

Pacific Journal of Mathematics

DERIVATIONS OF $A W^*$ -ALGEBRAS ARE INNER

DORTE OLESEN

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Using the theory of spectral subspaces associated with a group of isometries of a Banach space it is proved that each derivation of an AW^* -algebra is inner. This constructive method of proof yields a generator b for the case of a skew-adjoint derivation which is seen to be the unique positive generator such that $\|bp\| = \|\delta|Ap\|$ for each central projection p in the AW^* -algebra A .

Introduction. The problem of whether derivations of AW^* -algebras are inner was first studied by I. Kaplansky in [9] and settled in the affirmative for the case of a type I algebra. Later the result was extended to type III algebras and type II factors by G. A. Elliott, and to type II_1 algebras with central trace by J. C. Deel, ([3], [4]). It is not known whether this covers all cases.

The purpose of the present note is to show that each derivation of an AW^* -algebra is inner, avoiding type classification. The method employed is a modification of the one developed by W. B. Arveson in [1], where he proves the corresponding theorem for W^* -algebras. (See also Borchers [2].)

Specifically, we prove that the group of $*$ -automorphisms e^{itb} , where δ is a derivation of the AW^* -algebra A satisfying the condition $\delta(a^*) = -(\delta(a))^*$, is implemented by a unitary group e^{itb} with b a positive element of A .

In § 2 we prove a lemma which establishes a sufficient condition that an element of a C^* -algebra belong to a spectral subspace of the group $u_t \cdot u_t^*$, where $u_t = \int_{\alpha}^{\beta} e^{itz} dp(x)$ with $p(x)$ a given increasing family of projections on $[\alpha, \beta]$. This lemma is a corollary of [1, Theorem 2.3], formulated to suit the present context.

In § 3, we use Lemma 1 and the fact that each subset of an AW^* -algebra has a largest left-annihilating projection inside the algebra to construct an implementing group of unitaries for e^{itb} . The constructive method of proof yields a generator b for δ , which is seen to be the unique positive generator for δ such that $\|bp\| = \|\delta|Ap\|$ for each projection p in the center of A , an observation not made in [1].

I want to thank G. K. Pedersen for his constant encouragement, and G. A. Elliott for his critical comments.

1. Notation. The notation is taken from [1]. For a brief recapitulation, let us look at the special case in which we are interested,

where α_t is a norm-continuous one-parameter group of isometries of a Banach space X . For each f in $L^1(\mathbf{R})$ let $\pi_\alpha(f)$ denote the bounded operator on X given by

$$\pi_\alpha(f) = \int \alpha_t f(t) dt,$$

where the integral exists in the Bochner sense. With $\hat{f}(s) = \int f(t)e^{ist} dt$ and $-\infty \leq t \leq w \leq \infty$ we denote by $R^\alpha(t, w)$ the norm-closed subspace in X generated by the vectors $\pi_\alpha(f)x$ where $x \in X$ and \hat{f} has compact support in (t, w) . Note that since every norm-closed convex set in X is $\sigma(X, X')$ -closed, with X' the dual of X , these subspaces are in fact identical to the ones defined in [1]. The spectral subspace associated with $[t, w]$ is

$$M^\alpha[t, w] = \bigcap_{n \in \mathbf{N}} R^\alpha\left(t - \frac{1}{n}, w + \frac{1}{n}\right).$$

It follows immediately from this definition that

$$\bigcap_{s < t} M^\alpha[s, w] = M^\alpha[t, w]$$

and that the spectral subspaces are invariant under α_t . As shown in [1] we have

$$M^\alpha[t, w] = \{x \in X \mid \pi_\alpha(f)x = 0 \ \forall f \in I_0[t, w]\}$$

where $I_0[t, w]$ denotes the set of function f in $L^1(\mathbf{R})$ such that \hat{f} has support disjoint from $[t, w]$. The existence of an approximate unit (f_λ) in $L^1(\mathbf{R})$ where (\hat{f}_λ) consist of functions with compact support ensures that the above relation also holds if we define $I_0[t, w]$ to be those L^1 -functions f such that \hat{f} has compact support disjoint from $[t, w]$.

