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**ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY
ELEMENTARY SURGERY ALONG A KNOT**

LOUISE ELIZABETH MOSER

ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY ELEMENTARY SURGERY ALONG A KNOT

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Elementary surgery along a knot has been used in an attempt to construct a counterexample to the Poincaré Conjecture. Certain classes of knots have been examined, but no counterexample has yet been found. Another, and perhaps as interesting a question, is whether $S^2 \times S^1$ can be obtained by elementary surgery along a knot. In this paper the question is answered in the negative for knots with nontrivial Alexander polynomial, for composite knots, and for a large class of knots with trivial Alexander polynomial—the simply doubled knots.

By a knot we will mean a polygonal simple closed curve in the 3-sphere S^3 . A solid torus T is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of T is a torus, a 2-manifold homeomorphic to $S^1 \times S^1$. A meridian of T is a simple closed curve on $\text{Bd } T$ which bounds a disk in T but is not homologous to zero on $\text{Bd } T$. A meridional disk of T is a disk D in T such that $D \cap \text{Bd } T = \text{Bd } D$, and $\text{Bd } D$ is a meridian of T . A longitude of T is a simple closed curve on $\text{Bd } T$ which is transverse to a meridian of T and is null-homologous in $\overline{S^3 - T}$.

The basic construction, elementary surgery along a knot, is now described: Let N be a regular neighborhood of a knot K , m an oriented meridional curve on $\text{Bd } N$, and l an oriented curve on $\text{Bd } N$ which is transverse to m and bounds an orientable surface in $\overline{S^3 - N}$. Let T be a solid torus and let $h: T \rightarrow N$ be a homeomorphism. Then S^3 is homeomorphic to $\overline{S^3 - N} \cup_{h|_{\text{Bd } T}} T$. Now let $h_1: \text{Bd } T \rightarrow \text{Bd } N$ be a homeomorphism with the property that $h^{-1} \cdot h_1: \text{Bd } T \rightarrow \text{Bd } T$ does not extend to a homeomorphism of T onto T . Let $M^3 = \overline{S^3 - N} \cup_{h_1} T$, then we say that M^3 is obtained from S^3 by performing an elementary surgery along K .

Consider now the fundamental group of the complement of the knot $\pi_1(\overline{S^3 - N})$ with base point $m \cap l$, where m and l are considered as elements of $\pi_1(\overline{S^3 - N}) = G$. Then the coset $\bar{m} = mG'$ generates the commutator quotient group $G/G' = H_1(\overline{S^3 - N})$, and the longitude l is in the second commutator subgroup G'' . The fundamental group of M^3 is obtained by adjoining the relation $l^p = m^q$ to $\pi_1(\overline{S^3 - N})$ where $pl - qm$ is the image under h_1 of the boundary of a meridional disk of T , p and q are relatively prime, and $p > 0$. The first homology group of M^3 is generated by \bar{m} with the relation $\bar{m}^q = 1$.

Thus if M^3 is homeomorphic to $S^2 \times S^1$, then $\pi_1(M^3) \simeq H_1(M^3) \simeq Z$. Hence, $q = 0$ and $p = 1$; that is, a longitudinal surgery is performed in which the image of the boundary of a meridional disk is a longitude. It should be noted that a longitudinal surgery along a trivial knot does yield $S^2 \times S^1$. In the following theorem we give a necessary condition that a surgered manifold be homeomorphic to $S^2 \times S^1$.

THEOREM 1. *If a manifold homeomorphic to $S^2 \times S^1$ results from elementary surgery along a knot K , then the Alexander polynomial of K is trivial.*

Proof. If a surgered manifold M^3 is homeomorphic to $S^2 \times S^1$, then a longitudinal surgery must have been performed. The fundamental group of M^3 is obtained by adding the relation $l = 1$ to $\pi_1(\overline{S^3 - N}) = G$. In other words, $\pi_1(M^3)$ is the quotient group of G by the normal closure of the subgroup generated by l ; denote this subgroup by $(l)^\circ$. Now since $l \in G''$ and G'' is a characteristic subgroup of G' , it follows that $(l)^\circ \leq G'' \leq G'$. Thus if G'' is a proper subgroup of G' , then $\pi_1(M^3) \neq Z$ and M^3 is not homeomorphic to $S^2 \times S^1$. But G'' is a proper subgroup of G' if and only if the Alexander polynomial of K is nontrivial [1]. This establishes Theorem 1.

