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LOUISE E. MOSER

Elementary surgery along a knot has been used in an attempt to construct a counterexample to the Poincaré Conjecture. Certain classes of knots have been examined, but no counterexample has yet been found. Another, and perhaps as interesting a question, is whether $S^2 \times S^1$ can be obtained by elementary surgery along a knot. In this paper the question is answered in the negative for knots with nontrivial Alexander polynomial, for composite knots, and for a large class of knots with trivial Alexander polynomial—the simply doubled knots.

By a knot we will mean a polygonal simple closed curve in the 3-sphere S^3. A solid torus T is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of T is a torus, a 2-manifold homeomorphic to $S^1 \times S^1$. A meridian of T is a simple closed curve on Bd T which bounds a disk in T but is not homologous to zero on Bd T. A meridianal disk of T is a disk D in T such that $D \cap \text{Bd}\, T = \text{Bd}\, D$, and Bd D is a meridian of T. A longitude of T is a simple closed curve on Bd T which is transverse to a meridian of T and is null-homologous in $\overline{S^3 - T}$.

The basic construction, elementary surgery along a knot, is now described: Let N be a regular neighborhood of a knot K, m an oriented meridianal curve on Bd N, and l an oriented curve on Bd N which is transverse to m and bounds an orientable surface in $\overline{S^3 - N}$. Let T be a solid torus and let $h: T \to N$ be a homeomorphism. Then S^3 is homeomorphic to $\overline{S^3 - N} \cup_{h|\text{Bd}\, T} T$. Now let $h_1: \text{Bd}\, T \to \text{Bd}\, N$ be a homeomorphism with the property that $h^{-1} \cdot h_1: \text{Bd}\, T \to \text{Bd}\, T$ does not extend to a homeomorphism of T onto T. Let $M^3 = \overline{S^3 - N} \cup_{h_1} T$, then we say that M^3 is obtained from S^3 by performing an elementary surgery along K.

Consider now the fundamental group of the complement of the knot $\pi_1(\overline{S^3 - N})$ with base point $m \cap l$, where m and l are considered as elements of $\pi_1(\overline{S^3 - N}) = G$. Then the coset $\bar{m} = mG'$ generates the commutator quotient group $G/G' = H_1(\overline{S^3 - N})$, and the longitude l is in the second commutator subgroup G''. The fundamental group of M^3 is obtained by adjoining the relation $l^p = m^q$ to $\pi_1(\overline{S^3 - N})$ where $pl - qm$ is the image under h_1 of the boundary of a meridianal disk of T, p and q are relatively prime, and $p > 0$. The first homology group of M^3 is generated by $\bar{m}$ with the relation $\bar{m}^q = 1$.

Thus if M^3 is homeomorphic to $S^2 \times S^1$, then $\pi_1(M^3) \simeq H_1(M^3) \simeq Z$. Hence, $q = 0$ and $p = 1$; that is, a longitudinal surgery is performed in which the image of the boundary of a meridianal disk is a longitude. It should be noted that a longitudinal surgery along a trivial knot does yield $S^2 \times S^1$. In the following theorem we give a necessary condition that a surgered manifold be homeomorphic to $S^2 \times S^1$.

THEOREM 1. *If a manifold homeomorphic to $S^2 \times S^1$ results from elementary surgery along a knot K, then the Alexander polynomial of K is trivial.*

Proof. If a surgered manifold M^3 is homeomorphic to $S^2 \times S^1$, then a longitudinal surgery must have been performed. The fundamental group of M^3 is obtained by adding the relation $l = 1$ to $\pi_1(\overline{S^3 - N}) = G$. In other words, $\pi_1(M^3)$ is the quotient group of G by the normal closure of the subgroup generated by l; denote this subgroup by $(l)^c$. Now since $l \in G''$ and G'' is a characteristic subgroup of G', it follows that $(l)^c \leqq G'' \leqq G'$. Thus if G'' is a proper subgroup of G', then $\pi_1(M^3) \neq Z$ and M^3 is not homeomorphic to $S^2 \times S^1$. But G'' is a proper subgroup of G' if and only if the Alexander polynomial of K is nontrivial [1]. This establishes Theorem 1.