THEOREM 2.3. [1] states the following relation: Let α_t, β_t be groups of isometries on X . Denote by φ_t the group on $B(X)$ such that $\varphi_t(a) = \alpha_t \cdot a \cdot \beta_t^{-1}$, all a in $B(X)$. Then

$$aM^\beta[t, \infty) \subseteq M^\alpha[s + t, \infty) \forall t \iff a \in M^\varphi[s, \infty).$$

2. On some unitary groups. Let $t \rightarrow p(t)$ be an increasing projection-valued map from \mathbf{R} into the C^* -algebra A , and assume that there exist α and β in \mathbf{R} , $\alpha \leq \beta$, such that $p(t) = 0$ for all $t \leq \alpha$ and $p(t) = 1$ for all $t \geq \beta$. Let f be a continuous map from $[\alpha, \beta]$ into C . Put

$$s_\pi(f, p) = \sum_{i=1}^n f(t_i)(p(u_i) - p(u_{i-1}))$$

where π denotes the division $\alpha = u_0 \leq u_1 \leq \dots \leq u_n = \beta$ and $t_i \in [u_{i-1}, u_i]$. Then by a well-known theorem the limit of s_π exists and is

$$\int_\alpha^\beta f(t)dp(t) = \lim_{|\pi| \rightarrow 0} s_\pi$$

where $|\pi| = \max |u_i - u_{i-1}|$. Take $f(t) = e^{itx}$ with x in \mathbf{R} , and set

$$\int_\alpha^\beta e^{itx}dp(t) = u_x.$$

Then $x \rightarrow u_x$ is a norm-continuous group of unitary elements of A . In the case where $p(t) = 1$ for all $t > \beta$, u_x as above denotes the common value of the integrals from α to $\beta + \varepsilon$, all $\varepsilon > 0$.

LEMMA 1. Let $u_x = \int_\alpha^\beta e^{itx}dp(t)$ and put $\varphi_x = u_x \cdot u_x^*$. Then for a in A and s in \mathbf{R}

$$p(t + s)a p(t) = p(t + s)a \forall t \in \mathbf{R} \implies a \in M^\varphi[s, \infty).$$

Proof. Assume A to be represented faithfully on a Hilbert space H . By Stone's theorem we know the existence of a unique increasing left-continuous spectral measure $q(t)$ such that

$$u_x = \int e^{itx}dq(t),$$

and from the relations

$$M^u[t_0, \infty) \subset R^u(t, \infty) \subset [(1 - q(t))H] \subset M^u[t, \infty)$$

for all $t_0 > t$ we see that

$$M^u[t, \infty) = [(1 - q(t))H].$$

Now $p(t)$ tends strongly to $q(t_0)$ as $t \nearrow t_0$, and so $p(t + s)a(1 - p(t))$ tends strongly to $q(t_0 + s)a(1 - q(t_0))$ for all a in A . From this it follows that if a satisfies the hypothesis of the lemma, it also satisfies the relation

$$q(t + s)a q(t) = q(t + s)a \forall t \in \mathbf{R},$$

but this is equivalent to

$$aM^u[t, \infty) \subset M^u[t + s, \infty) \forall t \in \mathbf{R},$$

which by [1, Theorem 2.3] implies that $a \in M^\varphi[s, \infty)$.

3. Construction of the generator for δ . Recall that a C^* -algebra A is an AW^* -algebra, (see [8]) if for any subset S of A

there is a unique projection p in A such that

$$\{a \in A \mid as = 0 \forall s \in S\} = Ap .$$

p is called the left-annihilating projection of S .

THEOREM 2. *If δ is a derivation of the AW*-algebra A there is an element b in A such that $\delta = ad_b$. If $\delta = -\delta^*$, b can be chosen positive and with norm equal to the norm of δ .*

Proof. Since each derivation δ has a unique decomposition $\delta = \delta_1 + i\delta_2$, with $\delta_i = \delta_i^*$, it suffices to prove the last statement.