So now we consider a large class of nontrivial knots with trivial Alexander polynomial—the simply doubled knots. A simply doubled knot or a doubled knot without twists is defined as follows: Let T_0 be a standardly embedded solid torus in S^3 with meridian m_0 and longitude l_0 . Let J be a self-linking simple closed curve in T_0 (as shown in Figure 1 for the trefoil) and let T_1 be a regular neighborhood of J in T_0 with meridian m_1 and longitude l_1 . Let K be a nontrivial knot in S^3 , $N(K)$ a regular neighborhood of K with meridian m and longitude l which bounds an orientable surface in $\overline{S^3 - N(K)}$. Let $f: T_0 \rightarrow N(K)$ be a homeomorphism with the property that $f(m_0) = m$ and $f(l_0) = l$, then we say that K is simply doubled to obtain $f(J)$.

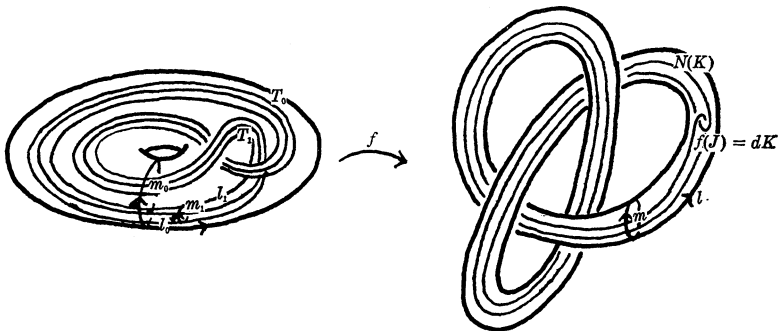


FIGURE 1.

The doubled knot $f(J)$ we will denote by dK .

Consider now the fundamental group of $\overline{T_0 - T_1}$ with base point $m_0 \cap l_0$; let $G_1 = \pi_1(\overline{T_0 - T_1})$ and let $G(K) = \pi_1(\overline{S^3 - N(K)})$. By van Kampen's theorem, the group of the double of K , $G(dK) = \pi_1(\overline{S^3 - N(dK)})$, is the free product with amalgamation $G(K) * G_1$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_1 determined by $l = l_0$ and $m = m_0$. Furthermore, G_1 is generated by l_0 and m_1 subject to the relation $[l_0, m_0] = 1$ where $[x, y] = xyx^{-1}y^{-1}$, $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$, and $l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}]$. See [2].

THEOREM 2. *Elementary surgery along a doubled knot does not yield $S^2 \times S^1$.*

Proof. Perform a longitudinal surgery along dK by replacing the regular neighborhood $f(T_1)$ of dK by a solid torus T_2 to obtain $M^3 = \overline{S^3 - f(T_1)} \cup_h T_2$ where $h: \text{Bd } T_2 \rightarrow \text{Bd } f(T_1)$ is a homeomorphism which takes a meridian of T_2 to the longitude $f(l_1)$ of $f(T_1)$.

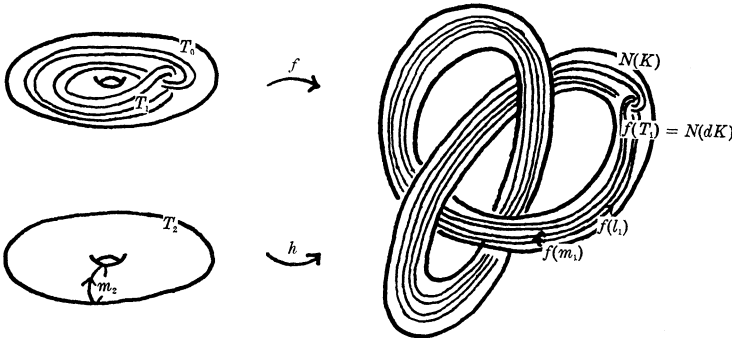


FIGURE 2.

