question, is whether $S^2 \times S^1$ can be obtained by elementary surgery along a knot. In this paper the question is answered in the negative for knots with nontrivial Alexander polynomial, for composite knots, and for a large class of knots with trivial Alexander polynomial—the simply doubled knots.

By a knot we will mean a polygonal simple closed curve in the 3-sphere S^3 . A solid torus T is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of T is a torus, a 2-manifold homeomorphic to $S^1 \times S^1$. A meridian of T is a simple closed curve on $\text{Bd } T$ which bounds a disk in T but is not homologous to zero on $\text{Bd } T$. A meridional disk of T is a disk D in T such that $D \cap \text{Bd } T = \text{Bd } D$, and $\text{Bd } D$ is a meridian of T . A longitude of T is a simple closed curve on $\text{Bd } T$ which is transverse to a meridian of T and is null-homologous in $\overline{S^3 - T}$.

The basic construction, elementary surgery along a knot, is now described: Let N be a regular neighborhood of a knot K , m an oriented meridional curve on $\text{Bd } N$, and l an oriented curve on $\text{Bd } N$ which is transverse to m and bounds an orientable surface in $\overline{S^3 - N}$. Let T be a solid torus and let $h: T \rightarrow N$ be a homeomorphism. Then S^3 is homeomorphic to $\overline{S^3 - N} \cup_{h|_{\text{Bd } T}} T$. Now let $h_1: \text{Bd } T \rightarrow \text{Bd } N$ be a homeomorphism with the property that $h^{-1} \cdot h_1: \text{Bd } T \rightarrow \text{Bd } T$ does not extend to a homeomorphism of T onto T . Let $M^3 = \overline{S^3 - N} \cup_{h_1} T$, then we say that M^3 is obtained from S^3 by performing an elementary surgery along K .

Consider now the fundamental group of the complement of the knot $\pi_1(\overline{S^3 - N})$ with base point $m \cap l$, where m and l are considered as elements of $\pi_1(\overline{S^3 - N}) = G$. Then the coset $\bar{m} = mG'$ generates the commutator quotient group $G/G' = H_1(\overline{S^3 - N})$, and the longitude l is in the second commutator subgroup G'' . The fundamental group of M^3 is obtained by adjoining the relation $l^p = m^q$ to $\pi_1(\overline{S^3 - N})$ where $pl - qm$ is the image under h_1 of the boundary of a meridional disk of T , p and q are relatively prime, and $p > 0$. The first homology group of M^3 is generated by \bar{m} with the relation $\bar{m}^q = 1$.

Thus if M^3 is homeomorphic to $S^2 \times S^1$, then $\pi_1(M^3) \simeq H_1(M^3) \simeq Z$. Hence, $q = 0$ and $p = 1$; that is, a longitudinal surgery is performed in which the image of the boundary of a meridional disk is a longitude. It should be noted that a longitudinal surgery along a trivial knot does yield $S^2 \times S^1$. In the following theorem we give a necessary condition that a surgered manifold be homeomorphic to $S^2 \times S^1$.

THEOREM 1. *If a manifold homeomorphic to $S^2 \times S^1$ results from elementary surgery along a knot K , then the Alexander polynomial of K is trivial.*

Proof. If a surgered manifold M^3 is homeomorphic to $S^2 \times S^1$, then a longitudinal surgery must have been performed. The fundamental group of M^3 is obtained by adding the relation $l = 1$ to $\pi_1(\overline{S^3 - N}) = G$. In other words, $\pi_1(M^3)$ is the quotient group of G by the normal closure of the subgroup generated by l ; denote this subgroup by $(l)^c$. Now since $l \in G''$ and G'' is a characteristic subgroup of G' , it follows that $(l)^c \leq G'' \leq G'$. Thus if G'' is a proper subgroup of G' , then $\pi_1(M^3) \neq Z$ and M^3 is not homeomorphic to $S^2 \times S^1$. But G'' is a proper subgroup of G' if and only if the Alexander polynomial of K is nontrivial [1]. This establishes Theorem 1.

So now we consider a large class of nontrivial knots with trivial Alexander polynomial—the simply doubled knots. A simply doubled knot or a doubled knot without twists is defined as follows: Let T_0 be a standardly embedded solid torus in S^3 with meridian m_0 and longitude l_0 . Let J be a self-linking simple closed curve in T_0 (as shown in Figure 1 for the trefoil) and let T_1 be a regular neighborhood of J in T_0 with meridian m_1 and longitude l_1 . Let K be a nontrivial knot in S^3 , $N(K)$ a regular neighborhood of K with meridian m and longitude l which bounds an orientable surface in $\overline{S^3 - N(K)}$. Let $f: T_0 \rightarrow N(K)$ be a homeomorphism with the property that $f(m_0) = m$ and $f(l_0) = l$, then we say that K is simply doubled to obtain $f(J)$.