Let $\delta = -\delta^*$. Denote by α_t the *-automorphism group $e^{it\delta}$, and let $p(t)$ be the left-annihilating projection of the spectral subspace $M^\alpha[t, \infty)$. The map $t \rightarrow p(t)$ taking \mathbf{R} into the fixed-point algebra $M^\alpha[0]$ is increasing. As $1 \in M^\alpha[0]$, we have $p(0) = 0$. The claim $p(t) = 1$ for $t > \|\delta\|$ is seen as follows: We want to prove that whenever $f \in L^1(\mathbf{R})$ such that \hat{f} has compact support in $(\|\delta\| + \varepsilon, \infty)$, then $\pi_\alpha(f) = 0$, or equivalently that for all $g \in L^1(\mathbf{R})$ where \hat{g} has compact support in $(0, \infty)$, $\pi_\alpha(g \cdot e^{-i(\|\delta\| + \varepsilon)\cdot}) = 0$.

Now g extends to an H^1 function in the lower half plane if we define

$$g(z) = \frac{1}{2\pi} \int_0^\infty \hat{g}(t)e^{-itz} dt$$

and for the L^1 -norms of $x \rightarrow g_y(x) = g(x + iy)$, y fixed, we have

$$\|g_y\|_1 \leq \|g\|_1$$

(see [6], p. 124–128 and p. 131). Now

$$\begin{aligned} g_y(x) = g(x + iy) &= \frac{1}{2\pi} \int_0^\infty \hat{f}(t + \|\delta\| + \varepsilon)e^{-i(x+iy)t} dt \\ &= \frac{1}{2\pi} \int_{\|\delta\| + \varepsilon}^\infty \hat{f}(w)e^{-izw} e^{+iz(\|\delta\| + \varepsilon)} dw \\ &= e^{+iz(\|\delta\| + \varepsilon)} f(z) = e^{+iz\|\delta\|} e^{-y(\|\delta\| + \varepsilon)} f_y(x) \end{aligned}$$

so

$$\|g_y\|_1 = e^{-y(\|\delta\| + \varepsilon)} \|f_y\|_1$$

from which it follows that

$$e^{-y(\|\delta\| + \varepsilon)} \|f_y\|_1 \leq \|f\|_1$$

and so we get (see [2])

$$\begin{aligned} \left\| \int f(x)\alpha_x dx \right\| &= \left\| \int f(x + iy)\alpha_{x+iy} dx \right\| \\ &\leq e^{y(\|\delta\|+e)} \|f\|_1 \cdot \|e^{i(x+iy)\delta}\| \leq e^{y(\|\delta\|+e)} \|f\|_1 \cdot e^{-y\|\delta\|} \\ &= e^{ye} \|f\|_1 \longrightarrow 0 \quad \text{as } y \longrightarrow -\infty. \end{aligned}$$

According to § 2, the group $u_t = \int e^{itx} dp(x)$ is well-defined. We want to show that it implements α_t , i.e., that

$$\alpha_t = u_t \cdot u_t^* .$$

Denoting the right side by φ_t , it suffices to see that

$$M^\alpha[t, \infty) \subseteq M^\varphi[t, \infty) \quad \forall t \in \mathbf{R} .$$

Indeed, as φ_t and α_t are both norm-continuous one-parameter groups of self-adjoint (i.e., adjoint-preserving) operators on A , the group $\beta_t(\gamma) = \varphi_t \cdot \gamma \cdot \alpha_t^{-1}$, $\gamma \in B(A)$, is a norm-continuous adjoint-preserving group on $B(A)$. It follows that $M^\beta[t, \infty)^* = M^\beta(-\infty, -t]$ for all t in \mathbf{R} , so whenever a self-adjoint element γ in $B(A)$ belongs to $M^\beta[t, \infty)$, γ belongs to $M^\beta[t, -t]$. We know by [1, Theorem 2.3] that the inclusion $M^\alpha[t, \infty) \subseteq M^\varphi[t, \infty)$ implies that $id \in M^\beta[0, \infty)$. The preceding argument shows that id is then in $M^\beta[0]$, so $\varphi_t id \alpha_t^{-1} = id$ for all t , thus $\varphi_t = \alpha_t$.