So now we consider a large class of nontrivial knots with trivial Alexander polynomial—the simply doubled knots. A simply doubled knot or a doubled knot without twists is defined as follows: Let T_0 be a standardly embedded solid torus in S^3 with meridian m_0 and longitude l_0. Let J be a self-linking simple closed curve in T_0 (as shown in Figure 1 for the trefoil) and let T_1 be a regular neighborhood of J in T_0 with meridian m_1 and longitude l_1. Let K be a nontrivial knot in S^3, $N(K)$ a regular neighborhood of K with meridian m and longitude l which bounds an orientable surface in $\overline{S^3 - N(K)}$. Let $f\colon T_0 \to N(K)$ be a homeomorphism with the property that $f(m_0) = m$ and $f(l_0) = l$, then we say that K is simply doubled to obtain $f(J)$.

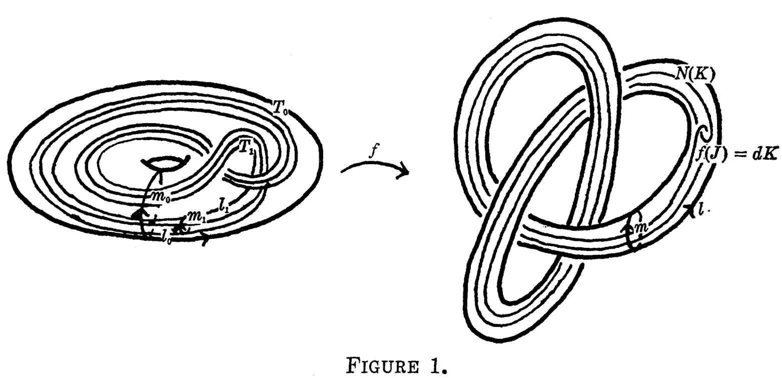

FIGURE 1.

The doubled knot $f(J)$ we will denote by dK.

Consider now the fundamental group of $\overline{T_0 - T_1}$ with base point $m_0 \cap l_0$; let $G_1 = \pi_1(\overline{T_0 - T_1})$ and let $G(K) = \pi_1(\overline{S^3 - N(K)})$. By van Kampen's theorem, the group of the double of K, $G(dK) = \pi_1(\overline{S^3 - N(dK)})$, is the free product with amalgamation $G(K)*G_1$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_1 determined by $l = l_0$ and $m = m_0$. Furthermore, G_1 is generated by l_0 and m_1 subject to the relation $[l_0, m_0] = 1$ where $[x, y] = xyx^{-1}y^{-1}$, $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$, and $l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}]$. See [2].

THEOREM 2. *Elementary surgery along a doubled knot does not yield* $S^2 \times S^1$.

Proof. Perform a longitudinal surgery along dK by replacing the regular neighborhood $f(T_1)$ of dK by a solid torus T_2 to obtain $M^3 = \overline{S^3 - f(T_1)} \cup_h T_2$ where $h: \text{Bd } T_2 \to \text{Bd } f(T_1)$ is a homeomorphism which takes a meridian of T_2 to the longitude $f(l_1)$ of $f(T_1)$.

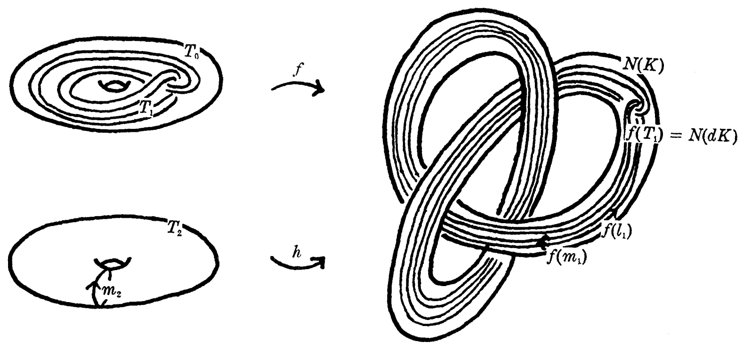

FIGURE 2.