Now instead of first replacing $N(K)$ by T_0 and then replacing $N(dK) = f(T_1)$ by T_2 , first replace T_1 by T_2 and then replace $N(K)$ by T_0 . Then by van Kampen's theorem, the fundamental group of M^3 is the free product with amalgamation $G(K) * G_2$ with the identification of subgroups (l, m) of $G(K)$ and (l_0, m_0) of G_2 where G_2 is obtained from G_1 by adding the relation $l_1 = 1$. The group G_2 has the following presentation: $G_2 = (l_0, m_1 \mid [l_0, m_0] = 1, m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}], l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}] = 1)$. If we add the relation $m_1 l_0 = l_0^{-1} m_1$ to G_2 , then $m_1^{-1} l_0 = l_0^{-1} m_1^{-1}$, and it follows that $m_0 = l_0^{-1} m_1 l_0 m_1^{-1} l_0^{-1} m_1^{-1} l_0 m_1 = l_0^{-4}$ and $l_1 = m_1^{-1} l_0 m_1 l_0^{-1} m_1^{-1} l_0^{-1} m_1 l_0 = 1$. Thus the relations $[l_0, m_0] = 1$ and $l_1 = 1$ are consequences of the relation $m_1 l_0 = l_0^{-1} m_1$, and the group $\bar{G}_2 = (\bar{l}_0, \bar{m}_1 \mid \bar{m}_1 \bar{l}_0 = \bar{l}_0^{-1} \bar{m}_1)$ is a quotient group of G_2 . Now the properties of \bar{G}_2 are well-known: \bar{G}_2 is torsion-free and $\bar{l}_0 \neq 1$. Hence, $\bar{m}_0 = \bar{l}_0^{-4} \neq 1$ in \bar{G}_2 , $m_0 \neq 1$ in G_2 , and $m_0 \neq 1$ in $\pi_1(M^3)$. But $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$.

Thus $\pi_1(M^3)$ is not abelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This completes the proof of Theorem 2.

Finally we consider composite knots. A knot K is a composite of nontrivial knots K_1 and K_2 if there is a 2-sphere S^2 and an arc α in S^2 such that (1) $S^2 \cap K = \{x, y\}$ ($x \neq y$) (2) α is an arc from x to y (3) $((\text{Int } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_1 (4) $((\text{Ext } S^2) \cap K) \cup \alpha$ is a knot of the same type as K_2 . The composite knot K is denoted by $K_1 \# K_2$.

If m_i is a meridian of K_i and l_i is a longitude of K_i ($i = 1, 2$), then the group of the composite knot, $G(K_1 \# K_2) = \pi_1(\overline{S^3 - N(K)})$, is the free product with amalgamation $G(K_1) * G(K_2)$ with the identification of subgroups (m_i) of $G(K_1)$ and (m_2) of $G(K_2)$ determined by $m_1 = m_2$. A longitude for $K_1 \# K_2$ is $l = l_1 l_2$. See [3]. By Theorem 1 it suffices to consider composite knots with trivial Alexander polynomial. Such a knot is the composite of two knots each with trivial Alexander polynomial. The following theorem will be proved, however, for arbitrary composite knots.

THEOREM 3. *Elementary surgery along a composite knot does not yield $S^2 \times S^1$.*

Proof. Perform a longitudinal surgery along $K_1 \# K_2$. The fundamental group of the surgered manifold M^3 is obtained by adding the relation $l = 1$ or $l_1 = l_2^{-1}$ to $G(K_1 \# K_2)$. Thus $\pi_1(M^3)$ can be considered as the free product with amalgamation $G(K_1) * G(K_2)$ with the identification of subgroups (l_1, m_1) of $G(K_1)$ and (l_2, m_2) of $G(K_2)$ determined by $l_1 = l_2^{-1}$ and $m_1 = m_2$. Since K_i is nontrivial, $l_i \neq 1$ in $G(K_i)$, and so $l_i \neq 1$ in $\pi_1(M^3)$. But l_i is in the commutator subgroup of $G(K_i)$, so also in the commutator subgroup of $\pi_1(M^3)$. Hence $\pi_1(M^3)$ is nonabelian, and M^3 is not homeomorphic to $S^2 \times S^1$. This establishes Theorem 3.

We conclude with the following conjecture: $S^2 \times S^1$ cannot be obtained by elementary surgery along any nontrivial knot. The proof of this conjecture like the proof of the conjecture, that elementary surgery along a nontrivial knot does not yield a counterexample to the Poincaré Conjecture, seems very difficult.

REFERENCES

1. R. Crowell, *The group G'/G'' of a knot group G* , Duke Math. J., **30** (1963), 349-354.
2. ———, *The annihilator of a knot module*, Proc. Amer. Math. Soc., **15** (1960), 696-700.
3. L. Neuwirth, *Knot Groups*, Ann. Math. Stud. 56, Princeton: Princeton Univ. Press, 1965.

4. M. H. A. Newman and J. H. C. Whitehead, *On the group of a certain linkage*, Quart. J. Math., **8** (1937), 14-21.
5. Dieter Noga, *Über den Aussenraum von Produktknoten und die Bedeutung der Fixgruppen*, Math. Zeitschr., **101** (1967), 131-141.
6. J. H. C. Whitehead, *On doubled knots*, J. London Math. Soc., **12** (1937), 63-71.

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