Using the multiplicative property of α_t a rather straightforward calculation shows that for all t and s in \mathbf{R}

$$R^\alpha(t, \infty)R^\alpha(s, \infty) \subseteq R^\alpha(t + s, \infty) .$$

Indeed, for $f, g \in L^1(\mathbf{R})$ such that \hat{f}, \hat{g} have compact support

$$\begin{aligned} \pi_\alpha(f)x\pi_\alpha(g)y &= \iint f(t)g(u)\alpha_t(x\alpha_{u-t}y)dtdu \\ &= \iint f(t)g(w+t)\alpha_t(x\alpha_wy)dt dw \\ &= \int \left(\int f(t)g_w(t)\alpha_t(x\alpha_wy)dt \right) dw \\ &= \int \left(\int (\hat{f} * \hat{g}_w)^\vee(t)\alpha_t(x\alpha_wy)dt \right) dw \\ &= \int z_w dw . \end{aligned}$$

So if $\text{supp } \hat{f} \subset (t, \infty)$ and $\text{supp } \hat{g} \subset (s, \infty)$ we have $z_w \subset R^\alpha(t + s, \infty)$, as $\text{supp } \hat{f} * \hat{g}_w \subset (t + s, \infty)$ ($g_w(t) = g(t + w)$), so $\hat{g}_w(s) = e^{-isw}\hat{g}(s)$.

From this it follows immediately that for all t and s in \mathbf{R}

$$M^\alpha[t, \infty)M^\alpha[s, \infty) \subseteq M^\alpha[t + s, \infty) ,$$

so if $a \in M^\alpha[t, \infty)$ and $d \in M^\alpha[s, \infty)$ we get that

$$p(t + s)ad = 0 .$$

However, this implies that $p(t + s)a$ belongs to the left-annihilator of $M^\alpha[s, \infty)$, thus

$$p(t + s)a p(s) = p(t + s)a .$$

The desired conclusion now follows from Lemma 1.

The generator for δ thus constructed is

$$b = \int_0^{||\delta||} t dp(t) .$$

It is obvious that $||b|| \leq ||\delta||$. On the other hand, $b - (||b||/2) \cdot 1$ is also a generator for δ , and $||b - (||b||/2) \cdot 1|| = ||b||/2$. So we get that $||b|| = ||\delta||$.

In [5] it is shown that for each inner derivation δ of an AW^* -algebra A there is a unique generator a of norm $||\delta||/2$ such that

$$||\alpha p|| = \frac{1}{2} ||\delta | Ap||$$

for each projection p in the center of A . This generalizes a result in [7] concerning self-adjoint derivations of von Neumann algebras. Here we have the following result:

PROPOSITION 3. *With $\delta = -\delta^*$, the element b in A as constructed above is the unique positive generator for δ such that*

$$||bp|| = ||\delta | Ap||$$

for each projection p in the center C of A . If $\delta = ad_c$, $c \geq 0$, then $c \geq b \geq 0$, so b is the minimal positive generator for δ .

Proof. Let p denote a central projection in A . We want to see that $||bp|| = ||\delta | Ap||$. Since $\alpha_t = e^{it\delta}$ leaves C pointwise invariant it follows from the definition of spectral subspaces that

$$M^\alpha[t, w] \cap Ap = M^{\alpha p}[t, w] ,$$

where αp denotes the group of automorphisms of Ap obtained by restricting the α_t 's to Ap . Consequently the construction carried out in the proof above will produce bp as a generator for δ on Ap . Thus

$$||bp|| = ||\delta | Ap|| .$$

Now assume c to be another positive generator for δ . Using nothing but the fact that b is a positive generator for δ satisfying the above condition on the norm we can prove $c \geq b$, as follows: Since

b and c are both positive generators for δ , the difference $b - c$ is in C . Suppose λ was a positive scalar in $sp(b - c)$. Given a sufficiently small $\varepsilon > 0$ we could then find a nonzero projection p in C such that

$$(b - c)p \geq \varepsilon p .$$

But as cp is a positive generator for $\delta \upharpoonright Ap$ we have, arguing as before

$$\|bp\| = \|\delta \upharpoonright Ap\| \leq \|cp\| ,$$

and combining we get

$$0 \leq cp \leq bp - \varepsilon p \leq (\|bp\| - \varepsilon)p \leq (\|cp\| - \varepsilon)p ,$$

a contradiction. Therefore, $sp(b - c) \subseteq (-\infty, 0)$, i.e., $c - b \geq 0$. The uniqueness of the positive generator satisfying the above norm condition follows from the fact that it is the smallest positive generator.

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Received April 3, 1973.

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The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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