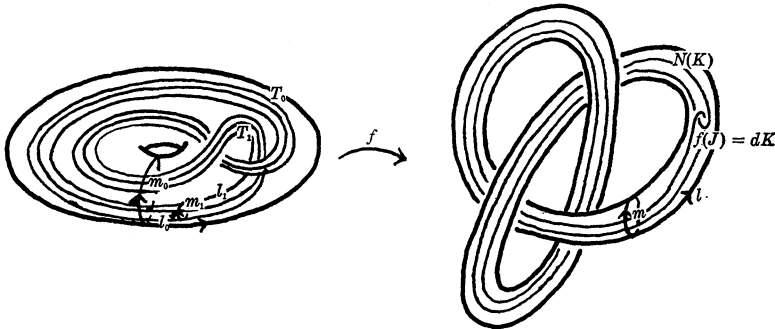


FIGURE 1.

The doubled knot $f(J)$ we will denote by dK .

Consider now the fundamental group of $\overline{T_0 - T_1}$ with base point $m_0 \cap l_0$; let $G_1 = \pi_1(\overline{T_0 - T_1})$ and let $G(K) = \pi_1(\overline{S^3 - N(K)})$. By van Kampen's theorem, the group of the double of K , $G(dK) = \pi_1(\overline{S^3 - N(dK)})$, is the free product with amalgamation $G(K)*G_1$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_1 determined by $l = l_0$ and $m = m_0$. Furthermore, G_1 is generated by l_0 and m_1 subject to the relation $[l_0, m_0] = 1$ where $[x, y] = xyx^{-1}y^{-1}$, $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$, and $l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}]$. See [2].

THEOREM 2. *Elementary surgery along a doubled knot does not yield $S^2 \times S^1$.*

Proof. Perform a longitudinal surgery along dK by replacing the regular neighborhood $f(T_1)$ of dK by a solid torus T_2 to obtain $M^3 = \overline{S^3 - f(T_1)} \cup_h T_2$ where $h: \text{Bd } T_2 \rightarrow \text{Bd } f(T_1)$ is a homeomorphism which takes a meridian of T_2 to the longitude $f(l_1)$ of $f(T_1)$.

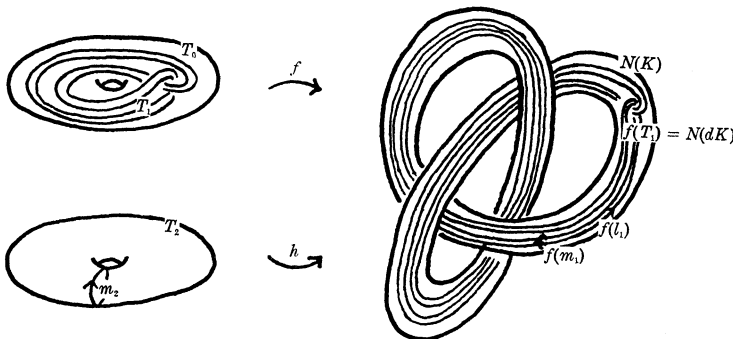


FIGURE 2.

Now instead of first replacing $N(K)$ by T_0 and then replacing $N(dK) = f(T_1)$ by T_2 , first replace T_1 by T_2 and then replace $N(K)$ by T_0 . Then by van Kampen's theorem, the fundamental group of M^3 is the free product with amalgamation $G(K)*G_2$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_2 where G_2 is obtained from G_1 by adding the relation $l_1 = 1$. The group G_2 has the following presentation: $G_2 = \langle l_0, m_1 \mid [l_0, m_0] = 1, m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}], l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}] = 1 \rangle$. If we add the relation $m_1 l_0 = l_0^{-1} m_1$ to G_2 , then $m_1^{-1} l_0 = l_0^{-1} m_1^{-1}$, and it follows that $m_0 = l_0^{-1} m_1 l_0 m_1^{-1} l_0^{-1} m_1^{-1} l_0 m_1 = l_0^{-4}$ and $l_1 = m_1^{-1} l_0 m_1 l_0^{-1} m_1^{-1} l_0^{-1} m_1 l_0 = 1$. Thus the relations $[l_0, m_0] = 1$ and $l_1 = 1$ are consequences of the relation $m_1 l_0 = l_0^{-1} m_1$, and the group $\bar{G}_2 = \langle \bar{l}_0, \bar{m}_1 \mid \bar{m}_1 \bar{l}_0 = \bar{l}_0^{-1} \bar{m}_1 \rangle$ is a quotient group of G_2 . Now the properties of \bar{G}_2 are well-known: \bar{G}_2 is torsion-free and $\bar{l}_0 \neq 1$. Hence, $\bar{m}_0 = \bar{l}_0^{-4} \neq 1$ in \bar{G}_2 , $m_0 \neq 1$ in G_2 , and $m_0 \neq 1$ in $\pi_1(M^3)$. But $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$.