Now instead of first replacing $N(K)$ by T_0 and then replacing $N(dK) = f(T_1)$ by T_2, first replace T_1 by T_2 and then replace $N(K)$ by T_0. Then by van Kampen's theorem, the fundamental group of M^3 is the free product with amalgamation $G(K)*G_2$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_2 where G_2 is obtained from G_1 by adding the relation $l_1 = 1$. The group G_2 has the following presentation: $G_2 = (l_0, m_1 \mid [l_0, m_0] = 1, \quad m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}], \quad l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}] = 1)$. If we add the relation $m_1 l_0 = l_0^{-1} m_1$ to G_2, then $m_1^{-1} l_0 = l_0^{-1} m_1^{-1}$, and it follows that $m_0 = l_0^{-1} m_1 l_0 m_1^{-1} l_0^{-1} m_1^{-1} l_0 m_1 = l_0^{-4}$ and $l_1 = m_1^{-1} l_0 m_1 l_0^{-1} m_1^{-1} l_0^{-1} m_1 l_0 = 1$. Thus the relations $[l_0, m_0] = 1$ and $l_1 = 1$ are consequences of the relation $m_1 l_0 = l_0^{-1} m_1$, and the group $\bar{G}_2 = (\bar{l}_0, \bar{m}_1 \mid \bar{m}_1 \bar{l}_0 = \bar{l}_0^{-1} \bar{m}_1)$ is a quotient group of G_2. Now the properties of $\bar{G}_2$ are well-known: $\bar{G}_2$ is torsion-free and $\bar{l}_0 \neq 1$. Hence, $\bar{m}_0 = \bar{l}_0^{-4} \neq 1$ in $\bar{G}_2$, $m_0 \neq 1$ in G_2, and $m_0 \neq 1$ in $\pi_1(M^3)$. But $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$.

Thus $\pi_1(M^3)$ is not abelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This completes the proof of Theorem 2.

Finally we consider composite knots. A knot K is a composite of nontrivial knots K_1 and K_2 if there is a 2-sphere S^2 and an arc α in S^2 such that (1) $S^2 \cap K = \{x, y\}$ $(x \neq y)$ (2) α is an arc from x to y (3) $((\text{Int } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_1 (4) $((\text{Ext } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_2. The composite knot K is denoted by $K_1 \# K_2$.

If m_i is a meridian of K_i and l_i is a longitude of K_i $(i = 1, 2)$, then the group of the composite knot, $G(K_1 \# K_2) = \pi_1(\overline{S^3 - N(K)})$, is the free product with amalgamation $G(K_1)*G(K_2)$ with the identification of subgroups (m_1) of $G(K_1)$ and (m_2) of $G(K_2)$ determined by $m_1 = m_2$. A longitude for $K_1 \# K_2$ is $l = l_1 l_2$. See [3]. By Theorem 1 it suffices to consider composite knots with trivial Alexander polynomial. Such a knot is the composite of two knots each with trivial Alexander polynomial. The following theorem will be proved, however, for arbitrary composite knots.

THEOREM 3. *Elementary surgery along a composite knot does not yield* $S^2 \times S^1$.

Proof. Perform a longitudinal surgery along $K_1 \# K_2$. The fundamental group of the surgered manifold M^3 is obtained by adding the relation $l = 1$ or $l_1 = l_2^{-1}$ to $G(K_1 \# K_2)$. Thus $\pi_1(M^3)$ can be considered as the free product with amalgamation $G(K_1)*G(K_2)$ with the identification of subgroups (l_1, m_1) of $G(K_1)$ and (l_2, m_2) of $G(K_2)$ determined by $l_1 = l_2^{-1}$ and $m_1 = m_2$. Since K_i is nontrivial, $l_i \neq 1$ in $G(K_i)$, and so $l_i \neq 1$ in $\pi_1(M^3)$. But l_i is in the commutator subgroup of $G(K_i)$, so also in the commutator subgroup of $\pi_1(M^3)$. Hence $\pi_1(M^3)$ is nonabelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This establishes Theorem 3.

We conclude with the following conjecture: $S^2 \times S^1$ cannot be obtained by elementary surgery along any nontrivial knot. The proof of this conjecture like the proof of the conjecture, that elementary surgery along a nontrivial knot does not yield a counterexample to the Poincaré Conjecture, seems very difficult.

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