Thus $\pi_1(M^3)$ is not abelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This completes the proof of Theorem 2.

Finally we consider composite knots. A knot K is a composite of nontrivial knots K_1 and K_2 if there is a 2-sphere S^2 and an arc α in S^2 such that (1) $S^2 \cap K = \{x, y\}$ ($x \neq y$) (2) α is an arc from x to y (3) $((\text{Int } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_1 (4) $((\text{Ext } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_2 . The composite knot K is denoted by $K_1 \# K_2$.

If m_i is a meridian of K_i and l_i is a longitude of K_i ($i = 1, 2$), then the group of the composite knot, $G(K_1 \# K_2) = \pi_1(\overline{S^3 - N(K)})$, is the free product with amalgamation $G(K_1) * G(K_2)$ with the identification of subgroups (m_1) of $G(K_1)$ and (m_2) of $G(K_2)$ determined by $m_1 = m_2$. A longitude for $K_1 \# K_2$ is $l = l_1 l_2$. See [3]. By Theorem 1 it suffices to consider composite knots with trivial Alexander polynomial. Such a knot is the composite of two knots each with trivial Alexander polynomial. The following theorem will be proved, however, for arbitrary composite knots.

THEOREM 3. *Elementary surgery along a composite knot does not yield $S^2 \times S^1$.*

Proof. Perform a longitudinal surgery along $K_1 \# K_2$. The fundamental group of the surgered manifold M^3 is obtained by adding the relation $l = 1$ or $l_1 = l_2^{-1}$ to $G(K_1 \# K_2)$. Thus $\pi_1(M^3)$ can be considered as the free product with amalgamation $G(K_1) * G(K_2)$ with the identification of subgroups (l_1, m_1) of $G(K_1)$ and (l_2, m_2) of $G(K_2)$ determined by $l_1 = l_2^{-1}$ and $m_1 = m_2$. Since K_i is nontrivial, $l_i \neq 1$ in $G(K_i)$, and so $l_i \neq 1$ in $\pi_1(M^3)$. But l_i is in the commutator subgroup of $G(K_i)$, so also in the commutator subgroup of $\pi_1(M^3)$. Hence $\pi_1(M^3)$ is nonabelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This establishes Theorem 3.

We conclude with the following conjecture: $S^2 \times S^1$ cannot be obtained by elementary surgery along any nontrivial knot. The proof of this conjecture like the proof of the conjecture, that elementary surgery along a nontrivial knot does not yield a counterexample to the Poincaré Conjecture, seems very difficult.

REFERENCES

1. R. Crowell, *The group G'/G'' of a knot group G* , Duke Math. J., **30** (1963), 349-354.
2. ———, *The annihilator of a knot module*, Proc. Amer. Math. Soc., **15** (1960), 696-700.
3. L. Neuwirth, *Knot Groups*, Ann. Math. Stud. 56, Princeton: Princeton Univ. Press, 1965.

4. M. H. A. Newman and J. H. C. Whitehead, *On the group of a certain linkage*, Quart. J. Math., **8** (1937), 14-21.
5. Dieter Noga, *Über den Aussenraum von Produktknoten und die Bedeutung der Fixgruppen*, Math. Zeitschr., **101** (1967), 131-141.
6. J. H. C. Whitehead, *On doubled knots*, J. London Math. Soc., **12** (1937), 63-71.

Received April 19, 1974.

CALIFORNIA STATE UNIVERSITY, HAYWARD

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1973 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Pacific Journal of Mathematics

Vol. 53, No. 2

April, 1974

Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i>	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i>	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i>	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i>	347
Stephen LaVern Campbell, <i>Linear operators for which T^*T and TT^* commute. II</i>	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i>	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i>	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i>	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in 2^X and $C(X)$</i>	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i>	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i>	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i>	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i>	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i>	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i>	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ...	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i>	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i>	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i>	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot</i>	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i>	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i>	539
Dorte Olesen, <i>Derivations of AW^*-algebras are inner</i>	555
Dorte Olesen and Gert Kjærgaard Pedersen, <i>Derivations of C^*-algebras have semi-continuous generators</i>	563
Duane O'Neill, <i>On conjugation cobordism</i>	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i>	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of $L^1(\mu; E)$</i>	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i>	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i>	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i>	627
Carl E. Swenson, <i>Direct sum subset decompositions of Z</i>	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i>	635
Robert S. Wilson, <i>Representations of finite rings</i>	643